TRANSPORT THEORY

1. An Exact Solution of the Equations of Radiative Transfer / *C. E. Siewert (Middle East Technical Univ., Turkey), P. F. Zweifel (U of Mich.)*

The equations of radiative transfer can, in certain cases, be reduced to equations similar to the equations of neutron transport and they can be solved by techniques similar to those from neutron problems[1]. When the absorption coefficient K_ν, for photons of frequency ν is independent of ν, one speaks of the 'gray case' and it is identical with the one-speed neutron problem. We consider a generalization of the gray case, the 'Uniform Picket Fence Model.' Here K_ν takes two values, K_1 and K_2. Assuming that the frequency spectrum can be divided into ranges $\Delta\nu_i$ over which the Planck function

$$B_\nu(T(z)) = \frac{2h^3}{c^2}\left[e^{h\nu/kT(z)} - 1\right]^{-1} \tag{1}$$

is essentially constant, and that in each $\Delta\nu_i$ the fractional range over which K_ν has the value K_i is constant, we obtain equations for the photon energy density similar to the two-group equations[2]. Thus, we have

$$\mu \frac{\partial}{\partial x} \underset{\sim}{\Psi}(x_1\mu) + \underset{\sim}{\Sigma}\underset{\sim}{\Psi}(x_1\mu) = \underset{\sim}{C}\int_{-1}^{1} \underset{\sim}{\Psi}(x_1\mu')d\mu' . \tag{2}$$

Here $\underset{\sim}{\Psi}(x_1\mu)$ has two components $\psi_i(x_1\mu)$ defined as the integral of the energy density over the range $\delta\nu_i$ in which K_ν has the value K_i . Also

$$(\underset{\sim}{\Sigma})_{ij} = \frac{K_i}{K_2}\,\delta_{ij} \triangleq \sigma_i\delta_{ij} \tag{3}$$

and

$$(\underset{\sim}{C})_{ij} = \frac{1}{2}(\sigma_i\sigma_j w_i)(\sigma_1 w_1 + w_2)^{-1}. \tag{4}$$

Here w_i is the ratio of the integral of $B_\nu(T(z))$ over $\delta\nu_i$ to the integral over all ν. The local temperature is $T(z)$. The physical assumptions above guarantee that w_i are independent of position z. Finally, x is the optical thickness $= K_2 \int^z \rho(z')dz'$. Note that det. $\underset{\sim}{C} = 0$.

To solve Eq. (2), we proceed as in Ref. 1 by assuming eigensolutions of the form $\underset{\sim}{F}(\eta,\mu)e^{-x/\eta}$. We find two discrete modes ($\eta_0 = \pm\,\infty$) and two sets of singular continuum modes on the range $-1 \le \eta \le 1$. For $-\frac{1}{\sigma_1} \le \eta \le \frac{1}{\sigma_1}$ the continuum modes are two-fold degenerate. Using these modes we prove half-range completeness and orthogonality.

The weight function is

$$\underset{\sim}{W}(\mu) = \begin{bmatrix} \sigma_1\gamma(\mu/\sigma_1) & 0 \\ 0 & \gamma(\mu) \end{bmatrix}, \tag{5}$$

where $\gamma(\mu)$ has a similar definition as in Ref. 1, although the dispersion function (whose zeros give the discrete eigenvalues) is used,

$$\Omega(z) = 1 - 2ZC_{11}\tanh^{-1}\left(\frac{1}{\sigma_1 z}\right) - 2ZC_{22}\tanh^{-1}\left(\frac{1}{z}\right). \tag{6}$$

In the orthogonality relation the adjoint eigensolution is used (solution to Eq. (2) with C_{ij} replaced by C_{ji}). We also prove X-function identities appropriate to the two-group problem.

Using these results, typical half-range problems can be solved. For the Milne problem, the extrapolated endpoint is

$$X_0 = \frac{3\sigma_1}{2(\sigma_1 w_1 + w_2)}\left\{w_2\int_0^1 \frac{\mu^2}{X(-\mu)}d\mu + \sigma_1^2 w_1\int_0^{1/\sigma_1}\frac{\mu^2}{X(-\mu)}d\mu\right\}, \tag{7}$$

where $X(z)$ is Case's X-function (thought of as a functual of $\Omega(z)$). We also have explicit results for the temperature distribution and the law of darkening (emergent angular distribution) which are too lengthy to be included in this summary.

The case det. $\underset{\sim}{C} \ne 0$ is of great interest in neutron transport. We have succeeded in proving full-range orthogonality and completeness and have solved the infinite-medium Green's function. However, half-space problems, involving half-range completeness, do not appear to be solvable in closed form. We have, though, been able to obtain an explicit solution for the half-space Green's function for det. $\underset{\sim}{C} = 0$.

A slightly restricted version of the Milne problem has already been solved by Stewart[3] using another technique. However, our method is simpler; it readily solves other half-space problems (e.g. Albedo problem) and is easily generalized to more than two groups. This generalization is reported elsewhere.

1. CASE, K. M., *Ann. Phys.*, (N. Y.), 9, 1 (1960).
2. ŻELAZNY, R., and A. KUSZELL, *Ann. Phys.*, (N. Y.), 16, 81 (1961).
3. STEWART, J. C., unpublished.