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Radiative Transfer in a Free-Electron Atmosphere

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The equations of radiative transfer for an electron-scattering atmosphere, as developed by Chandrasekhar, are solved using Case's method of singular eigensolutions. The eigenfunctions of the set of coupled transport equations have been found. There are two discrete eigenvectors and two linearly independent, degenerate, singular, continuum eigensolutions. These eigenvectors are shown to form a complete basis set for the expansion of arbitrary two-component vectors defined on the half-range. In addition, all of the necessary adjoint functions have been developed so that all expansion coefficients can be obtained by taking scalar products. As an example of the method, the Milne problem is solved and explicit results are obtained for the two components of the polarized radiation field at *any optical depth* in the stellar medium.

I. INTRODUCTION

In a classic paper on radiative transfer published in 1946 (*1*), Chandrasekhar explicitly formulated the equations of transfer for the two components of the polarized radiation field in a free-electron stellar atmosphere. The Thomson scattering of radiation by free electrons was recognized to be an important mechanism in the transfer of energy in a certain class of stars. This fact made necessary a more detailed description of the scattering laws in the formulation of the basic equations. In (*1*), Chandrasekhar provided the new theory and presented an approximate solution for the outgoing angular distribution of the polarized light. In a following work, (*2*), by passing to the infinite limit in a Wick–Chandrasekhar discrete-ordinate procedure, he was able to solve exactly for the laws of darkening in the Milne problem.

As a consequence of the method of singular eigensolutions, developed and introduced by Case (*3*) in 1960, a large class of problems in both radiative-transfer and neutron-transport theory have become amenable to exact, closed-form solution [as for example in Refs. (*4*)–(*8*)].

In this paper, using the basic techniques of normal mode expansion that were initiated by Case, we are able to solve exactly and rigorously the equations of transfer formulated in Ref. 1. Thus our solution to the Milne problem is considerably more general than the ones given earlier (*1*), (*2*), in that we are able to determine the angular distribution of the two components of the radiation intensity at *any optical distance* into the medium. The method of analysis is given in detail because it is intended to serve as a basis for other problems which may be described by the same physical model.[1] The completeness and orthogonality theorems proved here for these eigenvectors are quite general and may be useful in many similar problems.[2] Once these theorems are established, the solutions to actual physical problems follow directly and can be formulated in a simple and canonical manner.

Section II contains a brief review of the physical model considered. A more detailed discussion is given by Chandrasekhar (*1*), (*10*). Section III is devoted to the solution of the homogeneous vector transfer equation and in Section IV these eigensolutions are shown to obey a general half-range completeness theorem. In Section V we present the adjoint functions and prove that they are, in fact, orthogonal to the normal basis set on the half-range. Thus, equipped with the completeness theorem and the adjoint functions, we solve, in Section VI, the classical Milne problem. Although the Milne problem is of primary concern here, the method for solving any of the standard half-space problems is indicated.

We believe that our method has merit because (a) it is simpler than the discrete-ordinate procedure taken in its infinite limit, (b) it gives the complete solution throughout the entire medium, and (c) it is easily adapted to the other half-space problems.

II. THE EQUATIONS OF TRANSFER

In this section we formulate the equations that describe the flow of radiant energy through a semi-infinite half-space that is considered to represent a stellar atmosphere. The source of radiation is assumed to be located at an infinite depth within the medium so that no extraneous source terms will appear in the balance equations. Further, in this model it is assumed that the only mechanism for interactions between the radiation field and the stellar medium is that of Thomson scattering by free electrons.[3] In the scattering medium considered, the radiation field is adequately described by two perpendicularly polarized intensities, $\Psi_1(x, \mu)$ and $\Psi_2(x, \mu)$. Here x is the optical distance into the medium measured in units of the Thomson scattering coefficient and μ is the direction cosine of the directed pencil of radiation. We measure μ from the *inward normal* of the free

[1] The albedo problem and the half-space Green's function are solved in a forthcoming paper by O. J. Smith and C. E. Siewert.

[2] Simmons (*9*) has recently investigated the full-range completeness and orthogonality theorems.

[3] Since the Thomson scattering cross section is independent of energy, we need not consider the intensity of the radiation to be frequency dependent. Thus the intensities that we use are the integral sums over all frequencies.

surface.[4] The intensities, $\Psi_1(x, \mu)$ and $\Psi_2(x, \mu)$, refer respectively to the states of polarization where the electric vector vibrates along and perpendicularly to the principal meridian. Thus $2\pi\Psi_i(x, \mu)\, dx\, d\mu$ represents the total radiant energy of polarization state i contained in optical position dx at x and in solid angle $d\mathbf{\Omega} = 2\pi\, d\mu$ about $\mathbf{\Omega}$.

Writing the two balance equations for a differential element of phase space, we have

$$\mu \frac{\partial}{\partial x} \Psi_1(x, \mu) + \Psi_1(x, \mu) = S_{11}(x, \mu) + S_{12}(x, \mu) \tag{1a}$$

and

$$\mu \frac{\partial}{\partial x} \Psi_2(x, \mu) + \Psi_2(x, \mu) = S_{21}(x, \mu) + S_{22}(x, \mu). \tag{1b}$$

Here $S_{ij}(x, \mu)$ represents the scattering source term from state j into state i. Unlike the assumption often made in the theory of neutron transport, the scattering here cannot be taken to be isotropic. Chandrasekhar, in Ref. *1*, investigates these source terms for the Thomson scattering kernel; we do not repeat his analysis here but simply state the results:

$$S_{11}(x, \mu) = 3/8 \int_{-1}^{1} \Psi_1(x, \mu')[2(1 - \mu'^2) + \mu^2(3\mu'^2 - 2)]\, d\mu', \tag{2a}$$

$$S_{12}(x, \mu) = 3/8\, \mu^2 \int_{-1}^{1} \Psi_2(x, \mu')\, d\mu', \tag{2b}$$

$$S_{21}(x, \mu) = 3/8 \int_{-1}^{1} \Psi_1(x, \mu')\mu'^2\, d\mu', \tag{2c}$$

and

$$S_{22}(x, \mu) = 3/8 \int_{-1}^{1} \Psi_2(x, \mu')\, d\mu'. \tag{2d}$$

Thus the two equations of transfer are

$$\mu \frac{\partial}{\partial x} \Psi_1(x, \mu) + \Psi_1(x, \mu) = \frac{3}{8} \int_{-1}^{1} \Psi_1(x, \mu') \cdot [2(1 - \mu'^2) + \mu^2(3\mu'^2 - 2)]\, d\mu' + \frac{3}{8}\mu^2 \int_{-1}^{1} \Psi_2(x, \mu')\, d\mu', \tag{3a}$$

and

$$\mu \frac{\partial}{\partial x} \Psi_2(x, \mu) + \Psi_2(x, \mu) = \frac{3}{8} \int_{-1}^{1} \Psi_1(x, \mu')\mu'^2\, d\mu' + \frac{3}{8} \int_{-1}^{1} \Psi_2(x, \mu')\, d\mu'. \tag{3b}$$

[4] It is customary in astrophysics to measure this direction from the outward normal rather than from the inward one. We select the latter here because Case's method of normal modes, which we use, is developed in this notation. We also use the symbols $\Psi_1(x, \mu)$ and $\Psi_2(x, \mu)$ rather than the corresponding $I_l(x, \mu)$ and $I_r(x, \mu)$.

We prefer to write Eqs. (3) in matrix notation; defining

$$\mathbf{K}(\mu, \mu') = \frac{3}{4}\begin{bmatrix} 2(1 - \mu'^2)(1 - \mu^2) + \mu'^2\mu^2 & \mu^2 \\ \mu'^2 & 1 \end{bmatrix}, \tag{4}$$

we have

$$\mu \frac{\partial}{\partial x} \mathbf{\Psi}(x, \mu) + \mathbf{\Psi}(x, \mu) = \frac{1}{2}\int_{-1}^{1} \mathbf{K}(\mu, \mu')\mathbf{\Psi}(x, \mu')\, d\mu'. \tag{5}$$

Here $\mathbf{\Psi}(x, \mu)$ is a two-component vector with elements $\Psi_1(x, \mu)$ and $\Psi_2(x, \mu)$, and $\mathbf{K}(\mu, \mu')$ is the transfer matrix; we note $\tilde{\mathbf{K}}(\mu', \mu) = \mathbf{K}(\mu, \mu')$. The superscript tilde denotes the transpose operation. We now turn to the solution of Eq. (5).

III. EIGENVALUES AND EIGENFUNCTIONS

Noting that Eq. (5) possesses translational invariance, we are led to solutions of the type

$$\mathbf{\Psi}_\eta(x, \mu) = e^{-x/\eta}\mathbf{\Phi}(\eta, \mu), \tag{6}$$

where the permissible values of η and the functions $\mathbf{\Phi}(\eta, \mu)$ are to be determined. Substituting the *ansatz* above into Eq. (5), we obtain

$$(\eta - \mu)\mathbf{\Phi}(\eta, \mu) = \frac{\eta}{2}\int_{-1}^{1} \mathbf{K}(\mu, \mu')\mathbf{\Phi}(\eta, \mu')\, d\mu'. \tag{7}$$

If Eq. (7) is multiplied by $P_\beta(\mu)$, the Legendre polynomial of order β, and integrated over μ from -1 to 1, the following set of equations results:

$$\begin{aligned} \eta\mathbf{M}_\beta(\eta) - \frac{1}{2\beta + 1}[(\beta + 1)\mathbf{M}_{\beta+1}(\eta) + \beta\mathbf{M}_{\beta-1}(\eta)] \\ = \frac{\eta}{2}[\mathbf{A}_1\mathbf{M}_0(\eta)\delta_{\beta 0} + \mathbf{A}_2\mathbf{M}_2(\eta)\delta_{\beta 0} + \mathbf{A}_3\mathbf{M}_0(\eta)\delta_{\beta 2} + \mathbf{A}_4\mathbf{M}_2(\eta)\delta_{\beta 2}] \\ \cdot \frac{2}{2\beta + 1}, \qquad \beta = 0, 1, 2, \cdots . \end{aligned} \tag{8}$$

Here we have made use of the fact that $\mathbf{K}(\mu, \mu')$ can be written in bilinear form (*9*), i.e.,

$$\mathbf{K}(\mu, \mu') = \mathbf{A}_1 + \mathbf{A}_3P_2(\mu) + P_2(\mu')[\mathbf{A}_2 + \mathbf{A}_4P_2(\mu)], \tag{9}$$

where

$$\begin{aligned} \mathbf{A}_1 &= \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, & \mathbf{A}_2 &= \frac{1}{4}\begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix}, \\ \mathbf{A}_3 &= \frac{1}{4}\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}, \quad \text{and} & \mathbf{A}_4 &= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{10}$$

In addition to the orthogonality properties of the Legendre polynomials, we have used the relation

$$\mu P_\beta(\mu) = \frac{(\beta + 1)P_{\beta+1}(\mu) + \beta P_{\beta-1}(\mu)}{2\beta + 1}. \tag{11}$$

Also, we have defined

$$\mathbf{M}_\beta(\eta) \triangleq \int_{-1}^{1} d\mu P_\beta(\mu)\mathbf{\Phi}(\eta, \mu). \tag{12}$$

Equation (7) can be written explicitly in terms of the $\mathbf{M}_\beta(\eta)$; we find

$$(\eta - \mu)\mathbf{\Phi}(\eta, \mu) = \frac{\eta}{2}\{[\mathbf{A}_1 + \mathbf{A}_3 P_2(\mu)]\mathbf{M}_0(\eta) + [\mathbf{A}_2 + \mathbf{A}_4 P_2(\mu)]\mathbf{M}_2(\eta)\}. \tag{13}$$

The vector $\mathbf{M}_2(\eta)$ can be found in terms of $\mathbf{M}_0(\eta)$ from Eq. (8) and the result can be substituted into Eq. (13). The equation for $\mathbf{\Phi}(\eta, \mu)$ takes a somewhat more tractable form:[5]

$$(\eta - \mu)\mathbf{\Phi}(\eta, \mu) = (\eta/2)\mathbf{K}_\eta(\mu)\mathbf{M}_0(\eta), \tag{14}$$

where

$$\mathbf{K}_\eta(\mu) = \begin{bmatrix} 1 - P_2(\mu) & \dfrac{1}{4 - 3\eta^2}[1 + (2 - 3\eta^2)P_2(\mu)] \\ 0 & \dfrac{3(1 - \eta^2)}{4 - 3\eta^2} \end{bmatrix}. \tag{15}$$

The vector $\mathbf{M}_0(\eta)$ in the right-hand side of Eq. (14) is as yet unknown; it is the normalization of the eigenvectors as indicated by Eq. (12). Since the factor $(\eta - \mu)$ multiplying $\mathbf{\Phi}(\eta, \mu)$ can vanish for values of η equal to μ, we must consider the solution of Eq. (14) for two distinct regions of the eigenvalue η.

If we restrict η not to lie on the real line $[-1, 1]$, the solution for $\mathbf{\Phi}(\eta, \mu)$ can be written as

$$\mathbf{\Phi}(\eta, \mu) = \frac{\eta}{2}\frac{1}{\eta - \mu}\mathbf{K}_\eta(\mu)\mathbf{M}_0(\eta). \tag{16}$$

Integrating the above over μ from -1 to 1, we find

$$\mathbf{M}_0(\eta) = (\eta/2)\mathbf{K}_\eta\mathbf{M}_0(\eta), \tag{17}$$

where

$$\mathbf{K}_\eta \triangleq \int_{-1}^{1} \mathbf{K}_\eta(\mu)\frac{d\mu}{\eta - \mu}, \qquad \eta \notin [-1, 1]. \tag{18}$$

[5] The factor $(4 - 3\eta^2)^{-1}$ in $\mathbf{K}_\eta(\mu)$ is to be noted since for $\eta \notin [-1, 1]$ the possibility of a singularity appears to be present. It turns out, however, that $\eta = \pm\infty$ are the only discrete eigenvalues.

Since Eq. (17) is a set of homogeneous equations for the two unknown components of $\mathbf{M}_0(\eta)$, we require that

$$\det\left[\mathbf{I} - \frac{\eta}{2}\mathbf{K}_\eta\right] = 0. \tag{19}$$

Expanding Eq. (19) we obtain the condition,

$$3(1 - \eta^2)[1 - \eta T(1/\eta)] = 1, \qquad \eta \notin [-1, 1], \tag{20}$$

i.e., the discrete eigenvalues are the zeros of the dispersion function

$$\Omega(z) \triangleq -1 + 3(1 - z^2)[1 - zT(1/z)]. \tag{21}$$

Here we use the abbreviation $T(x)$ for $\tanh^{-1}(x)$. Also, the symbol $\mathbf{I}$ denotes the unit matrix. It is obvious that $\Omega(z)$ has zeros occurring in $\pm$ and conjugate pairs. In Appendix A, we prove that $\Omega(z)$ has only two zeros, which are easily seen to be $\eta = \pm\infty$. Taking the limit $\eta \to \infty$ in Eq. (16), we note that $\mathbf{\Phi}(\eta, \mu)$ is independent of μ. It is, in fact, the constant vector,

$$\mathbf{\Phi}_+(\eta, \mu) = \mathbf{\Psi}_+(x, \mu) = \begin{bmatrix}1\\1\end{bmatrix}. \tag{22}$$

Because of the degeneracy of these discrete solutions, the one-dimsenional representation, $e^{-x/\eta}$, is no longer adequate. Since we are dealing with a twofold degeneracy, the two-dimensional representation is appropriate. It can be generated by the basis set (7),

$$Y_1(x, \eta) = e^{-x/\eta} \to 1, \tag{23a}$$

$$Y_2(x, \eta) = xe^{-x/\eta} \to x, \tag{23b}$$

where the arrows indicate the limit for $\eta \to \infty$. The appropriate linear combination which satisfies Eq. (5) is found to be

$$\mathbf{\Psi}_-(x, \mu) = (x - \mu)\begin{bmatrix}1\\1\end{bmatrix}. \tag{24}$$

The two discrete solutions are thus obtained. It should be noted that $\mathbf{\Psi}_+(x, \mu)$ satisfies Eqs. (5) and (7), whereas $\mathbf{\Psi}_-(x, \mu)$ is a solution of Eq. (5) *only*.

We now proceed to find the solutions to Eq. (14) for $\eta \in [-1, 1]$. These are the so-called continuum eigensolutions and they are, in fact, distributions rather than functions in L^2 space. The general solution for $\mathbf{\Phi}(\eta, \mu)$, $\eta \in [-1, 1]$, can thus be written

$$\mathbf{\Phi}(\eta, \mu) = \left[\frac{\eta}{2}\frac{P}{\eta - \mu} + \lambda(\eta)\delta(\eta - \mu)\right]\mathbf{K}_\eta(\mu)\mathbf{M}_0(\eta). \tag{25}$$

Here the symbol P denotes that all integrals involving these solutions are to be done in the Cauchy principal value sense. Integrating Eq. (25), we find

$$\left[\mathbf{I} - \frac{\eta}{2}\mathbf{K}_\eta^P + \lambda(\eta)\mathbf{K}_\eta(\eta)\right]\mathbf{M}_0(\eta) = 0, \tag{26}$$

where

$$\mathbf{K}_\eta^P \triangleq P\int_{-1}^{1} \mathbf{K}_\eta(\mu)\,\frac{d\mu}{\eta-\mu}. \tag{27}$$

We find

$$\mathbf{K}_\eta^P = \begin{bmatrix} 3\eta + 3(1-\eta^2)T(\eta) & \dfrac{1}{4-3\eta^2}[-3\eta(2-3\eta^2) + 9\eta^2(1-\eta^2)T(\eta)] \\ 0 & \dfrac{6(1-\eta^2)T(\eta)}{4-3\eta^2} \end{bmatrix}. \tag{28}$$

Since Eq. (26) is a homogeneous equation for $\mathbf{M}_0(\eta)$, we set the determinant of the coefficient matrix equal to zero. This determines two allowable $\lambda(\eta)$ functions which in turn determine two linearly independent $\mathbf{M}_0(\eta)$. Thus we find a twofold degeneracy in the continuum solutions. A similar degeneracy was found by Siewert and Zweifel (*4*), (*5*). We choose to write the two degenerate continuum solutions in the form

$$\mathbf{\Phi}_1(\eta,\mu) = \begin{bmatrix} \dfrac{3\eta}{2}(1-\mu^2)\dfrac{P}{\eta-\mu} + \lambda_1(\eta)\delta(\eta-\mu) \\ 0 \end{bmatrix} \tag{29a}$$

and

$$\mathbf{\Phi}_2(\eta,\mu) = \begin{bmatrix} -\dfrac{3\eta}{2}(\eta+\mu) \\ \dfrac{3\eta}{2}(1-\eta^2)\dfrac{P}{\eta-\mu} + \lambda_2(\eta)\delta(\eta-\mu) \end{bmatrix}, \tag{29b}$$

where

$$\lambda_1(\eta) = -1 + 3(1-\eta^2)[1-\eta T(\eta)] \tag{30a}$$

and

$$\lambda_2(\eta) = 1 + 3(1-\eta^2)[1-\eta T(\eta)]. \tag{30b}$$

All of the eigensolutions are thus determined. In the next section we prove that these eigenvectors obey a rather remarkable completeness theorem.

IV. COMPLETENESS OF EIGENSOLUTIONS

THEOREM I. *The eigensolutions* $\mathbf{\Phi}_+$, $\mathbf{\Phi}_1(\eta, \mu)$ *and* $\mathbf{\Phi}_2(\eta, \mu)$ *are complete on the half-range,* $\mu \in [0, 1]$, *in the sense that an arbitrary two-component vector* $\mathbf{\Psi}(\mu)$ *defined for* $0 \leqq \mu \leqq 1$ *can be expanded in the form*

$$\mathbf{\Psi}(\mu) = A_+\mathbf{\Phi}_+ + \int_0^1 \alpha(\eta)\mathbf{\Phi}_1(\eta, \mu)\, d\eta + \int_0^1 \beta(\eta)\mathbf{\Phi}_2(\eta, \mu)\, d\eta. \tag{31}$$

To prove the theorem, we show that a solution to the singular integral equation above exists. This is done by solving Eq. (31) using the methods of Muskhelishvili (*11*). We begin by attempting to expand an arbitrary vector $\mathbf{\Psi}'(\mu)$ in terms of the continuum modes alone. We will find that this expansion is valid as long as a certain restriction is placed on $\mathbf{\Psi}'(\mu)$. This restriction is easily removed by simply introducing the discrete term. Let us therefore consider

$$\mathbf{\Psi}'(\mu) = \int_0^1 \alpha(\eta)\mathbf{\Phi}_1(\eta, \mu)\, d\eta + \int_0^1 \beta(\eta)\mathbf{\Phi}_2(\eta, \mu)\, d\eta, \qquad \mu \in [0, 1]. \tag{32}$$

Substituting the expressions for the $\mathbf{\Phi}_i(\eta, \mu)$ given by Eqs. (29), we obtain the following two equations:

$$\begin{aligned}\Psi_1'(\mu) = \int_0^1 \alpha(\eta) \left[\frac{3\eta}{2}(1 - \mu^2)\frac{P}{\eta - \mu} + \lambda_1(\eta)\delta(\eta - \mu)\right] d\eta \\ - \frac{3}{2}\int_0^1 \beta(\eta)\eta(\eta + \mu)\, d\eta\end{aligned} \tag{33}$$

and

$$\Psi_2'(\mu) = \int_0^1 \beta(\eta)\left[\frac{3\eta}{2}(1 - \eta^2)\frac{P}{\eta - \mu} + \lambda_2(\eta)\delta(\eta - \mu)\right] d\eta. \tag{34}$$

Since Eq. (34) contains only one unknown, $\beta(\eta)$, our procedure will be to solve firstly for $\beta(\eta)$; this result can then be used in Eq. (33), thus leaving $\alpha(\eta)$ as the only unknown. Integrating the delta function term in Eq. (34), we find

$$\Psi_2'(\mu) = \frac{3}{2}\int_0^1 \beta(\eta)\eta(1 - \eta^2)\frac{P}{\eta - \mu}\, d\eta + \lambda_2(\mu)\beta(\mu). \tag{35}$$

We now introduce the auxiliary function

$$N_2(z) \triangleq \frac{1}{2\pi i}\int_0^1 \eta(1 - \eta^2)\beta(\eta)\frac{d\eta}{\eta - z}. \tag{36}$$

Taking the limit of $N_2(z)$ as z approaches the branch cut $[0, 1]$ from above and below, we obtain the boundary values,

$$N_2^{\pm}(\mu) = \frac{1}{2\pi i} P \int_0^1 \beta(\eta)\eta(1-\eta^2)\frac{d\eta}{\eta-\mu} \pm \frac{\mu}{2}(1-\mu^2)\beta(\mu). \quad (37)$$

Thus

$$N_2^{+}(\mu) - N_2^{-}(\mu) = \mu(1-\mu^2)\beta(\mu) \quad (38a)$$

and

$$N_2^{+}(\mu) + N_2^{-}(\mu) = \frac{1}{\pi i} P \int_0^1 \beta(\eta)\eta(1-\eta^2)\frac{d\eta}{\eta-\mu}. \quad (38b)$$

It is also convenient to introduce the function

$$\Lambda(z) \triangleq 1 + 3(1-z^2)\left[1 + \frac{z}{2}\int_{-1}^{1}\frac{d\eta}{\eta - z}\right]. \quad (39)$$

Thus $\Lambda(z)$ is analytic in the complex plane cut from -1 to 1 on the real axis and its boundary values are

$$\Lambda^{\pm}(\mu) = 1 + 3(1-\mu^2)\left[1 - \mu T(\mu) \pm \frac{\pi i}{2}\mu\right]. \quad (40)$$

Adding and subtracting these boundary values, we find

$$\lambda_2(\mu) = \frac{\Lambda^{+}(\mu) + \Lambda^{-}(\mu)}{2} \quad (41a)$$

and

$$3\pi i\mu(1-\mu^2) = \Lambda^{+}(\mu) - \Lambda^{-}(\mu). \quad (41b)$$

With the aid of Eqs. (38) and (41), we now write Eq. (35) as

$$(1-\mu^2)\mu\Psi_2'(\mu) = N_2^{+}(\mu)\Lambda^{+}(\mu) - N_2^{-}(\mu)\Lambda^{-}(\mu). \quad (42)$$

The $N_2(z)$ function has its branch cut from 0 to 1, whereas the $\Lambda(z)$ branch cut is $[-1, 1]$; thus the right-hand side of Eq. (42) is not the difference of the boundary values of an analytic function on the same domain. We therefore introduce a new function $Y(z)$ which is to be analytic in the complex plane cut on the line $[0, 1]$ and whose boundary values satisfy the ratio condition, i.e.,

$$\frac{Y^{+}(\mu)}{Y^{-}(\mu)} = \frac{\Lambda^{+}(\mu)}{\Lambda^{-}(\mu)}, \qquad \mu \in [0, 1]. \quad (43)$$

Equation (43) is a homogeneous Hilbert problem (*11*); the solution is

$$Y(z) = \exp\left[\frac{1}{\pi}\int_0^1 \arg \Lambda^{+}(\mu)\frac{d\mu}{\mu - z}\right]. \quad (44)$$

In terms of the $Y(z)$ function, Eq. (42) can be written as

$$(1 - \mu^2)\gamma_2(\mu)\Psi_2'(\mu) = N_2^+(\mu)Y^+(\mu) - N_2^-(\mu)Y^-(\mu), \tag{45}$$

where

$$\gamma_2(\mu) \triangleq \mu[Y^+(\mu)/\Lambda^+(\mu)]. \tag{46}$$

The right-hand side of Eq. (45) is now written in terms of analytic functions with a common branch cut. The solution for this special case of the inhomogeneous Hilbert problem is

$$N_2(z) = \frac{1}{2\pi i Y(z)} \int_0^1 (1 - \mu^2)\gamma_2(\mu)\Psi_2'(\mu) \frac{d\mu}{\mu - z}. \tag{47}$$

The fact that the arg $\Lambda^+(\mu)$ is zero at *both* of the end points of the cut [0, 1] insures that the properties of $Y(z)$ are correct as it stands (*11*). Since $Y(z)$ is nonvanishing in the cut plane, $N_2(z)$ is obviously analytic in the cut plane; this is in agreement with our original definition of $N_2(z)$ in Eq. (36). Also $N_2(z)$ vanishes as $1/z$ for large z. This property also is as prescribed by Eq. (36). We therefore conclude that $N_2(z)$ as given by Eq. (47) is correct. It follows that $\beta(\eta)$ is known for it can be obtained from the boundary values of Eq. (47) according to the form of Eq. (38a).

We now turn to the solution of Eq. (33). Since $\beta(\eta)$ is known, $\alpha(\eta)$ is the quantity to be determined. Integrating the delta function term, we write Eq. (33) as

$$\Psi_1'(\mu) + g(\mu) = \frac{3}{2}(1 - \mu^2) \int_0^1 \eta\alpha(\eta) \frac{P}{\eta - \mu} d\eta + \lambda_1(\mu)\alpha(\mu), \tag{48}$$

where

$$g(\mu) \triangleq 3/2 \int_0^1 \eta(\eta + \mu)\beta(\eta)\, d\eta. \tag{49}$$

The dispersion function $\Omega(z)$, given by Eq. (21), can alternately be written

$$\Omega(z) = -1 + 3(1 - z^2)\left[1 + \frac{z}{2}\int_{-1}^1 \frac{d\mu}{\mu - z}\right]. \tag{50}$$

The boundary values of $\Omega(z)$ on the branch cut $[-1, 1]$ satisfy

$$\Omega^\pm(\mu) = -1 + 3(1 - \mu^2)\left[1 - \mu T(\mu) \pm \frac{\pi i}{2}\mu\right], \tag{51}$$

$$\frac{\Omega^+(\mu) + \Omega^-(\mu)}{2} = \lambda_1(\mu), \tag{52}$$

and

$$\Omega^+(\mu) - \Omega^-(\mu) = 3\pi i\mu(1 - \mu^2). \tag{53}$$

Continuing to parallel the solution for $\beta(\eta)$, we introduce

$$N_1(z) \triangleq \frac{1}{2\pi i} \int_0^1 \eta\alpha(\eta) \frac{d\eta}{\eta - z}, \tag{54}$$

such that

$$N_1^+(\mu) - N_1^-(\mu) = \mu\alpha(\mu) \tag{55}$$

and

$$N_1^+(\mu) + N_1^-(\mu) = \frac{1}{\pi i} P \int_0^1 \eta\alpha(\eta) \frac{d\eta}{\eta - \mu}. \tag{56}$$

With these facts we are able to write Eq. (48) in the form

$$\mu[\Psi_1'(\mu) + g(\mu)] = N_1^+(\mu)\Omega^+(\mu) - N_1^-(\mu)\Omega^-(\mu). \tag{57}$$

Here again the branch cut of $\Omega(z)$ is not the same as that of the $N_1(z)$ function. We therefore define a function $X(z)$ analytic in the complex plane cut from 0 to 1 and whose boundary values satisfy

$$\frac{X^+(\mu)}{X^-(\mu)} = \frac{\Omega^+(\mu)}{\Omega^-(\mu)}, \qquad \mu \in [0, 1]. \tag{58}$$

The solution to Eq. (58) is

$$X(z) = \frac{1}{1 - z} \exp\left[\frac{1}{\pi} \int_0^1 \arg \Omega^+(\mu) \frac{d\mu}{\mu - z}\right] (1 - z)^{-1}. \tag{59}$$

Here the factor $(1 - z)^{-1}$ had to be introduced in $X(z)$ because $\arg \Omega^+(\mu) \to \pi$ as $\mu \to 1$, i.e., the end point 1 is a special type (*11*).

In the usual manner, the solution to Eq. (57) is

$$N_1(z) = \frac{1}{2\pi i X(z)} \int_0^1 \gamma_1(\mu)[\Psi_1'(\mu) + g(\mu)] \frac{d\mu}{\mu - z}, \tag{60}$$

where

$$\gamma_1(\mu) \triangleq \mu[X^+(\mu)/\Omega^+(\mu)]. \tag{61}$$

Inspection of Eq. (60) indicates that $N_1(z)$ has the proper analytic properties in the finite plane. However, since $X(z) \sim 1/z$ as z approaches infinity, $N_1(z)$ does not have the correct behavior at infinity, for arbitrary $\Psi_1'(\mu)$. This is corrected by introducing the discrete mode.[6] We thus write

$$\Psi_i'(\mu) = \Psi_i(\mu) - A_+, \tag{62}$$

[6] It is indeed fortunate that the $Y(z)$ function did not require this modification since we would have had to introduce the discrete solution in the $\beta(\eta)$ equation. Thus when we needed the discrete solution in the $\alpha(\eta)$ equation the coefficient A_+ would have already been determined.

where the $\Psi_i'(\mu)$ are the components of the arbitrary function to be expanded; and if we require that

$$\int_0^1 \gamma_1(\mu)[\Psi_1'(\mu) + g(\mu)]\, d\mu = 0, \tag{63}$$

$N_1(z)$ is assured of vanishing as $1/z$ for large z. Therefore the coefficient A_+ is given by

$$A_+ = \frac{\int_0^1 \gamma_1(\mu)[\Psi_1(\mu) + g(\mu)]\, d\mu}{\int_0^1 \gamma_1(\mu)\, d\mu}. \tag{64}$$

The expansion coefficient, $\alpha(\eta)$, can be determined from $N_1(z)$ as given by Eq. (60). Thus all of the expansion coefficients have been found. The theorem is therefore proved.[7] We could explicitly obtain the coefficients from this completeness proof since it amounts to a solution of the initial equations. We prefer to use the orthogonality relations developed in the next section.

V. THE ADJOINT SOLUTIONS AND NORMALIZATION INTEGRALS

THEOREM II. *The eigensolutions $\mathbf{\Phi}_+$, $\mathbf{\Phi}_1(\eta, \mu)$ and $\mathbf{\Phi}_2(\eta, \mu)$ have corresponding half-range adjoint solutions, $\mathbf{\Phi}_+^\dagger$, $\mathbf{\Phi}_1^\dagger(\eta, \mu)$ and $\mathbf{\Phi}_2^\dagger(\eta, \mu)$, such that*

$$\int_0^1 \tilde{\mathbf{\Phi}}^\dagger(\eta', \mu)\mathbf{\Phi}(\eta, \mu)\, d\mu = 0, \qquad \eta \neq \eta'. \tag{65}$$

The adjoint solutions are given by Eqs. (66).

We have found the adjoint eigenvectors, $\mathbf{\Phi}^\dagger(\eta, \mu)$, without the necessity of developing the adjoint equation; the proof of the orthogonality theorem, therefore, is not given in the usual manner (*4*), (*5*). We will simply present our adjoint solutions and then proceed to show that they, in fact, satisfy Eq. (65). This procedure has merit since it will not only prove the theorem but will also provide the necessary normalization integrals.

In order that the adjoint solutions that we use do not appear to be apocalyptical, we briefly state their origin. As already mentioned in the completeness proof given in Section IV, the solution to the set of coupled, singular, integral equations yields results for the expansion coefficients A_+, $\alpha(\eta)$ and $\beta(\eta)$. We proceeded with the solution that is indicated in the completeness proof, and by performing many of the involved multiple integrals there, we were able to write the solutions for the expansion coefficients in the form of scalar products. This procedure suggested the use of the adjoint solutions as a way in which to present the results in canonical form. The manipulations were extremely lengthy and we choose not to reproduce them here since by verification we will show that our adjoint solutions are the correct ones. As might be expected, the fact that the continuum normal modes, $\mathbf{\Phi}_1(\eta, \mu)$ and $\mathbf{\Phi}_2(\eta, \mu)$, are degenerate introduces some complication. How-

[7] It is obvious that there is an analogous completeness theorem for the other half-range, $[-1, 0]$.

ever, the adjoint solutions that we use were constructed so that all expansion coefficients could be obtained by simply taking scalar products.

The adjoint solutions are

$$\mathbf{\Phi}_+^\dagger = \begin{bmatrix} \gamma_1(\mu) \\ -\gamma_2(\mu)[a + b\mu] \end{bmatrix}, \tag{66a}$$

$$\mathbf{\Phi}_1^\dagger(\eta, \mu) = \begin{bmatrix} \gamma_1(\mu)\left[\frac{3\eta}{2}(1-\eta^2)\frac{P}{\eta-\mu} + \lambda_1(\eta)\delta(\eta-\mu) + \frac{3\eta}{2}(c+\eta)\right] \\ \frac{15\eta}{2b}\gamma_2(\mu) \end{bmatrix}, \tag{66b}$$

and

$$\mathbf{\Phi}_2^\dagger(\eta, \mu) = \begin{bmatrix} \frac{3\eta}{2b}\gamma_1(\mu) \\ \gamma_2(\mu)\left[\frac{3\eta}{2}(1-\mu^2)\frac{P}{\eta-\mu} + \lambda_2(\eta)\delta(\eta-\mu) - \frac{3\eta}{2}(c+\mu)\right] \end{bmatrix}, \tag{66c}$$

where

$$a \triangleq X(1)Y(-1) + X(-1)Y(1), \tag{67}$$

$$b \triangleq X(1)Y(-1) - X(-1)Y(1), \tag{68}$$

and

$$c = \frac{X(1)Y(-1) + X(-1)Y(1)}{X(1)Y(-1) - X(-1)Y(1)}. \tag{69}$$

Let us show firstly that $\mathbf{\Phi}_+^\dagger$ is orthogonal to the two continuum modes that are given by Eqs. (29). Introducing the notation

$$\langle i \mid j\rangle \triangleq \int_0^1 \tilde{\mathbf{\Phi}}_i^\dagger(\eta', \mu)\mathbf{\Phi}_j(\eta, \mu)\, d\mu; \quad i, j = +, 1, 2, \tag{70}$$

we have

$$\langle + \mid 1\rangle = \int_0^1 \gamma_1(\mu)\left[\frac{3\eta}{2}(1-\mu^2)\frac{P}{\eta-\mu} + \lambda_1(\eta)\delta(\eta-\mu)\right] d\mu \tag{71}$$

or

$$\langle + \mid 1\rangle = \frac{3\eta}{2}\int_0^1 \gamma_1(\mu)(1-\mu^2)\frac{P}{\eta-\mu}\, d\mu + \lambda_1(\eta)\gamma_1(\eta). \tag{72}$$

The integral term in Eq. (72) can be evaluated from the sum of the boundary values of $X(z)$. (See Identity I in Appendix B.) Thus

$$\langle + \mid 1 \rangle = -\eta \left[\frac{X^+(\eta) + X^-(\eta)}{2} \right] + \lambda_1(\eta)\gamma_1(\eta). \tag{73}$$

This expression is easily shown to vanish by using the properties of the $X(z)$ and $\Lambda(z)$ functions given by Eqs. (58) and (52) and by noting the definition of $\gamma_1(\mu)$ [Eq. (61)]. In analogous fashion, for the second of the continuum modes we write

$$\langle + \mid 2 \rangle = \int_0^1 \tilde{\mathbf{\Phi}}_+^\dagger \mathbf{\Phi}_2(\eta, \mu)\, d\mu. \tag{74}$$

Expanding this, we obtain

$$\begin{aligned} \langle + \mid 2 \rangle = -\frac{3\eta}{2} \int_0^1 \gamma_1(\mu)(\eta + \mu)\, d\mu - \lambda_2(\eta)\gamma_2(\eta)[a + b\eta] \\ - \frac{3\eta}{2} (1 - \eta^2) \int_0^1 (a + b\mu)\gamma_2(\mu) \frac{P}{\eta - \mu}\, d\mu. \end{aligned} \tag{75}$$

Dividing the factor $(1 - \eta^2)$ in the integral term in Eq. (75) by $(\eta - \mu)$, we find

$$\begin{aligned} \langle + \mid 2 \rangle = -\frac{3\eta}{2} \int_0^1 \gamma_1(\mu)(\eta + \mu)\, d\mu - \lambda_2(\eta)\gamma_2(\eta)[a + b\eta] \\ + \frac{3\eta a}{2} \int_0^1 \gamma_2(\mu) \left[\eta + \mu - (1 - \mu^2) \frac{P}{\eta - \mu} \right] d\mu \\ + \frac{3\eta b}{2} \int_0^1 \gamma_2(\mu) \left[1 + \eta\mu - \eta(1 - \mu^2) \frac{P}{\eta - \mu} \right] d\mu. \end{aligned} \tag{76}$$

The principal-value integral in the above is evaluated in terms of the boundary values of $Y(z)$ with the aid of Identity V given in Appendix B. Thus in terms of the moments of $\gamma_i(\mu)$,

$$\gamma_i^j \triangleq \int_0^1 \gamma_i(\mu)\mu^j\, d\mu, \tag{77}$$

we find

$$\begin{aligned} \langle + \mid 2 \rangle = -\frac{3\eta}{2} [\eta\gamma_1^0 + \gamma_1^1] + \frac{3\eta a}{2} [\eta\gamma_2^0 + \gamma_2^1] - a\eta - b\eta^2 \\ + \frac{3\eta b}{2} [\eta\gamma_2^1 + \gamma_2^0]. \end{aligned} \tag{78}$$

In Appendix B the moments of $\gamma_i(\mu)$ have been evaluated in terms of $X(z)$ and $Y(z)$. Using these results, we find that $\langle + \mid 2 \rangle$ is identically zero. The normalization integral for the discrete mode,

$$\langle + \mid + \rangle = \int_0^1 \tilde{\mathbf{\Phi}}_+^\dagger \mathbf{\Phi}_+ \, d\mu, \tag{79}$$

is

$$\langle + \mid + \rangle = \gamma_1^0 - a\gamma_2^0 - b\gamma_2^1. \tag{80}$$

This in turn can be expressed in terms of $X(z)$ and $Y(z)$. We find

$$\langle + \mid + \rangle \triangleq N_+ = -\tfrac{2}{3}b. \tag{81}$$

To show that the two continuum adjoint solutions are correct is a very lengthy task and will not be given here. The evaluation of the various double principal-value integrals that are encountered must be done with extreme care; the Poincaré–Bertrand formula given by Kuščer, McCormick, and Summerfield (*12*) is a very useful relation in these calculations. The normalization integrals for the two continuum modes take a canonical form somewhat similar to those found in other transport problems (*4*), (*6*). In order to be complete, we state all of our results concerning the theorem below:

$$\langle i \mid j \rangle = 0; \qquad i \neq j = +, 1 \quad \text{or} \quad 2. \tag{82}$$

$$\langle + \mid + \rangle = -\tfrac{2}{3}[X(1)Y(-1) - X(-1)Y(1)], \tag{83a}$$

$$\langle 1 \mid 1 \rangle = S_1(\eta)\delta(\eta - \eta'), \tag{83b}$$

and

$$\langle 2 \mid 2 \rangle = S_2(\eta)\delta(\eta - \eta'). \tag{83c}$$

Here

$$S_1(\eta) = \gamma_1(\eta)\Omega^+(\eta)\Omega^-(\eta) \tag{84a}$$

and

$$S_2(\eta) = \gamma_2(\eta)\Lambda^+(\eta)\Lambda^-(\eta). \tag{84b}$$

Writing out $S_1(\eta)$ and $S_2(\eta)$ explicitly, we find

$$S_i(\eta) = \gamma_i(\eta)[\lambda_i^2(\eta) + \tfrac{9}{4}\pi^2\eta^2(1 - \eta^2)^2]. \tag{85}$$

Armed with the adjoint solutions and normalization integrals presented in this section and the completeness theorem of Section IV, we note that the solution of actual problems becomes a straightforward task. This is demonstrated in the next section where the classical Milne problem is solved rigorously.

VI. THE MILNE PROBLEM

Having developed the necessary formalism in the previous sections of this paper, we now illustrate the solution of a typical problem. We choose as our example the classical Milne problem, but we wish to emphasize that Theorems I and II are quite general and that other half-space problems can be solved in a very similar manner.

We seek the angular energy density $\boldsymbol{\Psi}_M(x, \mu)$ in a source-free half-space. In addition, we insist that $\boldsymbol{\Psi}_M(x, \mu)$ be of exponential order at infinity and that there be no incident radiation on the free surface. Thus the boundary conditions are stated as (*10*):

(a) $\boldsymbol{\Psi}_M(0, \mu) = 0, \mu > 0$ (zero re-entrant radiation)

(b) $\text{Lim}_{x\to\infty}\, e^{-x}\boldsymbol{\Psi}_M(x, \mu) = 0$.

The solution can be constructed from the normal modes of the transfer equation. Condition (b) requires that no $\boldsymbol{\Psi}_\eta(x, \mu)$ be included for $\eta \in [-1, 0]$. We write, therefore,

$$\boldsymbol{\Psi}_M(x, \mu) = A_+\boldsymbol{\Phi}_+ + A_-\boldsymbol{\Psi}_-(x, \mu) + \int_0^1 \alpha(\eta)e^{-x/\eta}\boldsymbol{\Phi}_1(\eta, \mu)\, d\eta + \int_0^1 \beta(\eta)e^{-x/\eta}\boldsymbol{\Phi}_2(\eta, \mu)\, d\eta. \tag{86}$$

The expression given above obviously satisfies Eq. (5). It also obeys the restriction imposed by the second boundary condition. The only remaining question is whether we can find the expansion coefficients in order that the first boundary condition be exactly incorporated. Setting $x = 0$ in Eq. (86), we must satisfy

$$0 = A_+\boldsymbol{\Phi}_+ + A_-\boldsymbol{\Psi}_-(0, \mu) + \int_0^1 \alpha(\eta)\boldsymbol{\Phi}_1(\eta, \mu)\, d\eta + \int_0^1 \beta(\eta)\boldsymbol{\Phi}_2(\eta, \mu)\, d\eta, \qquad \mu \geq 0. \tag{87}$$

Substituting the expression for $\boldsymbol{\Psi}(0, \mu)$ given by Eq. (24) and transposing it to the left-hand side, we obtain

$$\mu\begin{bmatrix}1\\1\end{bmatrix}A_- = A_+\boldsymbol{\Phi}_+ + \int_0^1 \alpha(\eta)\boldsymbol{\Phi}_1(\eta, \mu)\, d\eta + \int_0^1 \beta(\eta)\boldsymbol{\Phi}_2(\eta, \mu)\, d\eta, \qquad \mu \geq 0. \tag{88}$$

This is a half-range expansion of

$$\boldsymbol{\Psi}(\mu) = \mu\begin{bmatrix}1\\1\end{bmatrix}A_- \tag{89}$$

in terms of the half-range basis set. Theorem I therefore applies. The constant A_- we leave arbitrary since it depends on the normalization. Using the orthogonality relations established by Theorem II, we obtain the expansion coefficients immediately:

$$\frac{A_+}{A_-} = \frac{\int_0^1 \tilde{\Phi}_+^\dagger \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mu \, d\mu}{N_+}, \tag{90a}$$

$$\frac{\alpha(\eta)}{A_-} = \frac{\int_0^1 \tilde{\Phi}_1^\dagger(\eta, \mu) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mu \, d\mu}{S_1(\eta)}, \tag{90b}$$

and

$$\frac{\beta(\eta)}{A_-} = \frac{\int_0^1 \tilde{\Phi}_2^\dagger(\eta, \mu) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mu \, d\mu}{S_2(\eta)}. \tag{90c}$$

The normalization terms N_+, $S_1(\eta)$, and $S_2(\eta)$ are given in Eqs. (83). Of course, A_+, $\alpha(\eta)$, and $\beta(\eta)$ are not explicit until the integrals in Eqs. (90) are performed. These integrations are completely straightforward, although lengthy. We find

$$\frac{A_+}{A_-} = c + \frac{1}{2}[Y(1) - Y(-1)] + \frac{3}{4}\int_0^1 \frac{\mu^3}{Y(-\mu)} d\mu, \tag{91a}$$

$$\frac{\alpha(\eta)}{A_-} = 5\eta\{S_1(\eta)[X(1)Y(-1) - X(-1)Y(1)]\}^{-1}, \tag{91b}$$

and

$$\beta(\eta)/A_- = \eta(\eta - c)/S_2(\eta). \tag{91c}$$

Since the stellar atmosphere being considered is a conservative one, the net current through the medium is constant. We therefore normalize the solution by taking

$$2\int_{-1}^1 \mu \widetilde{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \Psi_{\mathrm{M}}(x, \mu) \, d\mu = -F, \tag{92}$$

where F is a constant.

Multiplying Eq. (86) by

$$\mu \widetilde{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \mu\tilde{\Phi}_+ \tag{93}$$

and integrating over μ from -1 to 1, we find

$$A_- = \tfrac{3}{8}F. \tag{94}$$

[The full-range scalar product of Eq. (86) by $\mu\tilde{\Phi}_+$ is easily evaluated by noting that the full-range adjoint solutions, $\Phi^\dagger$, are simply $\mu\Phi$. This is trivially proved from Eq. (7).]

The extrapolated endpoint, x_0, is defined as the distance from the boundary at which the asymptotic energy densities extrapolate to zero. It is

$$x_0 = A_+/A_- \tag{95}$$

and is already given by Eq. (91a).

The complete solution to the Milne problem takes the final form

$$\begin{aligned}\mathbf{\Psi}_{\mathrm{M}}(x, \mu) = \frac{3}{8} F \Bigg[(x_0 + x - \mu)\mathbf{\Phi}_+ - \frac{\sqrt{10}}{2} q \int_0^1 \frac{\eta e^{-x/\eta}}{S_1(\eta)} \mathbf{\Phi}_1(\eta, \mu)\, d\eta \\ + \int_0^1 \frac{\eta(\eta - c)e^{-x/\eta}}{S_2(\eta)} \mathbf{\Phi}_2(\eta, \mu)\, d\eta \Bigg],\end{aligned} \tag{96}$$

where

$$q \triangleq -\sqrt{10}[X(1)Y(-1) - X(-1)Y(1)]^{-1} \tag{97}$$

and c is given by Eq. (69).

The law of darkening can be reduced to a rather simple form by extending the method of Shure and Natelson (*13*) to evaluate the integrals over η. If we therefore consider $\mathbf{\Psi}_{\mathrm{M}}(0, \mu)$, $\mu < 0$, we have

$$\begin{aligned}\mathbf{\Psi}_{\mathrm{M}}(0, -\mu) = (A_+ + \mu A_-) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^1 \alpha(\eta) \begin{bmatrix} \dfrac{(1 - \mu^2)}{\eta + \mu} \\ 0 \end{bmatrix} \frac{3\eta}{2}\, d\eta \\ + \int_0^1 \frac{3\eta}{2} \beta(\eta) \begin{bmatrix} (\mu - \eta) \\ \dfrac{(1 - \eta^2)}{\eta + \mu} \end{bmatrix} d\eta, \qquad \mu \geq 0.\end{aligned} \tag{98}$$

Here the continuum part of the expansion becomes particularly simple because restricting μ to be negative makes the eigensolutions ($\eta > 0$) no longer singular. We find

$$\mathbf{\Psi}_{\mathrm{M}}(0, -\mu) = \frac{3}{8\sqrt{2}} F \begin{bmatrix} q \dfrac{\sqrt{5}}{X(-\mu)} \\ \dfrac{\sqrt{2}}{Y(-\mu)} (\mu + c) \end{bmatrix}, \qquad \mu \geq 0. \tag{99}$$

Here the law of darkening is expressed in terms of the two functions $X(-\mu)$ and $Y(-\mu)$.

For this special case, $x = 0$ and $\mu \leqq 0$, our result agrees with others (*2*). This comparison is facilitated by noting the relations between our X and Y functions and the H_l and H_r derived in Ref. (*10*):[8]

$$X(-\mu) = \sqrt{5}/H_l(\mu) \tag{100a}$$

[8] These results are obtained by comparing Identities *I* and *V* of Appendix B with the integral equations for H_l and H_r (*10*).

and

$$Y(-\mu) = \sqrt{2}/H_r(\mu). \tag{100b}$$

It is clear how other half-space problems could be solved. For example, consider the albedo problem where one has a source-free half-space with incident distribution

$$\mathbf{\Psi}_{\text{inc}}(\mu) = \begin{bmatrix} \delta(\mu - \mu_1) \\ \delta(\mu - \mu_2) \end{bmatrix}, \qquad \mu, \mu_i \geq 0. \tag{101}$$

The solution must not diverge at infinity, so we set

$$\mathbf{\Psi}_a(x, \mu) = A_+\mathbf{\Phi}_+ + \int_0^1 \alpha(\eta)e^{-x/\eta}\mathbf{\Phi}_1(\eta, \mu)\, d\eta + \int_0^1 \beta(\eta)e^{-x/\eta}\mathbf{\Phi}_2(\eta, \mu)\, d\eta. \tag{102}$$

Since

$$\mathbf{\Psi}_a(0, \mu) = \mathbf{\Psi}_{\text{inc}}(\mu), \qquad \mu \geqq 0, \tag{103}$$

the expansion coefficients are found as scalar products of adjoint functions with delta functions. We omit any details of this or other half-space problems in the present paper.

APPENDIX A. THE NUMBER OF DISCRETE ROOTS

The discrete eigenvalues $\eta_{i\pm}$ are defined as the zeros of the dispersion function

$$\Omega(z) = -1 + 3(1 - z^2)\left[1 + \frac{z}{2}\int_{-1}^{1} \frac{d\mu}{\mu - z}\right]. \tag{A1}$$

Here we verify that there are only two such zeros. (We already know that these zeros are $\eta_0 = \pm\infty$.) Since $\Omega(z)$ is analytic in the complex plane cut from -1 to 1 on the real axis and vanishes at infinity, the number of zeros is $(2\pi)^{-1}$ times the change in the argument of $\Omega(z)$ as a contour encircling the cut is traversed (*14*). Because $\Omega^+(\mu) = (\Omega^-(\mu))^*$ and $\Omega(z) = \Omega(-z)$, the change in the argument about the entire contour is four times the change in going from $0 + i\epsilon$ to $1 + i\epsilon$. Call this change $\Delta_+(0, 1)$.

From Eq. (51), we have

$$\Omega^+(\mu) = -1 + 3(1 - \mu^2)[1 - \mu T(\mu) + (\pi i/2)\mu]. \tag{A2}$$

From (A2) it is easily verified that

$$\Delta_+(0, 1) = \pi. \tag{A3}$$

The total change as the cut is encircled is therefore 4π, and thus we conclude that there are only two zeros of $\Omega(z)$ in the cut plane.

APPENDIX B. IDENTITIES

Identity I

$$X(z) = \frac{3}{2} \int_0^1 \gamma_1 (\mu) (1 - \mu^2) \frac{d\mu}{\mu - z}. \tag{B1}$$

The proof follows by writing Cauchy's theorem for $X(z)$, i.e.,

$$X(z) = \frac{1}{2\pi i} \oint X(z') \frac{dz'}{z' - z}, \tag{B2}$$

where the contour can be shrunk to include only the branch cut (the integrand vanishes at infinity). Thus we write

$$X(z) = \frac{1}{2\pi i} \int_0^1 [X^+(\mu) - X^-(\mu)] \frac{d\mu}{\mu - z}. \tag{B3}$$

Using Eqs. (53), (58), and (61), we obtain

$$X^+(\mu) - X^-(\mu) = 3\pi i \gamma_1(\mu)(1 - \mu^2). \tag{B4}$$

Entering Eq. (B2) with this result, we verify Identity I.

Identity II

$$X(z)X(-z) = \tfrac{5}{2}\Omega(z). \tag{B5}$$

For the proof, we note that the function

$$F(z) \triangleq \frac{2X(z)\, X(-z)}{5\Omega(z)} \tag{B6}$$

is an entire function because it is analytic in the cut plane and its discontinuity across the cut vanishes. It is thus a constant.

Letting z approach infinity, we find

$$\lim_{z\to\infty} F(z) = 1, \tag{B7}$$

because in the same limit

$$X(z)X(-z) \sim -1/z^2 \tag{B8}$$

and

$$\Omega(z) \sim -2/5z^2. \tag{B9}$$

This proves Identity II.

Identity III

$$X(z) = \sqrt{5} + \frac{15}{4} z \int_0^1 \frac{(1 - \mu^2)}{X(-\mu)} \frac{d\mu}{\mu - z}. \tag{B10}$$

This is a nonlinear integral equation for $X(z)$ which can be used by the ambitious to evaluate $X(z)$ numerically. It is obtained by combining Identities I and II and the trivial,

IDENTITY IV

$$\gamma_1(\mu) = \frac{5}{2}\frac{\mu}{X(-\mu)} \tag{B11}$$

obtained by taking the boundary values of Eq. (B5).

The following analogous set of identities for the $Y(z)$ function is stated without proof because its development follows exactly the same lines as those for $X(z)$:

IDENTITY V

$$Y(z) = 1 + \frac{3}{2}\int_0^1 \gamma_2(\mu)(1-\mu^2)\frac{d\mu}{\mu - z}. \tag{B12}$$

IDENTITY VI

$$Y(z)Y(-z) = \tfrac{1}{2}\Lambda(z). \tag{B13}$$

IDENTITY VII

$$Y(z) = \sqrt{2} + \frac{3}{4}z\int_0^1 \frac{(1-\mu^2)}{Y(-\mu)}\frac{d\mu}{\mu - z}. \tag{B14}$$

IDENTITY VIII

$$\gamma_2(\mu) = \frac{1}{2}\frac{\mu}{Y(-\mu)}. \tag{B15}$$

In addition to the foregoing set of identities, the following expressions for the various moments of the $\gamma_i(\mu)$ functions are useful. They were obtained by considering special cases of Identities I and V. Here

$$\gamma_i^j = \int_0^1 \gamma_i(\mu)\mu^j\, d\mu, \tag{B16}$$

and we find

$$\gamma_1^0 = \frac{X(-1) - X(1)}{3}, \qquad \gamma_1^1 = -\frac{X(1) + X(-1)}{3},$$
$$\gamma_1^2 = \frac{X(-1) - X(1) - 2}{3} \tag{B17}$$
$$\gamma_2^0 = \frac{Y(-1) - Y(1)}{3}, \quad \text{and} \quad \gamma_2^1 = \frac{2 - Y(1) - Y(-1)}{3}.$$

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