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ON SOLUTIONS OF A TRANSCENDENTAL EQUATION BASIC TO THE THEORY OF VIBRATING PLATES*

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Abstract. The theory of complex variables is used to develop exact closed-form solutions of the transcendental equation $a \tan \xi + \tanh \xi = 0$.

1. Introduction. As discussed by Leissa [1] and Marguerre [2], the study of the vibration of elastic plates invariably leads to eigenfunction expansions. In many such cases the required eigenvalues are established as the solutions of transcendental equations. One such problem is that of the dually clamped oscillating plate. Here we seek a solution to

$$(\nabla^4 - k^4)W(x, y) = 0, \tag{1}$$

with $W(\alpha, y) = W(0, y) = 0$, $W(x, 0) = f(x)$, and $W(x, \beta) = g(x)$. The solution for $W(x, y)$ can be established by separation of variables, with the x component expressed as

$$X(x) = \sinh\left(\frac{\gamma}{2}\right) \cos\left[\gamma\left(\frac{x}{\alpha} - \frac{1}{2}\right)\right] + \sin\left(\frac{\gamma}{2}\right) \cosh\left[\gamma\left(\frac{x}{\alpha} - \frac{1}{2}\right)\right], \tag{2a}$$

with

$$\tanh(\gamma/2) + \tan(\gamma/2) = 0, \tag{2b}$$

or

$$X(x) = \sinh\left(\frac{\gamma}{2}\right) \sin\left[\gamma\left(\frac{x}{\alpha} - \frac{1}{2}\right)\right] - \sin\left(\frac{\gamma}{2}\right) \sinh\left[\gamma\left(\frac{x}{\alpha} - \frac{1}{2}\right)\right], \tag{3a}$$

with

$$\tanh(\gamma/2) - \tan(\gamma/2) = 0. \tag{3b}$$

We wish here to investigate the transcendental equation

$$a \tan \xi + \tanh \xi = 0, \tag{4}$$

which clearly contains the foregoing as special cases.

2. General analysis: $|\xi| \leqq \pi/2$. In order to find the real and imaginary solutions of

$$a \tan \xi + \tanh \xi = 0, \qquad a \in (-\infty, \infty), \tag{5}$$

we first wish to introduce and study the sectionally analytic function

$$F(z) = \operatorname{Log}(z+1) - \operatorname{Log}(z-1) - i\frac{|a|}{a}[\log(z + i|a|) - \log(z - i|a|)]. \tag{6}$$

Here we use the standard notation $\operatorname{Log}(\zeta)$ to represent the principal branch of the log function, i.e.,

$$\operatorname{Log}(\zeta) = \ln|\zeta| + i \arg(\zeta), \qquad \arg(\zeta) \in (-\pi, \pi). \tag{7}$$

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For the functions $\log(z \pm i|a|)$ appearing in (6) we use branches of the log function such that

(8) $$\log(\zeta) = \ln|\zeta| + i\arg(\zeta), \qquad \arg(\zeta) \in \left(\frac{-\pi}{2}, \frac{3\pi}{2}\right).$$

With these choices of the log functions it is clear that $F(z)$ is analytic in the complex z plane cut from -1 to 1 along the real axis and from $-i|a|$ to $i|a|$ along the imaginary axis. It is a simple matter to show that

(9) $$F(z) = \frac{2}{z}(1+a) + \frac{2}{3z^3}(1-a^3) + O\left(\frac{1}{z^5}\right), \qquad \text{as } |z| \to \infty.$$

We now wish to use the argument principle [3] to establish the number of zeros of $F(z)$ inside the contours C_1 and C_2, shown in Fig. 1, as $R \to \infty$ and $\varepsilon \to 0$. Since in general $F(z)$ vanishes as $1/z$ as $|z| \to \infty$, we find that the argument of $F(z)$ decreases by 2π as the contour C_1 is traversed (in the positive sense). For the special case $a = -1$, $F(z)$ vanishes as $1/z^3$ as $|z| \to \infty$, and thus for this case the argument of $F(z)$ decreases by 6π as C_1 is traversed.

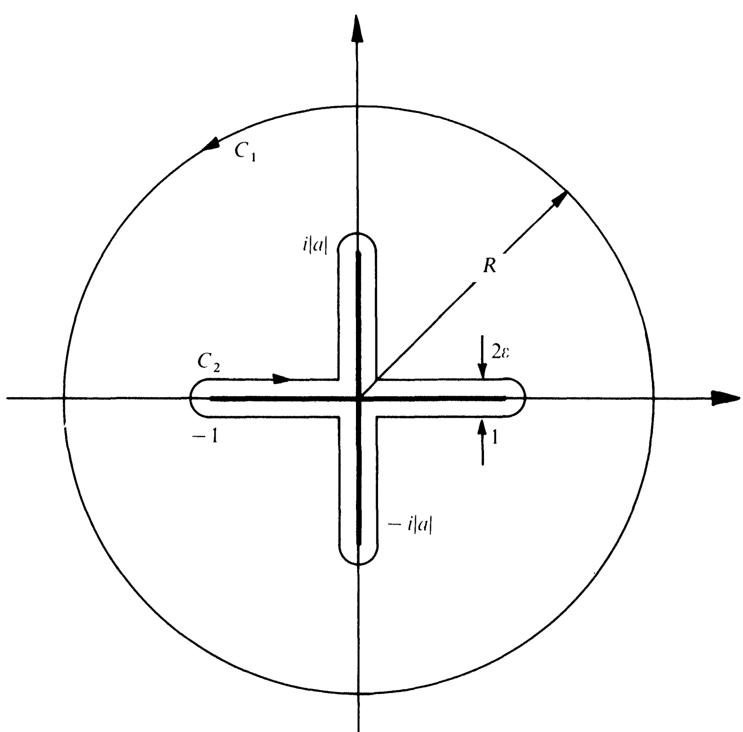

FIG. 1. *The contours C_1 and C_2.*

To compute the change in the argument of $F(z)$ as the contour C_2 is traversed, we first require the limiting values $F^{\pm}(x)$ of $F(z)$ as z approaches the cut $[-1, 1]$ from above (+) and below (−) and the limiting values $F^{\pm}(iy)$ as z approaches the cut $[-i|a|, i|a|]$ from the left (+) and the right (−). It is a relatively straightforward matter to show that

(10) $$F^{\pm}(x) = R(x) \mp i\pi, \qquad x \in [-1, 1],$$

where

(11) $$R(x)=2\tanh^{-1}(x)+\frac{|a|}{a}\left[\operatorname{sgn}(x)\pi-2\operatorname{Tan}^{-1}\left(\frac{x}{|a|}\right)\right].$$

We use here the convention that $\operatorname{Tan}^{-1}(x)$ denotes the principal branch of the arctan function. In a similar manner, we can compute the limiting values of $F(z)$ as z approaches the cut along the imaginary axis. We find

(12) $$F^{\pm}(iy)=\mp\frac{|a|}{a}\pi+iI_0(y),\qquad y\in[-|a|,|a|],$$

where

(13) $$I_0(y)=-\frac{2|a|}{a}\tanh^{-1}\left(\frac{y}{|a|}\right)+2\operatorname{Tan}^{-1}(y)-\operatorname{sgn}(y)\pi.$$

If we use Δ_2 to denote the change in the argument of $F(z)$ as the contour C_2 is traversed, in limit as $\varepsilon\to 0$, we can use (10) and (12) to deduce that $\Delta_2=2\pi$, for $a>0$, and that $\Delta_2=6\pi$ for $a<0$. (The special case $a=0$ clearly is not interesting.) We thus conclude that for $a>0$, $F(z)$ has only a zero at infinity; on the other hand, for $a<0$ we note that, in general, $F(z)$ has two zeros in the finite plane plus one zero at infinity. For the special case of $a=-1$, $F(z)$ clearly has only a triple zero at infinity.

In order to relate the zeros of $F(z)$ to the desired solutions of (5), let us first deduce the special forms of $F(z)$ for $z=x\in(-\infty,-1)\cup(1,\infty)$ and for $z=iy$, $y\in(-\infty,-|a|)\cup(|a|,\infty)$. Evaluating $F(z)$ on that part of the real axis that excludes the cut, we find

(14) $$F(x)=2\tanh^{-1}\left(\frac{1}{x}\right)+\frac{|a|}{a}\left[\operatorname{sgn}(x)\pi-2\operatorname{Tan}^{-1}\left(\frac{x}{|a|}\right)\right],\qquad x\in(-\infty,-1)\cup(1,\infty).$$

On that part of the imaginary axis that excludes the cut we find

(15)
$$F(iy)=i\left[\frac{-2|a|}{a}\tanh^{-1}\left(\frac{|a|}{y}\right)+2\operatorname{Tan}^{-1}(y)-\operatorname{sgn}(y)\pi\right],\qquad y\in(-\infty,-|a|)\cup(|a|,\infty).$$

If we now consider $a\in(-\infty,-1)$, we can deduce from (14) that $F(z)$ has (in addition to a zero at infinity) two real zeros $\pm x_0$, $x_0\in(1,\infty)$. It follows therefore that $\pm\hat{\xi}_0$, where

(16) $$\hat{\xi}_0=i\operatorname{Tan}^{-1}\left(\frac{|a|}{x_0}\right),\qquad a\in(-\infty,-1),$$

are two of the desired solutions of (5) for this case. Considering now $a\in(-1,0)$, we conclude that $F(z)$ has (in addition to a zero at infinity) two imaginary zeros $\pm iy_0$, $y_0\in(|a|,\infty)$. Thus we observe from (15) that $\pm\xi_0$, where

(17) $$\xi_0=\tanh^{-1}\left(\frac{|a|}{y_0}\right),\qquad a\in(-1,0),$$

are two of the desired solutions for the considered values of the parameter a. To summarize our conclusions thus far we note that for $a>0$, $F(z)$ has only a zero at infinity which corresponds to the trivial solution ($\xi_0=0$) of (5). For $a\in(-\infty,-1)$, $F(z)$ has two additional real zeros $\pm x_0$ which correspond to the imaginary solutions $\pm\hat{\xi}_0$, where $\hat{\xi}_0$ is given by (16). For $a\in(-1,0)$, $F(z)$ has, in addition to a zero at infinity, two imaginary zeros $\pm iy_0$ which correspond to the real solutions $\pm\xi_0$, where ξ_0 is given by (17).

We note from (5) that

$$\hat{\xi}_0(a) = i\xi_0\left(\frac{1}{a}\right), \qquad a \in (-\infty, -1), \tag{18}$$

and thus we need here only $\xi_0(a)$, $a \in (-1, 0)$, in order to establish the real and imaginary solutions of (5) such that $|\xi_0| \leqq \pi/2$.

If we now consider only $a < 0$ and let $\pm z_0$ denote the finite zeros of $F(z)$, then we note that the function

$$T(z) \triangleq \frac{F(z)}{z^2 - z_0^2} \tag{19}$$

is analytic in the complex plane cut along $L = [-1, 1] \cup [-i|a|, i|a|]$. In addition, $T(z)$ is nonvanishing in the finite plane, and the limiting values of $T(z)$ satisfy the Riemann–Hilbert problem [4]

$$T^+(\tau) = \left[\frac{F^+(\tau)}{F^-(\tau)}\right] T^-(\tau), \qquad \tau \in L. \tag{20}$$

It thus follows [4] that $T(z)$ can differ from any canonical solution of the Riemann–Hilbert problem by no more than a constant multiple. Thus we can write

$$\frac{F(z)}{z^2 - z_0^2} = KX(z), \tag{21}$$

where $X(z)$ is a canonical solution to the considered Riemann–Hilbert problem and K is a constant to be established. The desired canonical solution $X(z)$ can be constructed from the work of Muskhelishvili [4]; some care is required, however, to be sure that the "endpoint behavior" is correct. We find

$$X(z) = \frac{1}{z^3} \exp\left[\frac{2}{\pi}\int_0^1 x\theta_0(x)\frac{dx}{x^2 - z^2} + \frac{2}{\pi}\int_0^{|a|} y\phi_0(y)\frac{dy}{y^2 + z^2}\right], \tag{22}$$

where

$$\theta_0(x) = \tan^{-1}\left[\frac{-\pi}{R(x)}\right] \tag{23a}$$

and

$$\phi_0(y) = \tan^{-1}\left[\frac{-\pi}{I_0(y)}\right]. \tag{23b}$$

Here $\theta_0(x)$ and $\phi_0(y)$ are continuous, with $\theta_0(0) = \phi_0(0) = -3\pi/4$ and $\theta_0(1) = \phi_0(|a|) = 0$.

We can now substitute (22) into (21) and let $|z| \to \infty$ to find to find $K = 2(1+a)$, and thus we can solve (21) to obtain the general result

$$z_0^2 = z^2 - \frac{F(z)}{2(1+a)X(z)}, \qquad a < 0. \tag{24}$$

Equation (24) represents a general solution for the zeros of $F(z)$ and is valid for any value of z. We can let $|z| \to \infty$ in (24) to find the specific result

$$z_0^2 = \frac{2}{\pi}\int_0^{|a|} \phi_0(y) y\, dy - \frac{2}{\pi}\int_0^1 \theta_0(x) x\, dx - \frac{(1-a^3)}{3(1+a)}, \qquad a < 0. \tag{25}$$

It is clear that (25) can be used in

(26a) $$\xi_0 = \tanh^{-1}\left(\frac{|a|}{|z_0|}\right), \qquad a \in (-1, 0),$$

or

(26b) $$\xi_0 = i\,\mathrm{Tan}^{-1}\left(\frac{|a|}{|z_0|}\right), \qquad a \in (-\infty, -1),$$

to give exact analytical results for the desired solutions $(\pm\xi_0, |\xi_0| \leqq (\pi/2))$ of (5). In the next section we develop similar expressions for the solutions such that $|\xi_0| \geqq (\pi/2)$.

3. General analysis: $|\xi| \geqq (\pi/2)$. Here we wish to generalize the analysis of the previous section in order to find additional real and imaginary solutions of (5). If we let

(27) $$F_k(z) = F(z) + 2k\pi i, \qquad k = 1, 2, 3 \cdots,$$

then we conclude that $F_k(z)$ is analytic in the plane cut along L and has limiting values

(28) $$F_k^{\pm}(x) = R(x) + i(2k \mp 1)\pi, \qquad x \in [-1, 1],$$

and

(29) $$F_k^{\pm}(iy) = \mp\frac{|a|}{a}\pi + iI_k(y), \qquad y \in [-|a|, |a|].$$

Here $R(x)$ is given by (11) and

(30) $$I_k(y) = I_0(y) + 2k\pi.$$

If we use the argument principle again, we find that $F_k(z)$ has exactly one zero in the finite plane. Note that if we were to allow k to be negative we could write $F_{-k}(z) = -F_k(-z)$; thus the zeros corresponding to negative values of k are just the negative of the zeros corresponding to positive values of k. If now we evaluate $F_k(z)$ on the imaginary axis, but not on the cut, we find that

(31) $$F_k(iy) = F(iy) + 2k\pi i, \qquad y \in (-\infty, -|a|) \cup (|a|, \infty),$$

always has one simple zero, say y_k. It follows from (31) that $\pm\xi_k$, where

(32) $$\xi_k = \left(k - \frac{1}{2}\frac{|a|}{a}\right)\pi + \frac{|a|}{a}\mathrm{Tan}^{-1}(|y_k|), \qquad k = 1, 2, 3 \cdots,$$

are the additional real solutions of (5) that we seek. As in the previous section, we can generate the imaginary solutions $\pm\hat{\xi}_k$ of (5) by

(33) $$\hat{\xi}_k(a) = i\xi_k\left(\frac{1}{a}\right), \qquad a \in (-\infty, \infty).$$

We now observe that

(34) $$\frac{F_k(z)}{z - iy_k} = K_k X_k(z),$$

where K_k is a constant to be established and $X_k(z)$ is a canonical solution of the

Riemann–Hilbert problem defined by

$$X_k^+(\tau)=\left[\frac{F_k^+(\tau)}{F_k^-(\tau)}\right]X_k^-(\tau),\qquad \tau\in L. \tag{35}$$

We find that $X_k(z)$ can be written as

$$X_k(z)=\frac{1}{z-ia}\exp\left[\frac{1}{2\pi i}\int_0^1 [z\ln M_k(x)+2ix\theta_k(x)]\frac{dx}{x^2-z^2}+\frac{1}{\pi}\int_{-|a|}^{|a|}\phi_k(y)\frac{dy}{y+iz}\right], \tag{36}$$

where

$$M_k(x)=\frac{R^2(x)+(2k-1)^2\pi^2}{R^2(x)+(2k+1)^2\pi^2}, \tag{37}$$

$$\theta_k(x)=\tan^{-1}\left[\frac{-2\pi R(x)}{R^2(x)+\pi^2(4k^2-1)}\right], \tag{38}$$

and

$$\phi_k(y)=\tan^{-1}\left[\frac{(|a|/a)\pi}{I_k(y)}\right]. \tag{39}$$

The angle defined by (38) is continuous for $x\in(0,1)$, with $\theta_k(0)=\tan^{-1}(-|a|/a)/(2k^2)$. As y varies from $-|a|$ to $|a|$, the angle $\phi_k(y)$ varies from $-\pi\to 0$, for $a<0$, and from $0\to\pi$, for $a>0$; we note that $\phi_k(y)$ has a discontinuity at $y=0$.

If we now substitute (36) into (34) and let $|z|\to\infty$, we find that $K_k=2k\pi i$. Thus we can solve (34) to obtain the explicit result

$$y_k=-iz+\frac{F_k(z)}{2k\pi X_k(z)},\qquad a\in(-\infty,\infty). \tag{40}$$

Equation (40) is valid for any z, and thus can be substituted into (32) to give the remaining real solutions of (5). To obtain a specific form of (40), we can let $|z|\to\infty$ to find

$$y_k=a+\frac{1+a}{k\pi}+\frac{1}{2\pi}\int_0^1 \ln M_k(x)\,dx-\frac{|a|}{\pi}\int_{-1}^1 \phi_k(|a|x)\,dx. \tag{41}$$

4. Conclusions. We have successfully found all of the real and imaginary solutions of (5). The real solution corresponding to $k=0$ is given by (25) and (26a) for $a\in(-1,0)$, and the imaginary solutions are given by (25) and (26b) for $a\in(-\infty,-1)$. For $a>0$ there are no real or imaginary solutions corresponding to $k=0$. For $a\in(-\infty,\infty)$ and $k=1,2,3\cdots$, the real solutions of (5) are given by (32) and (41); the imaginary solutions are given by (33). Of course, if ξ is a solution, so is $-\xi$.

To be sure that our final results are free of errors, we have evaluated (25), (26), (32) and (41) numerically for various values of a and k; without difficulty solutions correct to six significant figures were obtained.

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