J. Quant. Spectrosc. Radiat. Transfer Vol. 21, pp. 35–39
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0022-4073/79/0101-0035/$02.00/0

ON USING THE F_N METHOD FOR POLARIZATION STUDIES IN FINITE PLANE-PARALLEL ATMOSPHERES

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(*Received* 25 *May*, 1978)

Abstract—The recently established F_N method for solving accurately problems in radiative transfer is used to reduce the azimuth-independent component of a basic polarization problem to a form readily amenable to numerical evaluation.

1. INTRODUCTION

SOME years ago we reported in a series of papers[1–4] a great deal of detailed analysis that was considered necessary to establish, for a semi-infinite half space, an "exact" solution of a basic polarization problem, with absorption, that was formulated by CHANDRASEKHAR.[5] Here we wish to use some of our previous exact analysis to develop the F_N approximation and thus to reduce the same problem, but for a finite, plane-parallel atmosphere, to a simple and concise computational form. We consider the vector equation of transfer,

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \frac{1}{2}\omega \mathbf{Q}(\mu) \int_{-1}^{1} \mathbf{Q}^T(\mu')\mathbf{I}(\tau, \mu')\, \mathrm{d}\mu', \tag{1}$$

developed by CHANDRASEKHAR[5] to study the scattering of polarized light. We use $\mathbf{I}(\tau, \mu)$ to denote a vector whose two components $I_l(\tau, \mu)$ and $I_r(\tau, \mu)$ are the azimuth-independent angular intensities for the two polarization states. Also, τ is the optical variable and μ is the direction cosine (as measured from the positive τ axis) of the propagating radiation. The albedo for single scattering is ω; by using

$$\mathbf{Q}(\mu) = \frac{3}{2}(c+2)^{-1/2} \begin{bmatrix} c\mu^2 + \frac{2}{3}(1-c) & (2c)^{1/2}(1-\mu^2) \\ \frac{1}{3}(c+2) & 0 \end{bmatrix}, \tag{2}$$

we allow a combination of Rayleigh and isotropic scattering.[5] Here we consider $\omega \in (0, 1]$ and allow the combination constant to be such that $c \in [0, 1]$. To keep our analysis general, we consider boundary conditions of the form

$$\mathbf{I}(0, \mu) = \mathbf{F}_1(\mu), \qquad \mu > 0, \tag{3a}$$

and

$$\mathbf{I}(\tau_0, -\mu) = \mathbf{F}_2(\mu), \qquad \mu > 0, \tag{3b}$$

where τ_0 is the optical thickness of the atmosphere, and $\mathbf{F}_1(\mu)$ and $\mathbf{F}_2(\mu)$ are considered given.

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2. ANALYSIS

We note[2] that an exact general solution of Eq. (1) can be written as

$$\mathbf{I}(\tau, \mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu)\,\mathrm{e}^{-\tau/\eta_0} + A(-\eta_0)\mathbf{\Phi}(-\eta_0, \mu)\,\mathrm{e}^{\tau/\eta_0} + \int_{-1}^{1} \mathbf{\Phi}(\eta, \mu)\mathbf{A}(\eta)\,\mathrm{e}^{-\tau/\eta}\,\mathrm{d}\eta, \tag{4}$$

where

$$\mathbf{\Phi}(\pm\eta_0, \mu) = \frac{1}{2}\,\omega\eta_0\left(\frac{1}{\eta_0 \mp \mu}\right)\mathbf{Q}(\mu)\mathbf{M}(\eta_0), \tag{5}$$

$$\mathbf{\Phi}(\eta, \mu) = \frac{1}{2}\,\omega\eta P v\left(\frac{1}{\eta - \mu}\right)\mathbf{Q}(\mu) + \delta(\eta - \mu)\,\mathbf{Q}^{-T}(\eta)\boldsymbol{\lambda}(\eta), \tag{6}$$

and $\pm\eta_0 \notin [-1, 1]$ are the zeros of $\Lambda(z) = \det \mathbf{\Lambda}(z)$. Here

$$\mathbf{\Lambda}(z) = \mathbf{I} + z\int_{-1}^{1} \mathbf{\Psi}(\mu)\frac{\mathrm{d}\mu}{\mu - z}, \tag{7}$$

$$\boldsymbol{\lambda}(\eta) = \mathbf{I} + \eta P\int_{-1}^{1} \mathbf{\Psi}(\mu)\frac{\mathrm{d}\mu}{\mu - \eta}, \tag{8}$$

$\mathbf{M}(\eta_0)$ is a null vector of $\mathbf{\Lambda}(\pm\eta_0)$, i.e.

$$\mathbf{M}(\eta_0) = \begin{bmatrix} \Lambda_{22}(\eta_0) \\ -\Lambda_{12}(\eta_0) \end{bmatrix}, \tag{9}$$

and the characteristic matrix is

$$\mathbf{\Psi}(\mu) = \frac{1}{2}\,\omega\mathbf{Q}^T(\mu)\mathbf{Q}(\mu). \tag{10}$$

In Eq. (4), the constants $A(\pm\eta_0)$ and the vector $\mathbf{A}(\eta)$ are expansion coefficients to be determined by the boundary conditions to yield an exact solution. Our goal here is not to find immediately the expansion coefficients. We wish instead to establish a system of singular integral equations for the surface quantities $\mathbf{I}(0, -\mu)$, $\mu > 0$, and $\mathbf{I}(\tau_0, \mu)$, $\mu > 0$. The full-range orthogonality relation[2]

$$\int_{-1}^{1} \mu\mathbf{\Phi}^T(\xi, \mu)\,\mathbf{\Phi}(\xi', \mu)\,\mathrm{d}\mu = 0, \qquad \xi \neq \xi', \tag{11}$$

can be used with Eq (4) to deduce that

$$\int_{-1}^{1} \mu\mathbf{\Phi}^T(\mp\xi, \mu)\,\mathbf{I}(0, \mu)\,\mathrm{d}\mu = \mathbf{N}(\mp\xi)\mathbf{A}(\mp\xi), \qquad \xi \in P, \tag{12a}$$

and

$$\int_{-1}^{1} \mu\mathbf{\Phi}^T(\mp\xi, \mu)\mathbf{I}(\tau_0, \mu)\,\mathrm{d}\mu = \mathbf{N}(\mp\xi)\,\mathbf{A}(\mp\xi)\,\mathrm{e}^{\pm\tau_0/\xi}, \qquad \xi \in P, \tag{12b}$$

where $\xi \in P \Rightarrow \xi = \eta_0$ or $\xi = \eta \in (0, 1)$ and $\mathbf{N}(\mp\xi)$ are full-range normalization factors, i.e.

$$\mathbf{N}(\pm\eta) = \pm\frac{1}{2}\,\omega\eta[\boldsymbol{\lambda}(\eta)\mathbf{\Psi}^{-1}(\eta)\boldsymbol{\lambda}(\eta) + \pi^2\eta^2\mathbf{\Psi}(\eta)] \tag{13a}$$

and

$$N(\pm\eta_0) = \pm\frac{1}{2}\,\omega\eta_0^2\Lambda_{22}(\eta_0)\frac{\mathrm{d}}{\mathrm{d}z}\,\Lambda(z)\bigg|_{z=\eta_0}. \tag{13b}$$

It is evident that we can eliminate the normalization factors between Eqs. (12) to find

$$\int_{-1}^{1} \mu \mathbf{\Phi}^T(\mp\xi, \mu)\mathbf{I}(0, \mu)\,d\mu = e^{\mp\tau_0/\xi} \int_{-1}^{1} \mu \mathbf{\Phi}^T(\mp\xi, \mu)\mathbf{I}(\tau_0, \mu)\,d\mu, \qquad \xi \in P, \tag{14}$$

or, more explicitly for $\xi \in P$,

$$\int_0^1 \mu \mathbf{\Phi}^T(\xi, \mu)\mathbf{I}(0, -\mu)\,d\mu + e^{-\tau_0/\xi} \int_0^1 \mu \mathbf{\Phi}^T(-\xi, \mu)\mathbf{I}(\tau_0, \mu)\,d\mu = \mathbf{L}_1(\xi) \tag{15a}$$

and

$$\int_0^1 \mu \mathbf{\Phi}^T(\xi, \mu)\,\mathbf{I}(\tau_0, \mu)\,d\mu + e^{-\tau_0/\xi} \int_0^1 \mu \mathbf{\Phi}^T(-\xi, \mu)\,\mathbf{I}(0, -\mu)\,d\mu = \mathbf{L}_2(\xi), \tag{15b}$$

where the known functions are

$$\mathbf{L}_1(\xi) = \int_0^1 \mu \mathbf{\Phi}^T(-\xi, \mu)\,\mathbf{F}_1(\mu)\,d\mu + e^{-\tau_0/\xi} \int_0^1 \mu \mathbf{\Phi}^T(\xi, \mu)\,\mathbf{F}_2(\mu)\,d\mu \tag{16a}$$

and

$$\mathbf{L}_2(\xi) = \int_0^1 \mu \mathbf{\Phi}^T(-\xi, \mu)\,\mathbf{F}_2(\mu)\,d\mu + e^{-\tau_0/\xi} \int_0^1 \mu \mathbf{\Phi}^T(\xi, \mu)\,\mathbf{F}_1(\mu)\,d\mu. \tag{16b}$$

Equations (15) represent a system of singular integral equations (and constraints) for the exit distributions $\mathbf{I}(0, -\mu)$, $\mu > 0$, and $\mathbf{I}(\tau_0, \mu)$, $\mu > 0$. The methods of MUSKHELISHVILI[6] could be used to convert Eqs. (15) to a system of Fredholm integral equations; however, we prefer here to use the F_N approximation. Thus, we let

$$\mathbf{I}(0, -\mu) = \sum_{\alpha=0}^{N} \mathbf{a}_\alpha \mu^\alpha, \qquad \mu > 0, \tag{17a}$$

and

$$\mathbf{I}(\tau_0, \mu) = \sum_{\alpha=0}^{N} \mathbf{b}_\alpha \mu^\alpha, \qquad \mu > 0, \tag{17b}$$

which can be substituted into Eqs. (15) to yield

$$\sum_{\alpha=0}^{N} [\mathbf{B}_\alpha^T(\xi)\mathbf{a}_\alpha + e^{-\tau_0/\xi}\,\mathbf{A}_\alpha^T(\xi)\mathbf{b}_\alpha] = \frac{2}{\omega\xi}\mathbf{L}_1(\xi) \tag{18a}$$

and

$$\sum_{\alpha=0}^{N} [\mathbf{B}_\alpha^T(\xi)\mathbf{b}_\alpha + e^{-\tau_0/\xi}\,\mathbf{A}_\alpha^T(\xi)\mathbf{a}_\alpha] = \frac{2}{\omega\xi}\mathbf{L}_2(\xi), \tag{18b}$$

where

$$\mathbf{B}_\alpha(\xi) = \frac{2}{\omega\xi}\int_0^1 \mu^{\alpha+1}\,\mathbf{\Phi}(\xi, \mu)\,d\mu \tag{19a}$$

and

$$\mathbf{A}_\alpha(\xi) = \frac{2}{\omega\xi}\int_0^1 \mu^{\alpha+1}\mathbf{\Phi}(-\xi, \mu)\,d\mu. \tag{19b}$$

We note that $\mathbf{B}_\alpha(\eta_0)$ and $\mathbf{A}_\alpha(\eta_0)$ are 2×1 vectors and the $\mathbf{B}_\alpha(\eta)$ and $\mathbf{A}_\alpha(\eta)$ are 2×2 matrices. For $\xi = \eta \in (0, 1)$, we find that we can use Eq. (6) in Eqs. (19) to obtain

$$\mathbf{A}_\alpha(\eta) = -\eta\mathbf{A}_{\alpha-1}(\eta) + \left(\frac{1}{\alpha+1}\right)\mathbf{Q}_0 + \left(\frac{1}{\alpha+3}\right)\mathbf{Q}_2 \tag{20a}$$

with

$$\mathbf{A}_0(\eta) = \mathbf{Q}_0 + \frac{1}{3}\mathbf{Q}_2 + \eta\left(\eta - \frac{1}{2}\right)\mathbf{Q}_2 - \eta \log\left(1 + \frac{1}{\eta}\right)\mathbf{Q}(\eta) \tag{20b}$$

and

$$\mathbf{B}_\alpha(\eta) = \eta\mathbf{B}_{\alpha-1}(\eta) - \left(\frac{1}{\alpha+1}\right)\mathbf{Q}_0 - \left(\frac{1}{\alpha+3}\right)\mathbf{Q}_2 \tag{21a}$$

with

$$\mathbf{B}_0(\eta) = \frac{2}{\omega}\mathbf{\Delta}(\eta) - \mathbf{Q}_0 - \frac{1}{3}\mathbf{Q}_2 + \eta\left(\eta - \frac{1}{2}\right)\mathbf{Q}_2 - \eta \log\left(1 + \frac{1}{\eta}\right)\mathbf{Q}(\eta). \tag{21b}$$

In establishing Eqs. (20) and (21), we have used $\mathbf{Q}(\mu)$ written as

$$\mathbf{Q}(\mu) = \mathbf{Q}_0 + \mu^2\mathbf{Q}_2 \tag{22}$$

and

$$\mathbf{\Delta}(\eta) = \mathbf{Q}^{-T}(\eta)\left[\mathbf{I} + \omega\eta^2\mathbf{Q}_2^T\left(\mathbf{Q}_0 + \frac{1}{3}\mathbf{Q}_2\right)\right]. \tag{23}$$

It is clear that the matrices $\mathbf{A}_\alpha(\eta)$ and $\mathbf{B}_\alpha(\eta)$ can be computed readily from the recursive relations given in Eqs. (20) and (21). Considering now the vectors required for $\xi = \eta_0$, we find that we may write

$$\mathbf{A}_\alpha(\eta_0) = -\eta_0\mathbf{A}_{\alpha-1}(\eta_0) + \left[\left(\frac{1}{\alpha+1}\right)\mathbf{Q}_0 + \left(\frac{1}{\alpha+3}\right)\mathbf{Q}_2\right]\mathbf{M}(\eta_0) \tag{24a}$$

with

$$\mathbf{A}_0(\eta_0) = \left[\mathbf{Q}_0 + \frac{1}{3}\mathbf{Q}_2 + \eta_0\left(\eta_0 - \frac{1}{2}\right)\mathbf{Q}_2 - \eta_0 \log\left(1 + \frac{1}{\eta_0}\right)\mathbf{Q}(\eta_0)\right]\mathbf{M}(\eta_0) \tag{24b}$$

and

$$\mathbf{B}_\alpha(\eta_0) = \eta_0\mathbf{B}_{\alpha-1}(\eta_0) - \left[\left(\frac{1}{\alpha+1}\right)\mathbf{Q}_0 + \left(\frac{1}{\alpha+3}\right)\mathbf{Q}_2\right]\mathbf{M}(\eta_0) \tag{25a}$$

with

$$\mathbf{B}_0(\eta_0) = \left[\frac{2}{\omega}\mathbf{\Delta}(\eta_0) - \mathbf{Q}_0 - \frac{1}{3}\mathbf{Q}_2 + \eta_0\left(\eta_0 - \frac{1}{2}\right)\mathbf{Q}_2 - \eta_0 \log\left(1 + \frac{1}{\eta_0}\right)\mathbf{Q}(\eta_0)\right]\mathbf{M}(\eta_0), \tag{25b}$$

and thus all of the known elements required in Eqs. (18) are now available in concise forms. In order to complete the F_N approximation, we use selected values, say $\{\xi_j\}$, of $\xi \in P$ and solve Eqs. (18) evaluated at $\{\xi_j\}$ for the constants $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$ required in Eqs. (17). We also introduce here one additional feature that was not necessary in the scalar version of the F_N method,[7–9] *viz.* we write

$$\mathbf{a}_0 = a_0\mathbf{M}(\eta_0) \quad \text{and} \quad \mathbf{b}_0 = b_0\mathbf{M}(\eta_0). \tag{26}$$

To obtain a simple scheme for selecting the $\{\xi_j\}$, we let $\xi_0 = \eta_0$, $\xi_1 = 0$, $\xi_2 = 1$ and space the remaining $N-2$ values of ξ_j equally distant in the interval $[0, 1]$. We now need simply to solve the following scalar system of $4N+2$ linear algebraic equations:

$$\sum_{\alpha=0}^{N} [\mathbf{B}_\alpha^T(\xi_j)\mathbf{a}_\alpha + e^{-\tau_0/\xi_j}\mathbf{A}_\alpha^T(\xi_j)\mathbf{b}_\alpha] = \frac{2}{\omega\xi_j}\mathbf{L}_1(\xi_j), \quad j = 0, 1, 2, \ldots, N, \tag{27a}$$

and

$$\sum_{\alpha=0}^{N} [\mathbf{B}_\alpha^T(\xi_j)\mathbf{b}_\alpha + e^{-\tau_0/\xi_j}\mathbf{A}_\alpha^T(\xi_j)\mathbf{a}_\alpha] = \frac{2}{\omega\xi_j}\mathbf{L}_2(\xi_j), \quad j = 0, 1, 2, \ldots, N. \tag{27b}$$

We note that, when $j = 0$, Eqs. (27) yield only 2 scalar equations and that, for each $j > 0$, Eqs. (27) yield 4 scalar equations. By expressing $\mathbf{I}(\tau, \mu)$ in terms of symmetric and antisymmetric functions, we note that the number of algebraic equations to be solved simultaneously can be reduced from $4N + 2$ to $2N + 1$.

3. CONCLUSIONS

As we have seen, by using the F_N method, we have been able to reduce our problem to one simply of solving the system of linear algebraic equations given by Eqs. (27). We consider the method to be particularly easy to use because all of the elements in the coefficient matrix in Eqs. (27) have been expressed in terms of polynomials and the log function. Of course, we have been principally concerned here with finding the surface quantities $\mathbf{I}(0, -\mu)$, $\mu > 0$, and $\mathbf{I}(\tau_0, \mu)$. $\mu > 0$; however, once the surface quantities are known, we can immediately find $\mathbf{I}(\tau, \mu)$ for all τ by again using the full-range orthogonality theorem.[2] Thus, setting $\tau = 0$ or $\tau = \tau_0$ in Eq. (4) and then multiplying by $\mu\mathbf{\Phi}^T(\pm\eta_0, \mu)$ and integrating over μ from -1 to 1, we find

$$A(\eta_0) = \frac{e^{\tau_0/\eta_0}}{N(\eta_0)}\left[\frac{1}{2}\,\omega\eta_0 \sum_{\alpha=0}^{N} \mathbf{B}_\alpha{}^T(\eta_0)\,\mathbf{b}_\alpha - \int_0^1 \mu\mathbf{\Phi}^T(-\eta_0, \mu)\mathbf{F}_2(\mu)\,d\mu\right] \tag{28a}$$

and

$$A(-\eta_0) = -\frac{1}{N(\eta_0)}\left[\int_0^1 \mu\mathbf{\Phi}^T(-\eta_0, \mu)\mathbf{F}_1(\mu)\,d\mu - \frac{1}{2}\,\omega\eta_0 \sum_{\alpha=0}^{N} \mathbf{B}_\alpha{}^T(\eta_0)\mathbf{a}_\alpha\right]. \tag{28b}$$

In a similar manner, we deduce that

$$\mathbf{A}(\eta) = \mathbf{N}^{-1}(\eta)\,e^{\tau_0/\eta}\left[\frac{1}{2}\,\omega\eta \sum_{\alpha=0}^{N} \mathbf{B}_\alpha{}^T(\eta)\mathbf{b}_\alpha - \int_0^1 \mu\mathbf{\Phi}^T(-\eta, \mu)\mathbf{F}_2(\mu)\,d\mu\right] \tag{29a}$$

and

$$\mathbf{A}(-\eta) = -\mathbf{N}^{-1}(\eta)\left[\int_0^1 \mu\mathbf{\Phi}^T(-\eta, \mu)\,\mathbf{F}_1(\mu)\,d\mu - \frac{1}{2}\,\omega\eta \sum_{\alpha=0}^{N} \mathbf{B}_\alpha{}^T(\eta)\mathbf{a}_\alpha\right] \tag{29b}$$

so that, when Eqs. (28) and (29) are used in Eq. (4), we have the complete solution.

In a recent study[9] of several basic problems defined by the scalar equation of transfer, the F_N method proved to be very easy to use, and the method consistently yielded results accurate to four significant figures for $N \leq 5$. We expect in the near future to be able to report on the numerical accuracy and the computer-time requirements of the F_N solution for the polarization problem discussed here. We also expect to be able to extend the F_N method and thus to be able to solve accurately the general Mie scattering model of the polarization problem.

Acknowledgement—The author is grateful to the Centre d'Etudes Nucléaires de Saclay for kind hospitality and partial support of this work, which was also supported in part by National Science Foundation grant ENG, 7709405.

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