The $F_N$ Method in Neutron-Transport Theory.
Part I: Theory and Applications

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The Placzek lemma is used to establish a system of singular integral equations and constraints that is solved uniquely for a half-space to yield the exact exit distribution. These singular integral equations and constraints are also used to develop a new approximation, the $F_N$ method, that yields concise and accurate results for the half-space and the finite slab.

I. INTRODUCTION

The Placzek lemma\(^1\) tells us that the solution to the half-space problem in neutron-transport theory defined (for $c < 1$) by

$$
\left( \mu \frac{\partial}{\partial x} + 1 - \frac{c}{2} \int_{-1}^{1} \ldots \ d\mu \right) \psi(x, \mu) = S(x, \mu) , \quad \mu x \in [0, \infty) , \quad (1a)
$$

$$
\psi(x, \mu) \rightarrow \psi_p(x, \mu) , \quad \text{as } x \rightarrow \infty , \quad (1b)
$$

and

$$
\psi(0, \mu) = f(\mu) , \quad \mu > 0 , \quad (1c)
$$

is related to the contrived infinite-medium problem defined by

$$
\left( \mu \frac{\partial}{\partial x} + 1 - \frac{c}{2} \int_{-1}^{1} \ldots \ d\mu \right) \psi_1(x, \mu)
= H(x)S(x, \mu) + \mu \psi(0, \mu) \delta(x) , \quad x \in (\infty, \infty) , \quad (2a)
$$

$$
\psi_1(x, \mu) \rightarrow \psi_p(x, \mu) , \quad \text{as } x \rightarrow \infty , \quad (2b)
$$

and

$$
\psi_1(-\infty, \mu) = 0 , \quad (2c)
$$

in the manner

$$
\psi_1(x, \mu) = H(x)\psi(x, \mu) . \quad (3)
$$

Here, $H(x)$ is the unit step function and $\psi_p(x, \mu)$ denotes a particular solution of Eq. (1a) appropriate to the inhomogeneous source $S(x, \mu)$. If we now let $G(x_0 \rightarrow x; \mu_0 \rightarrow \mu)$ denote the infinite medium Green's function, i.e.,

$$
\left( \mu \frac{\partial}{\partial x} + 1 - \frac{c}{2} \int_{-1}^{1} \ldots \ d\mu \right) G(x_0 \rightarrow x; \mu_0 \rightarrow \mu)
= \delta(x - x_0)\delta(\mu - \mu_0) \quad (4a)
$$

and

$$
G(x_0 \rightarrow \pm\infty; \mu_0 \rightarrow \mu) = 0 , \quad (4b)
$$

then $\psi_1(x, \mu)$ can be expressed as

$$
\psi_1(x, \mu) = \int_{-1}^{1} d\mu_0 \int_{-\infty}^{\infty} dx_0 G(x_0 \rightarrow x; \mu_0 \rightarrow \mu)
\times \left[ H(x_0)S(x_0, \mu_0) + \mu_0 \psi(0, \mu_0) \delta(x_0) \right] . \quad (5)
$$

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For $x \geq 0$, we can use Eq. (3) in Eq. (5) to find
\[
\psi(x, \mu) = \int_{-1}^{1} d\mu_{0} \left[ \int_{0}^{\infty} dx_{0} G(x_{0} \rightarrow x; \mu_{0} \rightarrow \mu) S(x_{0}, \mu_{0}) 
+ G(0 \rightarrow x; \mu_{0} \rightarrow \mu) \mu_{0} \psi(0, \mu_{0}) \right],
\] (6)
and in Eq. (6) we can let $x \rightarrow 0$ to obtain
\[
\psi(0, \mu) = S(\mu) + F(\mu) - \int_{0}^{1} d\mu_{0} G(0 \rightarrow 0^{+}; -\mu_{0} \rightarrow -\mu) \times \mu_{0} \psi(0, -\mu_{0}), \quad \mu \in (-1,1),
\] (7)
where the known functions are
\[
S(\mu) = \int_{-1}^{1} d\mu_{0} \int_{0}^{\infty} dx_{0} G(x_{0} \rightarrow 0; \mu_{0} \rightarrow \mu) S(x_{0}, \mu_{0}),
\] (8a)
and
\[
F(\mu) = \int_{0}^{1} d\mu_{0} G(0 \rightarrow 0^{+}; \mu_{0} \rightarrow -\mu) \mu_{0} f(\mu_{0}).
\] (8b)

Assuming that Eq. (7) has a solution, we can project the equation onto a basis formed by the functions $\phi(\xi, \mu)$ that are associated with the elementary solutions$^2$ of Eq. (1):
\[
\phi(\nu_{0}, \mu) = \frac{c\nu_{0}}{2} \frac{1}{\nu_{0} - \mu}
\] (9a)
and
\[
\phi(\nu, \mu) = \frac{c\nu}{2} P_{\nu} \left( \frac{1}{\nu - \mu} \right) + \lambda(\nu) \delta(\nu - \mu), \quad \nu \in (-1,1),
\] (9b)
where
\[
\lambda(\nu) = 1 - c\nu \tanh^{-1}(\nu),
\] (10)
and where $\pm \nu_{0}$ are the zeros of
\[
\Lambda(z) = 1 + \frac{c}{2} z \int_{-1}^{1} \frac{d\mu}{\mu - z}.
\] (11)
Thus, if we multiply Eq. (7) by $\mu \phi(-\xi, \mu), \xi = \nu_{0}$ or $\nu \in (0,1)$, and integrate over $\mu$ from $-1$ to $1$, we obtain
\[
\int_{0}^{1} \phi(\nu_{0}, \mu) \psi(0, -\mu) \mu_{0} d\mu = K(\nu_{0})
\] (12a)
and
\[
\int_{0}^{1} \phi(\nu, \mu) \psi(0, -\mu) \mu_{0} d\mu = K(\nu), \quad \nu \in (0,1),
\] (12b)
where the known functions are
\[
K(\xi) = \int_{0}^{1} \phi(-\xi, \mu) f(\mu) \mu_{0} d\mu
+ \int_{-1}^{1} d\mu \phi(-\xi, \mu) \int_{0}^{\infty} dx S(x, \mu) \exp(-x/\xi).
\] (13)
Equation (12b) is clearly a singular integral equation for the exit distribution $\psi(0, -\mu), \mu > 0$, and

Eq. (12a) is an integral constraint on the exit distribution. In Sec. II, we develop the unique solution of Eqs. (12). We note that multiplying Eq. (7) by $\mu \phi(\xi, \mu), \xi = \nu_{0}$ or $\nu \in (0,1)$, and integrating over $\mu$ from $-1$ to $1$ yields only a trivial identity.

II. EXACT RESULTS FOR THE EXIT DISTRIBUTION

We now wish to use the methods of Muskhelishvili$^3$ to solve Eqs. (12). If we let $\mu \psi(0, -\mu) = F(\mu)$, then we can consider Eq. (12b) to be written as
\[
\lambda(\nu) F(\nu) - \frac{c\nu}{2} \int_{0}^{1} F(\mu) \frac{d\mu}{\mu - \nu} = K(\nu), \quad \nu \in (0,1).
\] (14)
If we introduce
\[
N(z) = \frac{1}{2\pi i} \int_{0}^{1} F(\mu) \frac{d\mu}{\mu - z},
\] (15)
then Eq. (14) can be written as
\[
N^{+}(\nu) \Lambda^{+}(\nu) - N^{-}(\nu) \Lambda^{-}(\nu) = K(\nu),
\] (16)
where $N^{\pm}(\nu)$ and $\Lambda^{\pm}(\nu)$ represent the limiting values of the sectionally analytic functions $N(z)$ and $\Lambda(z)$ as the branch cuts are approached from above (+) and below (-). We find that Eq. (16) can be solved to yield
\[
N(z) = X(z) \left[ \frac{1}{2\pi i} \int_{0}^{1} \frac{K(\tau)}{X^{+}(\tau) \Lambda^{-}(\tau)} \frac{d\tau}{\tau - z} + B \right],
\] (17)
where $B$ is a constant to be determined by Eq. (12a). Since Eq. (17) represents the general solution of Eq. (16), it follows that Eqs. (12) have only one solution. In Eq. (17), the $X$ function is that introduced by Case$^2$:
\[
X(z) = \frac{1}{1 - z} \exp \left[ \frac{1}{\pi} \int_{0}^{1} \arg \Lambda^{\pm}(\tau) \frac{d\tau}{\tau - z} \right].
\] (18)
If we use Eq. (12a) to fix the constant $B$, then we can write Eq. (17) as
\[
N(z) = \frac{X(z)}{2\pi i} \left[ (\nu_{0} - z) \int_{0}^{1} \frac{K(\tau)}{X^{+}(\tau) \Lambda^{-}(\tau) (\nu_{0} - \tau)} \frac{d\tau}{\tau - z}
- \frac{2}{c\nu_{0} X(\nu_{0})} K(\nu_{0}) \right],
\] (19)
and since Eq. (15) yields
\[
F(\mu) = N^{+}(\mu) - N^{-}(\mu),
\] (20)
we can use Eq. (19) to find $F(\mu)$. Our final result is thus


\[ \psi(0,-\mu) = \int_0^1 R(\mu' - \mu)f(\mu')d\mu' + \int_0^\infty dx \int_{-1}^1 d\mu'S(x,\mu')L(x;\mu' - \mu) , \quad \mu > 0 , \]  

where

\[ R(\mu' - \mu) = \frac{c}{2} \mu' \frac{H(\mu)H(\mu')}{\mu' + \mu} \]

and

\[ L(x;\mu' - \mu) = H(\mu) \left[ \frac{\phi(-\nu_0,\mu')\phi(\nu_0,\mu) \exp(-x/\nu_0)}{H(\nu_0)N(\nu_0)} \right. 
+ \left. \int_0^\mu \frac{\phi(-\nu,\mu')\phi(\nu,\mu) \exp(-x/\nu)}{H(\nu)N(\nu)} d\nu \right] . \]

Here, \( H(x) \) is Chandrasekhar's \( H \) function,\(^4\)

\[ N(\nu_0) = \frac{c}{2} \nu_0 \left[ \frac{c}{\nu_0^2 - 1} - \frac{1}{\nu_0^2} \right] \]

and

\[ N(\nu) = \nu \left[ \lambda^2(\nu) + \frac{c^2\nu^2\beta^2}{4} \right] . \]

It is clear that our exact result given by Eq. (21) reduces to the well-known solution of Chandrasekhar\(^4\) when \( S(x,\mu) = 0 \).

### III. THE \( F_N \) METHOD

The \( C_N \) method\(^5,6\) of approximately solving problems in neutron-transport theory has proved to be an accurate and economical method. We wish to develop here a modified version of the \( C_N \) method (which we call the \( F_N \) method) that yields more concise equations that can be solved numerically even more efficiently. For the traditional \( C_N \) method, we use the representation

\[ \psi(0,-\mu) = \sum_{\alpha=0}^N a_\alpha \mu^\alpha , \quad \mu > 0 , \]

in Eq. (7) for either \( \mu > 0 \) or \( \mu < 0 \), i.e.,

\[ f(\mu) = S(\mu) + F(\mu) - \int_0^1 d\mu'G(0 \rightarrow 0^+; -\mu' - \mu)\mu'\psi(0,-\mu') , \quad \mu > 0 , \]

or

\[ \psi(0,-\mu) = S(-\mu) + F(-\mu) - \int_0^1 d\mu'G(0 \rightarrow 0^+; -\mu' - -\mu)\mu'\psi(0,-\mu') , \quad \mu > 0 . \]

Having substituted Eq. (25) into Eq. (26a) or (26b), we can multiply Eq. (26a) or (26b) by \( \mu^\beta, \beta = 0, 1, 2, \ldots, N \), and integrate over \( \mu \) from \( 0 \rightarrow 1 \) to obtain \( N + 1 \) linear algebraic equations to solve for the \( N + 1 \) coefficients \( a_\alpha \) required in Eq. (25).

To establish the \( F_N \) method, we wish to use Eq. (25) in Eqs. (12). We thus find

\[ \sum_{\alpha=0}^N a_\alpha B_\alpha(\xi) = \frac{2}{c\xi} K(\xi) , \quad \xi = \nu_0 \text{ or } \nu \in (0,1) . \]

Here,

\[ B_\alpha(\xi) = \frac{2}{c\xi} \int_0^1 \mu^{\alpha+1}\phi(\xi,\mu)d\mu \]

can easily be shown to satisfy

\[ B_\alpha(\xi) = \xi B_{\alpha-1}(\xi) - \frac{1}{\alpha + 1} , \quad \alpha > 1 , \]

with

\[ B_0(\xi) = \frac{2}{c} - \frac{1}{\xi} \log \left( 1 + \frac{1}{\xi} \right) . \]

It is clear that we can now choose \( N + 1 \) values of \( \xi \in \nu_0U[0,1] \), for example \( \xi_\beta \), and solve the set of algebraic equations

\[ \sum_{\alpha=0}^N a_\alpha B_\alpha(\xi_\beta) = \frac{2}{c\xi_\beta} K(\xi_\beta) , \quad \beta = 0, 1, 2, \ldots, N . \]

To illustrate explicitly the \( F_N \) method, we consider the half-space constant-source problem, i.e., we seek \( \psi(0,-\mu) \), where

\[ \mu \frac{\partial}{\partial x} \psi(x,\mu) + \psi(x,\mu) = \frac{c}{2} \int_{-1}^1 \psi(x,\mu')d\mu' + a , \]

\[ \psi(0,\mu) = b , \quad \mu > 0 , \]

and

\[ \psi(x,\mu) \rightarrow \frac{a}{1 - c} , \quad \text{as } x \rightarrow \infty . \]

Here, Eq. (13) yields

\[ \frac{2}{c\xi} K(\xi) = \frac{2a}{c} b + b \left[ 1 - \xi \log \left( 1 + \frac{1}{\xi} \right) \right] , \]

and thus Eq. (30) becomes (for this problem) simply

\[ \sum_{\alpha=0}^N a_\alpha B_\alpha(\xi_\beta) = \frac{2a}{c} b + b \left[ 1 - \xi_\beta \log \left( 1 + \frac{1}{\xi_\beta} \right) \right] , \]

\[ \beta = 0, 1, 2, \ldots, N . \]

To compare this approximation to exact results, we consider

\[ A^* = 2 \int_0^1 \psi(0,-\mu)d\mu = 2 \sum_{\alpha=0}^N \frac{a_\alpha}{\alpha + \frac{3}{2}} \]

---

for the two cases \( a = 1 \) and \( b = 0 \) and \( a = 0 \) and \( b = 1 \). Of course, the manner in which the \( \xi_\beta \)'s are chosen can affect the accuracy of the method. Some preliminary results of Grandjean and Siewert\(^7\) suggest that the \( F_N \) method can yield results for \( A^* \) accurate to at least three significant figures for \( N \leq 5 \). Grandjean and Siewert's calculations for \( c \in [0.1, 0.9] \) were based on an equal spacing scheme for choosing \( \xi_\beta \), i.e., \( \xi_0 = \nu_0, \xi_1 = 0, \xi_2 = 1 \), and the remaining \( \xi_\beta \) are spaced equally in the interval \([0,1]\).

IV. THE FINITE SLAB

Having developed the exact result and introduced the \( F_N \) method for the case of a semi-infinite half-space, we now wish to consider the more interesting case of a finite slab. Here, we seek to solve

\[
\left( \mu \frac{\partial}{\partial x} + 1 - \frac{c}{2} \int_{-1}^{1} \ldots d\mu \right) \psi(x, \mu) = S(x, \mu), \quad x \in [-a, a],
\]

subject to the boundary conditions

\[
\psi(-a, \mu) = f_1(\mu), \quad \mu > 0, \quad (37a)
\]

and

\[
\psi(a, -\mu) = f_2(\mu), \quad \mu > 0. \quad (37b)
\]

Here, the Placzek lemma\(^1\) allows us to write

\[
H_*(x) \psi(x, \mu) = \psi_1(x, \mu),
\]

where \( H_*(x) = 1 \) for \( x \in [-a, a] \), \( H_*(x) = 0 \) otherwise, and \( \psi_1(x, \mu) \) is the solution in an infinite medium of

\[
\left( \mu \frac{\partial}{\partial x} + 1 - \frac{c}{2} \int_{-1}^{1} \ldots d\mu \right) \psi_1(x, \mu)
\]

\[= H_* S(x, \mu) + \mu \psi_1(x, \mu) [\delta(x + a) - \delta(x - a)]. \quad (39)
\]

We can use the Green's function defined by Eqs. (4) to write

\[
\psi_1(x, \mu) = \int_{-1}^{1} d\mu_0 \left[ \int_{-1}^{1} dx_0 G(x_0 \to x; \mu_0 \to \mu) S(x_0, \mu_0)
\right.

\[+ G(-a \to x; \mu_0 \to \mu) \mu_0 \psi(-a, \mu_0)
\]

\[\left. - G(a \to x; \mu_0 \to \mu) \mu_0 \psi(a, \mu_0) \right]. \quad (40)
\]

If we now let \( x \to \pm a \) in Eq. (40), we obtain a system of equations for the exit distributions:

\[
\psi(a, \mu) = K(a^-, \mu) - \int_{0}^{1} G(a \to a^-; \mu_0 \to \mu) \mu_0 \psi(a, \mu_0) d\mu_0
\]

\[\quad - \int_{0}^{1} G(-a \to a^-; \mu_0 \to \mu) \mu_0 \psi(-a, \mu_0) d\mu_0.
\]

\[\psi(-a, \mu) = K(-a^+, \mu)
\]

\[- \int_{0}^{1} G(a \to -a; \mu_0 \to \mu) \mu_0 \psi(a, \mu_0) d\mu_0
\]

\[- \int_{0}^{1} G(-a \to -a^+; \mu_0 \to \mu) \mu_0 \psi(-a, \mu_0) d\mu_0,
\]

\[\psi(0, \mu) = K(0, \mu).
\]

\[\psi(x, \mu) = \int_{-1}^{1} dx_0 \int_{-1}^{1} d\mu_0 G(x_0 \to x; \mu_0 \to \mu) S(x_0, \mu_0)
\]

\[+ \int_{0}^{1} G(-a \to x; \mu_0 \to \mu) \mu_0 f_1(\mu_0) d\mu_0
\]

\[+ \int_{0}^{1} G(a \to x; \mu_0 \to \mu) \mu_0 f_2(\mu_0) d\mu_0. \quad (42)
\]

where the known functions \( K(\xi, \mu) \) are given by

\[
K(\xi, \mu) = \int_{-a}^{a} d\mu_0 \int_{-1}^{1} d\mu_0 G(x_0 \to \xi; \mu_0 \to \mu) S(x_0, \mu_0)
\]

\[+ \int_{0}^{1} G(-a \to \xi; \mu_0 \to \mu) \mu_0 f_1(\mu_0) d\mu_0
\]

\[+ \int_{0}^{1} G(a \to \xi; \mu_0 \to \mu) \mu_0 f_2(\mu_0) d\mu_0. \quad (42)
\]

To develop equations analogous to Eqs. (12) that were used for the half-space, we can multiply Eqs. (41) by \( \mu \phi(\nu \xi, \mu) \), \( \xi \in \nu_0 U(0,1) \), and integrate over \( \mu \) from -1 to 1 to obtain

\[
\int_{0}^{1} \phi(\xi, \mu) \mu \psi(-a, -\mu) d\mu
\]

\[+ \exp(-2a/\xi) \int_{0}^{1} \phi(-\xi, \mu) \mu \psi(a, \mu) d\mu
\]

\[= L_1(\xi), \quad \xi \in \nu_0 U(0,1), \quad (43a)
\]

and

\[
\int_{0}^{1} \phi(\xi, \mu) \mu \psi(a, \mu) d\mu
\]

\[+ \exp(-2a/\xi) \int_{0}^{1} \phi(-\xi, \mu) \mu \psi(-a, -\mu) d\mu
\]

\[= L_2(\xi), \quad \xi \in \nu_0 U(0,1), \quad (43b)
\]

where the known terms are

\[
L_1(\xi) = \int_{0}^{1} \phi(-\xi, \mu) \mu f_1(\mu) d\mu
\]

\[+ \exp(-2a/\xi) \int_{0}^{1} \phi(\xi, \mu) \mu f_2(\mu) d\mu
\]

\[+ \exp(-a/\xi) \int_{-1}^{1} d\mu \phi(-\xi, \mu) \int_{-a}^{a} dx S(x, \mu)
\]

\[\times \exp(-x/\xi), \quad (44a)
\]

and

\[
L_2(\xi) = \int_{0}^{1} \phi(-\xi, \mu) \mu f_2(\mu) d\mu
\]

\[+ \exp(-2a/\xi) \int_{0}^{1} \phi(\xi, \mu) \mu f_1(\mu) d\mu
\]

\[+ \exp(-a/\xi) \int_{-1}^{1} d\mu \phi(\xi, \mu) \int_{-a}^{a} dx S(x, \mu)
\]

\[\times \exp(x/\xi), \quad (44b)
\]

Equations (43) clearly represent a system of singular integral equations and constraints that can be regularized to give a system of Fredholm equations for the exit distributions \( \psi(-a, -\mu) \) and

\( \psi(a, \mu), \mu > 0. \) However, rather than pursue the “exact” solution of Eqs. (43), we wish to invoke the \( F_N \) approximation. Thus, we let

\[
\psi(-a, -\mu) = \sum_{a=0}^{N} a_a \mu^a, \quad \mu > 0 , \tag{45a}
\]

and

\[
\psi(a, \mu) = \sum_{a=0}^{N} b_a \mu^a , \quad \mu > 0 , \tag{45b}
\]

and upon entering Eqs. (45) into Eqs. (43), we find

\[
\sum_{a=0}^{N} \left[ a_a B_a(\xi_\beta) + \exp(-2a/\xi_\beta) b_a A_a(\xi_\beta) \right] = \frac{2}{c \xi_\beta} L_1(\xi_\beta) , \quad \beta = 0, 1, 2, \ldots, N , \tag{46a}
\]

and

\[
\sum_{a=0}^{N} \left[ b_a B_a(\xi_\beta) + \exp(-2a/\xi_\beta) a_a A_a(\xi_\beta) \right] = \frac{2}{c \xi_\beta} L_2(\xi_\beta) , \quad \beta = 0, 1, 2, \ldots, N . \tag{46b}
\]

Here, the \( B_\alpha(\xi) \)'s are as given by Eqs. (29) and

\[
A_\alpha(\xi) = \frac{2}{c \xi} \int_{0}^{1} \mu^{\alpha+1} \phi(-\xi, \mu) d\mu . \tag{47}
\]

Thus, we have

\[
A_0(\xi) = 1 - \xi \log \left( 1 + \frac{1}{\xi} \right) \tag{48a}
\]

and

\[
A_\alpha(\xi) = -\xi A_{\alpha-1}(\xi) + \frac{1}{\alpha + 1} , \quad \alpha \geq 1 . \tag{48b}
\]

Equations (46) are clearly \( 2(N + 1) \) linear algebraic equations for the \( 2(N + 1) \) unknowns \( a_\alpha \) and \( b_\alpha, \alpha = 0, 1, 2, \ldots, N. \)

As an explicit example of the \( F_N \) method, we consider the critical problem. Thus, for \( c > 1 \) and \( S(x, \mu) = f_1(\mu) = f_2(\mu) = 0, \) we seek the critical half-thickness, \( a. \) Since here \( \psi(x, \mu) = \psi(-x, -\mu), \) we note that \( a_\alpha = b_\alpha. \) Thus, we need consider only

\[
\sum_{\alpha=0}^{N} a_\alpha [ B_\alpha(\xi_\beta) + \exp(-2\alpha/\xi_\beta) A_\alpha(\xi_\beta) ] = 0 , \quad \beta = 0, 1, 2, \ldots, N . \tag{49}
\]

Equation (49) is of the form

\[
M(a)A = 0 , \tag{50}
\]

where the elements of \( A \) are \( a_\alpha, \alpha = 0, 1, 2, \ldots, N, \) and \( M(a) \) is an \( (N + 1) \times (N + 1) \) matrix. Of course, the value of \( a \) can be established immediately from \( \det M(a) = 0. \) Again, the work of Grandjean and Siewert\(^7\) suggests that Eq. (49), for \( c \in [1.1, 2.5], \) can yield a value of \( a \) accurate to at least three significant figures for \( N \leq 5. \)

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