NUCLEAR SCIENCE AND ENGINEERING: **69**, 156-160 (1979)

The F_N Method in Neutron-Transport Theory. Part I: Theory and Applications

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Received March 27, 1978
Accepted September 6, 1978

The Placzek lemma is used to establish a system of singular integral equations and constraints that is solved uniquely for a half-space to yield the exact exit distribution. These singular integral equations and constraints are also used to develop a new approximation, the F_N method, that yields concise and accurate results for the half space and the finite slab.

I. INTRODUCTION

The Placzek lemma[1] tells us that the solution to the half-space problem in neutron-transport theory defined (for $c < 1$) by

$$\left(\mu \frac{\partial}{\partial x} + 1 - \frac{c}{2} \int_{-1}^{1} \ldots d\mu\right) \psi(x,\mu) = S(x,\mu) \quad , \quad x \in [0,\infty) \quad , \tag{1a}$$

$$\psi(x,\mu) \to \psi_p(x,\mu) \quad , \quad \text{as } x \to \infty \quad , \tag{1b}$$

and

$$\psi(0,\mu) = f(\mu) \quad , \quad \mu > 0 \quad , \tag{1c}$$

is related to the contrived infinite-medium problem defined by

$$\left(\mu \frac{\partial}{\partial x} + 1 - \frac{c}{2} \int_{-1}^{1} \ldots d\mu\right) \psi_1(x,\mu) = H(x)S(x,\mu) + \mu\psi(0,\mu)\delta(x) \quad , \quad x \in (-\infty,\infty) \quad , \tag{2a}$$

$$\psi_1(x,\mu) \to \psi_p(x,\mu) \quad , \quad \text{as } x \to \infty \quad , \tag{2b}$$

and

$$\psi_1(-\infty,\mu) = 0 \quad , \tag{2c}$$

in the manner

$$\psi_1(x,\mu) = H(x)\psi(x,\mu) \quad . \tag{3}$$

Here, $H(x)$ is the unit step function and $\psi_p(x,\mu)$ denotes a particular solution of Eq. (1a) appropriate to the inhomogeneous source $S(x,\mu)$. If we now let $G(x_0 \to x;\ \mu_0 \to \mu)$ denote the infinite medium Green's function, i.e.,

$$\left(\mu \frac{\partial}{\partial x} + 1 - \frac{c}{2} \int_{-1}^{1} \ldots d\mu\right) G(x_0 \to x;\ \mu_0 \to \mu) = \delta(x - x_0)\delta(\mu - \mu_0) \tag{4a}$$

and

$$G(x_0 \to \pm\infty;\ \mu_0 \to \mu) = 0 \quad , \tag{4b}$$

then $\psi_1(x,\mu)$ can be expressed as

$$\psi_1(x,\mu) = \int_{-1}^{1} d\mu_0 \int_{-\infty}^{\infty} dx_0\, G(x_0 \to x;\ \mu_0 \to \mu) \times \left[H(x_0)S(x_0,\mu_0) + \mu_0\psi(0,\mu_0)\delta(x_0)\right] \quad . \tag{5}$$

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[1]K. M. CASE, F. de HOFFMANN, and G. PLACZEK, *Introduction to the Theory of Neutron Diffusion*, Vol. 1, U.S. Government Printing Office, Washington, D.C. (1953).

0029-5639/79/0002-0156$02.00/0

For $x \geq 0$, we can use Eq. (3) in Eq. (5) to find

$$\psi(x,\mu) = \int_{-1}^{1} d\mu_0 \left[\int_0^\infty dx_0 G(x_0 \to x; \mu_0 \to \mu) S(x_0,\mu_0) + G(0 \to x; \mu_0 \to \mu)\mu_0\psi(0,\mu_0)\right] , \quad (6)$$

and in Eq. (6) we can let $x \to 0$ to obtain

$$\psi(0,\mu) = S(\mu) + F(\mu) - \int_0^1 d\mu_0 G(0 \to 0^+; -\mu_0 \to \mu) \times \mu_0\psi(0,-\mu_0) , \quad \mu \in (-1,1) , \quad (7)$$

where the known functions are

$$S(\mu) = \int_{-1}^{1} d\mu_0 \int_0^\infty dx_0 G(x_0 \to 0; \mu_0 \to \mu) S(x_0,\mu_0) \quad (8a)$$

and

$$F(\mu) = \int_0^1 d\mu_0 G(0 \to 0^+; \mu_0 \to \mu)\mu_0 f(\mu_0) . \quad (8b)$$

Assuming that Eq. (7) has a solution, we can project the equation onto a basis formed by the functions $\phi(\xi,\mu)$ that are associated with the elementary solutions[2] of Eq. (1):

$$\phi(\nu_0,\mu) = \frac{c\nu_0}{2}\frac{1}{\nu_0 - \mu} \quad (9a)$$

and

$$\phi(\nu,\mu) = \frac{c\nu}{2} Pv\left(\frac{1}{\nu - \mu}\right) + \lambda(\nu)\delta(\nu - \mu) , \quad \nu \in (-1,1) , \quad (9b)$$

where

$$\lambda(\nu) = 1 - c\nu \tanh^{-1}(\nu) , \quad (10)$$

and where $\pm\nu_0$ are the zeros of

$$\Lambda(z) = 1 + \frac{c}{2} z \int_{-1}^{1} \frac{d\mu}{\mu - z} . \quad (11)$$

Thus, if we multiply Eq. (7) by $\mu\phi(-\xi,\mu)$, $\xi = \nu_0$ or $\nu \in (0,1)$, and integrate over μ from -1 to 1, we obtain

$$\int_0^1 \phi(\nu_0,\mu)\psi(0,-\mu)\mu d\mu = K(\nu_0) \quad (12a)$$

and

$$\int_0^1 \phi(\nu,\mu)\psi(0,-\mu)\mu d\mu = K(\nu) , \quad \nu \in (0,1) , \quad (12b)$$

where the known functions are

$$K(\xi) = \int_0^1 \phi(-\xi,\mu) f(\mu)\mu d\mu + \int_{-1}^{1} d\mu\phi(-\xi,\mu) \int_0^\infty dx S(x,\mu)\exp(-x/\xi) . \quad (13)$$

Equation (12b) is clearly a singular integral equation for the exit distribution $\psi(0,-\mu)$, $\mu > 0$, and Eq. (12a) is an integral constraint on the exit distribution. In Sec. II, we develop the unique solution of Eqs. (12). We note that multiplying Eq. (7) by $\mu\phi(\xi,\mu)$, $\xi = \nu_0$ or $\nu \in (0,1)$, and integrating over μ from -1 to 1 yields only a trivial identity.

[2]K. M. CASE, *Ann. Phys.*, 9, 1 (1960).

II. EXACT RESULTS FOR THE EXIT DISTRIBUTION

We now wish to use the methods of Muskhelishvili[3] to solve Eqs. (12). If we let $\mu\psi(0,-\mu) = F(\mu)$, then we can consider Eq. (12b) to be written as

$$\lambda(\nu)F(\nu) - \frac{c\nu}{2} p \int_0^1 F(\mu)\frac{d\mu}{\mu - \nu} = K(\nu) , \quad \nu \in (0,1) . \quad (14)$$

If we introduce

$$N(z) = \frac{1}{2\pi i}\int_0^1 F(\mu)\frac{d\mu}{\mu - z} , \quad (15)$$

then Eq. (14) can be written as

$$N^+(\nu)\Lambda^-(\nu) - N^-(\nu)\Lambda^+(\nu) = K(\nu) , \quad (16)$$

where $N^{\pm}(\nu)$ and $\Lambda^{\pm}(\nu)$ represent the limiting values of the sectionally analytic functions $N(z)$ and $\Lambda(z)$ as the branch cuts are approached from above (+) and below (-). We find that Eq. (16) can be solved to yield

$$N(z) = X(z)\left[\frac{1}{2\pi i}\int_0^1 \frac{K(\tau)}{X^+(\tau)\Lambda^-(\tau)}\frac{d\tau}{\tau - z} + B\right] , \quad (17)$$

where B is a constant to be determined by Eq. (12a). Since Eq. (17) represents the general solution of Eq. (16), it follows that Eqs. (12) have only one solution. In Eq. (17), the X function is that introduced by Case[2]:

$$X(z) = \frac{1}{1 - z}\exp\left[\frac{1}{\pi}\int_0^1 \arg \Lambda^+(\tau)\frac{d\tau}{\tau - z}\right] . \quad (18)$$

If we use Eq. (12a) to fix the constant B, then we can write Eq. (17) as

$$N(z) = \frac{X(z)}{2\pi i}\left[(\nu_0 - z)\int_0^1 \frac{K(\tau)}{X^+(\tau)\Lambda^-(\tau)(\nu_0 - \tau)}\frac{d\tau}{\tau - z} - \frac{2}{c\nu_0 X(\nu_0)} K(\nu_0)\right] , \quad (19)$$

and since Eq. (15) yields

$$F(\mu) = N^+(\mu) - N^-(\mu) , \quad (20)$$

we can use Eq. (19) to find $F(\mu)$. Our final result is thus

[3]N. I. MUSKHELISHVILI, *Singular Integral Equations*, Noordhoff, Groningen, Holland (1953).

$$\psi(0,-\mu) = \int_0^1 R(\mu' \to \mu) f(\mu')\, d\mu' + \int_0^\infty dx \int_{-1}^1 d\mu' S(x,\mu') L(x;\mu' \to \mu) \quad , \quad \mu > 0 \quad , \tag{21}$$

where

$$R(\mu' \to \mu) = \frac{c}{2}\mu' \frac{H(\mu)H(\mu')}{\mu' + \mu} \tag{22}$$

and

$$L(x;\mu' \to \mu) = H(\mu)\left[\frac{\phi(-\nu_0,\mu')\phi(\nu_0,\mu)\exp(-x/\nu_0)}{H(\nu_0)N(\nu_0)} + \int_0^1 \frac{\phi(-\nu,\mu')\phi(\nu,\mu)\exp(-x/\nu)}{H(\nu)N(\nu)}\, d\nu\right] \quad . \tag{23}$$

Here, $H(z)$ is Chandrasekhar's H function,[4]

$$N(\nu_0) = \frac{c}{2}\nu_0^3\left[\frac{c}{\nu_0^2 - 1} - \frac{1}{\nu_0^2}\right] \tag{24a}$$

and

$$N(\nu) = \nu\left[\lambda^2(\nu) + \frac{c^2\nu^2\pi^2}{4}\right] \quad . \tag{24b}$$

It is clear that our exact result given by Eq. (21) reduces to the well-known solution of Chandrasekhar[4] when $S(x,\mu) = 0$.

III. THE F_N METHOD

The C_N method[5,6] of approximately solving problems in neutron-transport theory has proved to be an accurate and economical method. We wish to develop here a modified version of the C_N method (which we call the F_N method) that yields more concise equations that can be solved numerically even more efficiently. For the traditional C_N method, we use the representation

$$\psi(0,-\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha \quad , \quad \mu > 0 \quad , \tag{25}$$

in Eq. (7) for either $\mu > 0$ or $\mu < 0$, i.e.,

$$f(\mu) = S(\mu) + F(\mu) - \int_0^1 d\mu' G(0 \to 0^+; -\mu' \to \mu)\mu'\psi(0,-\mu') \quad , \quad \mu > 0 \quad , \tag{26a}$$

or

$$\psi(0,-\mu) = S(-\mu) + F(-\mu) - \int_0^1 d\mu' G(0 \to 0^+; -\mu' \to -\mu)\mu'\psi(0,-\mu') \quad , \quad \mu > 0 \quad . \tag{26b}$$

[4] S. CHANDRASEKHAR, *Radiative Transfer*, Oxford University Press, London (1950).

[5] P. BENOIST and A. KAVENOKY, *Nucl. Sci. Eng.*, 32, 225 (1968).

[6] A. KAVENOKY, *Nucl. Sci. Eng.*, **65**, 209 (1978).

Having substituted Eq. (25) into Eq. (26a) or (26b), we can multiply Eq. (26a) or (26b) by μ^β, $\beta = 0, 1, 2, \ldots, N$, and integrate over μ from $0 \to 1$ to obtain $N + 1$ linear algebraic equations to solve for the $N + 1$ coefficients a_α required in Eq. (25).

To establish the F_N method, we wish to use Eq. (25) in Eqs. (12). We thus find

$$\sum_{\alpha=0}^{N} a_\alpha B_\alpha(\xi) = \frac{2}{c\xi} K(\xi) \quad , \quad \xi = \nu_0 \text{ or } \nu \in (0,1) \quad . \tag{27}$$

Here,

$$B_\alpha(\xi) = \frac{2}{c\xi}\int_0^1 \mu^{\alpha+1}\phi(\xi,\mu)\, d\mu \tag{28}$$

can easily be shown to satisfy

$$B_\alpha(\xi) = \xi B_{\alpha-1}(\xi) - \frac{1}{\alpha+1} \quad , \quad \alpha \geq 1 \quad , \tag{29a}$$

with

$$B_0(\xi) = \frac{2}{c} - 1 - \xi \log\left(1 + \frac{1}{\xi}\right) \quad . \tag{29b}$$

It is clear that we can now choose $N + 1$ values of $\xi \in \nu_0 U[0,1]$, for example ξ_β, and solve the set of algebraic equations

$$\sum_{\alpha=0}^{N} a_\alpha B_\alpha(\xi_\beta) = \frac{2}{c\xi_\beta} K(\xi_\beta) \quad , \quad \beta = 0, 1, 2, \ldots, N \quad . \tag{30}$$

To illustrate explicitly the F_N method, we consider the half-space constant-source problem, i.e., we seek $\psi(0,-\mu)$, where

$$\mu \frac{\partial}{\partial x}\psi(x,\mu) + \psi(x,\mu) = \frac{c}{2}\int_{-1}^1 \psi(x,\mu')\, d\mu' + a \quad , \tag{31}$$

$$\psi(0,\mu) = b \quad , \quad \mu > 0 \quad , \tag{32a}$$

and

$$\psi(x,\mu) \to \frac{a}{1-c} \quad , \quad \text{as } x \to \infty \quad . \tag{32b}$$

Here, Eq. (13) yields

$$\frac{2}{c\xi} K(\xi) = \frac{2a}{c} + b\left[1 - \xi \log\left(1 + \frac{1}{\xi}\right)\right] \quad , \tag{33}$$

and thus Eq. (30) becomes (for this problem) simply

$$\sum_{\alpha=0}^{N} a_\alpha B_\alpha(\xi_\beta) = \frac{2a}{c} + b\left[1 - \xi_\beta \log\left(1 + \frac{1}{\xi_\beta}\right)\right] \quad , \quad \beta = 0, 1, 2, \ldots, N \quad . \tag{34}$$

To compare this approximation to exact results, we consider

$$A^* = 2\int_0^1 \psi(0,-\mu)\mu\, d\mu = 2\sum_{\alpha=0}^{N} \frac{a_\alpha}{\alpha + 2} \tag{35}$$

for the two cases $a = 1$ and $b = 0$ and $a = 0$ and $b = 1$. Of course, the manner in which the ξ_β's are chosen can affect the accuracy of the method. Some preliminary results of Grandjean and Siewert[7] suggest that the F_N method can yield results for A^* accurate to at least three significant figures for $N \leq 5$. Grandjean and Siewert's calculations for $c \in [0.1, 0.9]$ were based on an equal spacing scheme for choosing ξ_β, i.e., $\xi_0 = \nu_0$, $\xi_1 = 0$, $\xi_2 = 1$, and the remaining ξ_β are spaced equally in the interval $[0,1]$.

IV. THE FINITE SLAB

Having developed the exact result and introduced the F_N method for the case of a semi-infinite half-space, we now wish to consider the more interesting case of a finite slab. Here, we seek to solve

$$\left(\mu \frac{\partial}{\partial x} + 1 - \frac{c}{2}\int_{-1}^{1} \ldots d\mu\right)\psi(x,\mu) = S(x,\mu) \ , \quad x \in [-a,a] \ , \tag{36}$$

subject to the boundary conditions

$$\psi(-a,\mu) = f_1(\mu) \ , \quad \mu > 0 \ , \tag{37a}$$

and

$$\psi(a,-\mu) = f_2(\mu) \ , \quad \mu > 0 \ . \tag{37b}$$

Here, the Placzek lemma[1] allows us to write

$$H_*(x)\psi(x,\mu) = \psi_1(x,\mu) \ , \tag{38}$$

where $H_*(x) = 1$ for $x \in [-a,a]$, $H_*(x) = 0$ otherwise, and $\psi_1(x,\mu)$ is the solution in an infinite medium of

$$\left(\mu \frac{\partial}{\partial x} + 1 - \frac{c}{2}\int_{-1}^{1} \ldots d\mu\right)\psi_1(x,\mu) = H_*S(x,\mu) + \mu\psi(x,\mu)[\delta(x+a) - \delta(x-a)] \ . \tag{39}$$

We can use the Green's function defined by Eqs. (4) to write

$$\psi_1(x,\mu) = \int_{-1}^{1} d\mu_0 \Big[\int_{-a}^{a} dx_0 G(x_0 \to x;\ \mu_0 \to \mu)S(x_0,\mu_0) + G(-a \to x;\ \mu_0 \to \mu)\mu_0\psi(-a,\mu_0) - G(a \to x;\ \mu_0 \to \mu)\mu_0\psi(a,\mu_0)\Big] \ . \tag{40}$$

If we now let $x \to \pm a$ in Eq. (40), we obtain a system of equations for the exit distributions:

$$\psi(a,\mu) = K(a^-,\mu) - \int_0^1 G(a \to a^-;\ \mu_0 \to \mu)\mu_0\psi(a,\mu_0)d\mu_0 - \int_0^1 G(-a \to a;\ -\mu_0 \to \mu)\mu_0\psi(-a,-\mu_0)d\mu_0 \tag{41a}$$

[7]P. GRANDJEAN and C. E. SIEWERT, *Nucl. Sci. Eng.*, **69**, 161 (1979).

and

$$\psi(-a,\mu) = K(-a^+,\mu) - \int_0^1 G(a \to -a;\ \mu_0 \to \mu)\mu_0\psi(a,\mu_0)d\mu_0 - \int_0^1 G(-a \to -a^+;\ -\mu_0 \to \mu)\mu_0\psi(-a,-\mu_0)d\mu_0 \ , \tag{41b}$$

where the known functions $K(\xi,\mu)$ are given by

$$K(\xi,\mu) = \int_{-a}^{a} dx_0 \int_{-1}^{1} d\mu_0 G(x_0 \to \xi;\ \mu_0 \to \mu)S(x_0,\mu_0) + \int_0^1 G(-a \to \xi;\ \mu_0 \to \mu)\mu_0 f_1(\mu_0)d\mu_0 + \int_0^1 G(a \to \xi;\ -\mu_0 \to \mu)\mu_0 f_2(\mu_0)d\mu_0 \ . \tag{42}$$

To develop equations analogous to Eqs. (12) that were used for the half space, we can multiply Eqs. (41) by $\mu\phi(\mp\xi,\mu)$, $\xi \in \nu_0 U(0,1)$, and integrate over μ from -1 to 1 to obtain

$$\int_0^1 \phi(\xi,\mu)\mu\psi(-a,-\mu)d\mu + \exp(-2a/\xi)\int_0^1 \phi(-\xi,\mu)\mu\psi(a,\mu)d\mu = L_1(\xi) \ , \quad \xi \in \nu_0 U(0,1) \ , \tag{43a}$$

and

$$\int_0^1 \phi(\xi,\mu)\mu\psi(a,\mu)d\mu + \exp(-2a/\xi)\int_0^1 \phi(-\xi,\mu)\mu\psi(-a,-\mu)d\mu = L_2(\xi) \ , \quad \xi \in \nu_0 U(0,1) \ , \tag{43b}$$

where the known terms are

$$L_1(\xi) = \int_0^1 \phi(-\xi,\mu)\mu f_1(\mu)d\mu + \exp(-2a/\xi)\int_0^1 \phi(\xi,\mu)\mu f_2(\mu)d\mu + \exp(-a/\xi)\int_{-1}^{1} d\mu\phi(-\xi,\mu)\int_{-a}^{a} dx S(x,\mu) \times \exp(-x/\xi) \tag{44a}$$

and

$$L_2(\xi) = \int_0^1 \phi(-\xi,\mu)\mu f_2(\mu)d\mu + \exp(-2a/\xi)\int_0^1 \phi(\xi,\mu)\mu f_1(\mu)d\mu + \exp(-a/\xi)\int_{-1}^{1} d\mu\phi(\xi,\mu)\int_{-a}^{a} dx S(x,\mu) \times \exp(x/\xi) \ . \tag{44b}$$

Equations (43) clearly represent a system of singular integral equations and constraints that can be regularized to give a system of Fredholm equations for the exit distributions $\psi(-a,-\mu)$ and

$\psi(a,\mu)$, $\mu > 0$. However, rather than pursue the "exact" solution of Eqs. (43), we wish to invoke the F_N approximation. Thus, we let

$$\psi(-a,-\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha \quad , \quad \mu > 0 \quad , \tag{45a}$$

and

$$\psi(a,\mu) = \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha \quad , \quad \mu > 0 \quad , \tag{45b}$$

and upon entering Eqs. (45) into Eqs. (43), we find

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\xi_\beta) + \exp(-2a/\xi_\beta) b_\alpha A_\alpha(\xi_\beta)] = \frac{2}{c\xi_\beta} L_1(\xi_\beta) \quad , \quad \beta = 0, 1, 2, \ldots, N \quad , \tag{46a}$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha(\xi_\beta) + \exp(-2a/\xi_\beta) a_\alpha A_\alpha(\xi_\beta)] = \frac{2}{c\xi_\beta} L_2(\xi_\beta) \quad , \quad \beta = 0, 1, 2, \ldots, N \quad . \tag{46b}$$

Here, the $B_\alpha(\xi)$'s are as given by Eqs. (29) and

$$A_\alpha(\xi) = \frac{2}{c\xi} \int_0^1 \mu^{\alpha+1} \phi(-\xi,\mu) d\mu \quad . \tag{47}$$

Thus, we have

$$A_0(\xi) = 1 - \xi \log\left(1 + \frac{1}{\xi}\right) \tag{48a}$$

and

$$A_\alpha(\xi) = -\xi A_{\alpha-1}(\xi) + \frac{1}{\alpha + 1} \quad , \quad \alpha \geq 1 \quad . \tag{48b}$$

Equations (46) are clearly $2(N + 1)$ linear algebraic equations for the $2(N + 1)$ unknowns a_α and b_α, $\alpha = 0, 1, 2, \ldots, N$.

As an explicit example of the F_N method, we consider the critical problem. Thus, for $c > 1$ and $S(x,\mu) = f_1(\mu) = f_2(\mu) = 0$, we seek the critical half-thickness, a. Since here $\psi(x,\mu) = \psi(-x,-\mu)$, we note that $a_\alpha = b_\alpha$. Thus, we need consider only

$$\sum_{\alpha=0}^{N} a_\alpha [B_\alpha(\xi_\beta) + \exp(-2a/\xi_\beta) A_\alpha(\xi_\beta)] = 0 \quad , \quad \beta = 0, 1, 2, \ldots, N \quad . \tag{49}$$

Equation (49) is of the form

$$\mathbf{M}(a)\mathbf{A} = \mathbf{0} \quad , \tag{50}$$

where the elements of $\mathbf{A}$ are a_α, $\alpha = 0, 1, 2, \ldots, N$, and $\mathbf{M}(a)$ is an $(N + 1) \times (N + 1)$ matrix. Of course, the value of a can be established immediately from $\det \mathbf{M}(a) = 0$. Again, the work of Grandjean and Siewert[7] suggests that Eq. (49), for $c \in [1.1, 2.5]$, can yield a value of a accurate to at least three significant figures for $N \leq 5$.

ACKNOWLEDGMENTS

The authors are grateful to P. Grandjean for communicating some of his numerical results concerning this work. One of the authors (CES) is grateful to the Centre d'Etudes Nucléaires de Saclay for kind hospitality and partial support of this work. This work was also supported in part by the National Science Foundation.