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The F_N Method in Neutron-Transport Theory. Part II: Applications and Numerical Results

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The recently developed F_N method is used to solve the half-space albedo problem and the half-space constant-source problem. In addition, the reflected and transmitted currents for the finite slab and the critical thickness of a multiplying slab are reported. As further tests of the method, the inverse problem for the finite slab is solved, and the flux distortion factor for typical two-media problems is computed. It is shown that the F_N method, although particularly concise, yields excellent numerical results for the problems considered.

I. INTRODUCTION

In a companion paper,[1] hereafter referred to as Part I, the F_N method of approximately solving problems in one-speed neutron-transport theory was introduced. Here, we wish to apply the F_N method to several basic problems and to evaluate the accuracy of the established solutions.

II. HALF-SPACE PROBLEMS

Although in Part I the F_N method was developed from the use of the Placzek lemma and thus resembled the C_N method,[2,3] we can now establish the same basic equations in a more direct manner. We first consider the half-space problem defined, for $c < 1$, by

$$\mu \frac{\partial}{\partial x} \psi(x,\mu) + \psi(x,\mu) = \frac{c}{2} \int_{-1}^{1} \psi(x,\mu')\, d\mu' + a \quad , \quad (1a)$$

$$\psi(0,\mu) = 1 - a \quad , \quad \mu > 0 \quad , \quad (1b)$$

and

$$\psi(x,\mu) \to \frac{a}{1-c} \quad , \quad \text{as } x \to \infty \quad . \quad (1c)$$

Clearly, $a = 1$ yields the usual constant-source problem,[4] whereas the half-space albedo problem is obtained by taking $a = 0$. If we express the solution in terms of the elementary solutions[4,5] and a particular solution, namely,

$$\psi(x,\mu) = A(\nu_0)\phi(\nu_0,\mu)\exp(-x/\nu_0) + \int_0^1 A(\nu)\phi(\nu,\mu)\exp(-x/\nu)\, d\nu + \frac{a}{1-c} \quad , \quad (2)$$

then, on noting that

$$\int_{-1}^{1} \left[\psi(x,\mu) - \frac{a}{1-c}\right] \mu\phi(-\xi,\mu)\, d\mu = 0 \quad , \quad \xi \in P \quad , \quad (3)$$

we find immediately, for $x = 0$, the singular integral equation and the constraint developed in Part I. Note that $\xi \in P \Rightarrow \xi = \nu_0$ or $\xi = \nu \in (0,1)$. Thus,

$$\frac{2}{c\xi} \int_0^1 \phi(\xi,\mu)\psi(0,-\mu)\mu\, d\mu = \frac{2a}{c} + (1-a)\left[1 - \xi \log\left(1 + \frac{1}{\xi}\right)\right] \quad , \quad \xi \in P \quad . \quad (4)$$

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[1]C. E. SIEWERT and P. BENOIST, *Nucl. Sci. Eng.*, **69**, 156 (1979).
[2]P. BENOIST and A. KAVENOKY, *Nucl. Sci. Eng.*, **32**, 225 (1968).
[3]A. KAVENOKY, *Nucl. Sci. Eng.*, **65**, 209 (1978).
[4]K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts (1967).
[5]K. M. CASE, *Ann. Phys.*, **9**, 1 (1960).

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Here

$$\phi(\nu_0, \mu) = \frac{c\nu_0}{2} \frac{1}{\nu_0 - \mu} \tag{5a}$$

and

$$\phi(\nu, \mu) = \frac{c\nu}{2} Pv\left(\frac{1}{\nu - \mu}\right) + \lambda(\nu)\delta(\nu - \mu) \quad , \quad \nu \in (-1,1) \quad , \tag{5b}$$

where

$$\lambda(\nu) = 1 - c\nu \tanh^{-1}(\nu) \tag{6}$$

and $\pm\nu_0$ are the zeros of

$$\Lambda(z) = 1 + \frac{cz}{2} \int_{-1}^{1} \frac{d\mu}{\mu - z} \quad . \tag{7}$$

As discussed in Part I, we can enter

$$\psi(0,-\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha \quad , \quad \mu > 0 \quad , \tag{8}$$

into Eq. (4) evaluated at selected values of ξ to obtain the F_N equations

$$\sum_{\alpha=0}^{N} a_\alpha B_\alpha(\xi_\beta) = \frac{2a}{c} + (1 - a) \left[1 - \xi_\beta \log\left(1 + \frac{1}{\xi_\beta}\right)\right] \quad , \quad \beta = 0, 1, 2, \ldots, N \quad , \tag{9}$$

where

$$B_\alpha(\xi) = \xi B_{\alpha-1}(\xi) - \frac{1}{\alpha + 1} \quad , \quad \alpha \geq 1 \quad , \tag{10a}$$

with

$$B_0(\xi) = \frac{2}{c} - 1 - \xi \log\left(1 + \frac{1}{\xi}\right) \quad . \tag{10b}$$

Here, we choose to use a simple scheme of picking the ξ_β in Eq. (9). Thus, we let $\xi_0 = \nu_0$, $\xi_1 = 0$, $\xi_2 = 1$, and the remaining ξ_β are spaced equally in the interval [0,1].

In Table I, we give our results for the computation of

$$A^* = 2 \int_0^1 \psi(0,-\mu)\mu \, d\mu = 2 \sum_{\alpha=0}^{N} \frac{a_\alpha}{\alpha + 2} \tag{11}$$

for the case $a = 0$. Table II contains A^* for $a = 1$. It is clear that, for the albedo problem, the F_5

TABLE I

A^* for $a = 0$

c	F_0	F_1	F_2	F_3	F_4	F_5	Exact
0.1	0.01676	0.02775	0.02375	0.02186	0.02177	0.02172	0.02170
0.2	0.03694	0.05730	0.04986	0.04651	0.04639	0.04630	0.04626
0.3	0.06160	0.08891	0.07900	0.07474	0.07461	0.07449	0.07445
0.4	0.09203	0.1235	0.1123	0.1076	0.1075	0.1074	0.1073
0.5	0.1299	0.1627	0.1514	0.1468	0.1467	0.1466	0.1465
0.6	0.1778	0.2094	0.1990	0.1950	0.1949	0.1948	0.1947
0.7	0.2406	0.2686	0.2600	0.2568	0.2567	0.2566	0.2566
0.8	0.3284	0.3503	0.3442	0.3420	0.3420	0.3419	0.3419
0.9	0.4691	0.4822	0.4791	0.4781	0.4781	0.4780	0.4780

TABLE II

A^* for $a = 1$

c	F_0	F_1	F_2	F_3	F_4	F_5	Exact
0.1	1.092	1.080	1.085	1.087	1.087	1.087	1.087
0.2	1.204	1.178	1.188	1.192	1.192	1.192	1.192
0.3	1.341	1.302	1.316	1.322	1.322	1.322	1.322
0.4	1.513	1.461	1.480	1.487	1.487	1.488	1.488
0.5	1.740	1.675	1.697	1.706	1.707	1.707	1.707
0.6	2.055	1.976	2.002	2.013	2.013	2.013	2.013
0.7	2.531	2.438	2.467	2.477	2.478	2.478	2.478
0.8	3.358	3.248	3.279	3.290	3.290	3.291	3.291
0.9	5.309	5.178	5.209	5.219	5.219	5.220	5.220

approximation yields results for A^* that are accurate, for the cases considered, to three significant figures and accurate to four significant figures for $c \geq 0.7$. Table II reveals that F_5 is accurate to four significant figures for all $a = 1$ cases considered. We conclude that the F_5 approximation is very good for these two applications, especially since the basic equations to be solved are so simple. We note that the exact results listed in Tables I and II were computed from

$$A^* = 1 - a + \frac{2}{(1-c)^{1/2}}\left[(1-a)(1-c) + a\right] \times \int_0^1 H(\mu)\mu\, d\mu \ , \tag{12}$$

where $H(\mu)$ is the Chandrasekhar H function.[6]

III. THE FINITE SLAB

We wish now to consider the finite-slab problem defined by

$$\mu \frac{\partial}{\partial x}\psi(x,\mu) + \psi(x,\mu) = \frac{c}{2}\int_{-1}^{1}\psi(x,\mu')\,d\mu' \ , \quad x \in [-\tau,\tau] \ , \tag{13a}$$

$$\psi(-\tau,\mu) = \mu^\beta \ , \quad \mu > 0 \ , \quad \beta = 0, 1, 2, \ldots \ , \tag{13b}$$

and

$$\psi(\tau,-\mu) = 0 \ , \quad \mu > 0 \ . \tag{13c}$$

Here, we write

$$\psi(x,\mu) = A(\nu_0)\phi(\nu_0,\mu)\exp(-x/\nu_0) + A(-\nu_0)\phi(-\nu_0,\mu)\exp(x/\nu_0) + \int_{-1}^{1} A(\nu)\phi(\nu,\mu)\exp(-x/\nu)\,d\nu \ , \tag{14}$$

and observe that

$$\int_{-1}^{1}\mu\phi(-\xi,\mu)\psi(\mp\tau,\mu)\,d\mu = A(-\xi)N(-\xi)\exp(\mp\tau/\xi) \ , \quad \xi \in P \tag{15}$$

and

$$\int_{-1}^{1}\mu\phi(\xi,\mu)\psi(\mp\tau,\mu)\,d\mu = A(\xi)N(\xi)\exp(\pm\tau/\xi) \ , \quad \xi \in P \ , \tag{16}$$

where the $N(\pm\xi)$'s are the full-range normalization factors.[4] We can now eliminate $A(\mp\xi)N(\mp\xi)$ between Eqs. (15) and (16) to develop the basic equations reported in Part I for the finite slab, i.e.,

$$\int_0^1 \phi(\xi,\mu)\mu\psi(-\tau,-\mu)\,d\mu + \exp(-2\tau/\xi)\int_0^1 \phi(-\xi,\mu)\mu\psi(\tau,\mu)\,d\mu = \int_0^1 \phi(-\xi,\mu)\mu^{\beta+1}\,d\mu \tag{17a}$$

and

$$\int_0^1 \phi(\xi,\mu)\mu\psi(\tau,\mu)\,d\mu + \exp(-2\tau/\xi)\int_0^1 \phi(-\xi,\mu)\mu\psi(-\tau,-\mu)\,d\mu = \exp(-2\tau/\xi)\int_0^1 \phi(\xi,\mu)\mu^{\beta+1}\,d\mu \ . \tag{17b}$$

Thus, we can enter

$$\psi(-\tau,-\mu) = \sum_{\alpha=0}^{N} a_\alpha\mu^\alpha \ , \quad \mu > 0 \tag{18a}$$

and

$$\psi(\tau,\mu) = \sum_{\alpha=0}^{N} b_\alpha\mu^\alpha \ , \quad \mu > 0 \tag{18b}$$

into Eqs. (17) evaluated at selected values of $\xi \in \nu_0 U(0,1)$ to obtain the F_N equations for this problem, i.e.,

$$\sum_{\alpha=0}^{N}\left[a_\alpha B_\alpha(\xi_j) + \exp(-2\tau/\xi_j)b_\alpha A_\alpha(\xi_j)\right] = A_\beta(\xi_j) \ , \quad \xi_j \in P \ , \tag{19a}$$

and

$$\sum_{\alpha=0}^{N}\left[b_\alpha B_\alpha(\xi_j) + \exp(-2\tau/\xi_j)a_\alpha A_\alpha(\xi_j)\right] = \exp(-2\tau/\xi_j)B_\beta(\xi_j) \ , \quad \xi_j \in P \ . \tag{19b}$$

Here, $B_\alpha(\xi)$ is again as given by Eqs. (10), whereas

$$A_\alpha(\xi) = -\xi A_{\alpha-1}(\xi) + \frac{1}{\alpha+1} \ , \quad \alpha \geq 1 \ , \tag{20a}$$

with

$$A_0(\xi) = 1 - \xi\log\left(1 + \frac{1}{\xi}\right) \ . \tag{20b}$$

Equations (19) for $j = 0, 1, 2, \ldots, N$ can readily be solved to yield a_α and b_α, and thus we are able to compute the albedo,

$$A^* = (\beta+2)\int_0^1 \psi(-\tau,-\mu)\mu\,d\mu = (\beta+2)\sum_{\alpha=0}^{N}\frac{a_\alpha}{\alpha+2} \ , \tag{21a}$$

and the transmission factor,

$$B^* = (\beta+2)\int_0^1 \psi(\tau,\mu)\mu\,d\mu = (\beta+2)\sum_{\alpha=0}^{N}\frac{b_\alpha}{\alpha+2} \ . \tag{21b}$$

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[6] S. CHANDRASEKHAR, *Radiative Transfer*, Oxford University Press, London (1950).

In Tables III and IV, we show typical results from our study, for $c \in [0.1, 0.9]$ and $2\tau = 1.0$, of the effect of β on the calculation of the albedo and the transmission factor. Again, we see that the F_5 approximation yields results accurate to three or four significant figures. In Table V, we list the value of c computed from

$$c = \frac{4\psi_\beta(-\tau)}{\left[\psi_0(-\tau) + \frac{1}{\beta+1}\right]^2 - [\psi_0(\tau)]^2} , \tag{22}$$

which is the solution of the inverse problem discussed elsewhere.[7] Here,

$$\psi_\alpha(-\tau) = \int_0^1 \psi(-\tau,-\mu)\mu^\alpha \, d\mu \tag{23a}$$

and

$$\psi_\alpha(\tau) = \int_0^1 \psi(\tau,\mu)\mu^\alpha \, d\mu \; . \tag{23b}$$

[7] C. E. SIEWERT, to appear in *J. Math. Phys.* (1978).

Thus, we have chosen a value of c, solved Eqs. (19) to find a_α and b_α, and then computed c from

$$c = 4 \sum_{\alpha=0}^{N} \frac{a_\alpha}{\alpha+\beta+1}\left[\left(\sum_{\alpha=0}^{N} \frac{a_\alpha}{\alpha+1} + \frac{1}{\beta+1}\right)^2 - \left(\sum_{\alpha=0}^{N} \frac{b_\alpha}{\alpha+1}\right)^2\right]^{-1} . \tag{24}$$

For interest, we give in Table VI the value of c computed in a similar manner by the C_N method.[2,3] The C_N results (for a given N) are clearly better, but we must remember that the F_N equations are much simpler.

In Tables VII and VIII, we list results typical of our study, for $\beta = 0$ and $c \in [0.1, 0.9]$, of the effect of the slab thickness on the accuracy of the F_N method. Tables IX and X show the effect of c, for $\beta = 0$ and $2\tau = 1.0$, on the accuracy of the method. We note that the F_5 approximation consistently yields results accurate to three or four significant figures.

TABLE III

A^* for $c = 0.8$ and $2\tau = 1.0$

β	F_0	F_1	F_2	F_3	F_4	F_5	Exact
0	0.2557	0.2896	0.2813	0.2803	0.2803	0.2802	0.2802
1	0.2326	0.2576	0.2578	0.2586	0.2586	0.2587	0.2587
2	0.2184	0.2413	0.2482	0.2469	0.2470	0.2471	0.2471
3	0.2088	0.2308	0.2449	0.2396	0.2398	0.2399	0.2399
4	0.2018	0.2235	0.2453	0.2348	0.2350	0.2350	0.2350

TABLE IV

B^* for $c = 0.8$ and $2\tau = 1.0$

β	F_0	F_1	F_2	F_3	F_4	F_5	Exact
0	0.4502	0.4142	0.4189	0.4161	0.4161	0.4162	0.4162
1	0.5050	0.4674	0.4512	0.4514	0.4516	0.4517	0.4516
2	0.5430	0.5034	0.4600	0.4723	0.4721	0.4721	0.4721
3	0.5712	0.5300	0.4541	0.4856	0.4853	0.4852	0.4852
4	0.5931	0.5505	0.4378	0.4938	0.4942	0.4942	0.4942

TABLE V

The Computed Value of c for $2\tau = 1$

β	F_0	F_1	F_2	F_3	F_4	F_5	Exact
0	0.7444	0.8245	0.8094	0.8021	0.8009	0.8005	0.8
1	0.9821	0.8165	0.8019	0.8009	0.8006	0.8006	0.8
2	1.1922	0.8414	0.7934	0.8038	0.8022	0.8010	0.8
3	1.4146	0.8711	0.7733	0.8081	0.8028	0.8009	0.8
4	1.6634	0.9010	0.7508	0.8146	0.8020	0.8007	0.8

TABLE VI

The Computed Value of c for $2\tau = 1$

β	C_0	C_1	C_2	C_3	C_4	C_5	Exact
0	0.7634	0.7923	0.7982	0.7995	0.7998	0.7999	0.8
1	0.9502	0.7962	0.7994	0.7999	0.7999	0.7999	0.8
2	1.070	0.7805	0.8040	0.7997	0.7995	0.7998	0.8
3	1.161	0.7640	0.8121	0.7980	0.7993	0.7998	0.8
4	1.235	0.7493	0.8223	0.7948	0.7996	0.7998	0.8

TABLE VII

A^* for $c = 0.8$ and $\beta = 0$

2τ	F_0	F_1	F_2	F_3	F_4	F_5	Exact
0.1	0.04798	0.1192	0.08763	0.06713	0.06563	0.06478	0.06493
0.5	0.1764	0.2240	0.2088	0.2051	0.2055	0.2057	0.2056
1.0	0.2557	0.2896	0.2813	0.2803	0.2803	0.2802	0.2802
2.0	0.3112	0.3359	0.3300	0.3282	0.3281	0.3280	0.3280
5.0	0.3281	0.3501	0.3440	0.3418	0.3418	0.3417	0.3417

TABLE VIII

B^* for $c = 0.8$ and $\beta = 0$

2τ	F_0	F_1	F_2	F_3	F_4	F_5	Exact
0.1	0.9168	0.8324	0.8703	0.8943	0.8958	0.8968	0.8966
0.5	0.6604	0.6046	0.6209	0.6229	0.6223	0.6219	0.6220
1.0	0.4502	0.4142	0.4189	0.4161	0.4161	0.4162	0.4162
2.0	0.2168	0.2002	0.1982	0.1971	0.1972	0.1973	0.1973
5.0	0.02558	0.02364	0.02259	0.02293	0.02292	0.02292	0.02292

TABLE IX

A^* for $2\tau = 1$ and $\beta = 0$

c	F_0	F_1	F_2	F_3	F_4	F_5	Exact
0.999	0.4274	0.4499	0.4454	0.4455	0.4456	0.4456	0.4455
0.99	0.4174	0.4405	0.4359	0.4360	0.4360	0.4360	0.4360
0.9	0.3305	0.3592	0.3530	0.3528	0.3528	0.3528	0.3527
0.8	0.2557	0.2896	0.2813	0.2803	0.2803	0.2802	0.2802
0.7	0.1969	0.2344	0.2243	0.2223	0.2222	0.2221	0.2221
0.6	0.1496	0.1889	0.1774	0.1746	0.1745	0.1744	0.1743
0.5	0.1110	0.1501	0.1380	0.1345	0.1344	0.1342	0.1342
0.4	0.07930	0.1157	0.1039	0.1001	0.1000	0.09990	0.09985
0.3	0.05325	0.08430	0.07396	0.07035	0.07024	0.07012	0.07007
0.2	0.03194	0.05481	0.04708	0.04417	0.04406	0.04397	0.04393
0.1	0.01449	0.02672	0.02258	0.02090	0.02082	0.02077	0.02075

TABLE X

B^* for $2\tau = 1$ and $\beta = 0$

c	F_0	F_1	F_2	F_3	F_4	F_5	Exact
0.999	0.5706	0.5481	0.5526	0.5525	0.5524	0.5525	0.5525
0.99	0.5629	0.5399	0.5445	0.5443	0.5443	0.5443	0.5444
0.9	0.4996	0.4706	0.4757	0.4747	0.4747	0.4747	0.4747
0.8	0.4502	0.4142	0.4189	0.4161	0.4161	0.4162	0.4162
0.7	0.4160	0.3723	0.3761	0.3710	0.3711	0.3711	0.3712
0.6	0.3929	0.3403	0.3432	0.3353	0.3354	0.3355	0.3355
0.5	0.3783	0.3154	0.3173	0.3064	0.3065	0.3066	0.3067
0.4	0.3705	0.2956	0.2967	0.2826	0.2827	0.2828	0.2830
0.3	0.3676	0.2796	0.2801	0.2626	0.2628	0.2630	0.2631
0.2	0.3675	0.2664	0.2666	0.2459	0.2460	0.2461	0.2463
0.1	0.3678	0.2551	0.2551	0.2316	0.2315	0.2317	0.2319

For the critical problem, we seek for $c > 1$ the value of a such that

$$\sum_{\alpha=0}^{N} a_\alpha[B_\alpha(\xi_j) + \exp(-2a/\xi_j)A_\alpha(\xi_j)] = 0 \quad , \qquad \xi_j \in P \ . \tag{25}$$

We have let $a_0 = 1$ and iterated between

$$\sum_{\alpha=1}^{N} a_\alpha[B_\alpha(\xi_j) + \exp(-2a/\xi_j)A_\alpha(\xi_j)] = -[B_0(\xi_j) + \exp(-2a/\xi_j)A_0(\xi_j)] \quad , \qquad \xi_j \in [0,1] \ , \tag{26a}$$

and

$$\sum_{\alpha=0}^{N} a_\alpha[B_\alpha(\nu_0) + \exp(-2a/\nu_0)A_\alpha(\nu_0)] = 0 \tag{26b}$$

to find the value of a. Our results are shown in Table XI.

IV. TWO-MEDIA PROBLEMS

In several recent publications,[8-10] the method of elementary solutions, the C_N method, and the integral-transform method were used to compute the flux distortion factor for two typical two-media problems. We now wish to demonstrate the merit of the F_N method when applied to these two problems. We consider region one, $x \in [-a,a]$, to be without an inhomogeneous source and to be characterized by c_1. For region two, $|x| > a$, the mean number of secondary neutrons per collision is c_2, and there is for problem A a constant source; for problem B, the angular flux is allowed to diverge as $x \to \infty$ in the same manner as the classical Milne problem.[4] Thus, we consider

$$\mu \frac{\partial}{\partial x} \psi_1(x,\mu) + \psi_1(x,\mu) = \frac{1}{2} c_1 \int_{-1}^{1} \psi_1(x,\mu')d\mu' \ , \qquad x \in [-a,a] \ , \tag{27}$$

and

$$\mu \frac{\partial}{\partial x} \psi_2(x,\mu) + \psi_2(x,\mu) = \frac{1}{2} c_2 \int_{-1}^{1} \psi_2(x,\mu')d\mu' + A \ , \qquad |x| > a \ , \tag{28}$$

subject to

$$\psi_1(a,\mu) = \psi_2(a,\mu) \quad , \qquad \mu > 0 \ , \tag{29a}$$

$$\psi_1(a,-\mu) = \psi_2(a,-\mu) \quad , \qquad \mu > 0 \ , \tag{29b}$$

$$\psi_2(x,\mu) \to \frac{A}{1-c_2} + B\phi_2(-\eta_0,\mu)\exp(x/\eta_0) \ , \qquad \text{as } x \to \infty \ , \tag{30a}$$

and

$$\psi_2(x,\mu) \to \frac{A}{1-c_2} \quad , \qquad \text{as } x \to -\infty \ . \tag{30b}$$

Here, we allow $A = 1$ and $B = 0$, which corresponds to the constant-source problem, and we allow $B = 1$ and $A = 0$, which corresponds to the case of an exponentially varying flux. We wish to compute the distortion factor,

$$\Delta = \left[\frac{2A}{1-c_2} + \frac{B}{a}\eta_0 \sinh(a/\eta_0)\right]^{-1} \times \frac{1}{2a}\int_{-a}^{a}\int_{-1}^{1} \psi_1(x,\mu)d\mu dx \ , \tag{31}$$

[8]C. E. SIEWERT, K. NESHAT, and J. S. PHELPS III, *Nucl. Sci. Eng.*, **64**, 884 (1977).

[9]V. C. BOFFI, V. G. MOLINARI, and G. SPIGA, *Nucl. Sci. Eng.*, **66**, 424 (1978).

[10]P. BENOIST, V. C. BOFFI, P. GRANDJEAN, A. KAVENOKY, V. G. MOLINARI, C. E. SIEWERT, and G. SPIGA, *Nucl. Sci. Eng.*, **68**, 217 (1978).

TABLE XI

The Critical Thickness

c	F_0	F_1	F_2	F_3	F_4	F_5	Exact
1.10	4.2776	4.1984	4.2231	4.2262	4.2264	4.2265	4.2266
1.30	1.8810	1.8278	1.8610	1.8754	1.8754	1.8753	1.8755
1.50	1.1911	1.1577	1.1874	1.2093	1.2100	1.2101	1.2101
1.70	0.8520	0.8343	0.8581	0.8826	0.8846	0.8852	0.8851
1.90	0.6507	0.6454	0.6633	0.6876	0.6907	0.6919	0.6919

which, after symmetrizing problem B, we can write as

$$\Delta = -\left[\frac{2aA(1-c_1)}{1-c_2} + 2(1-c_1)B\eta_0 \sinh(a/\eta_0)\right]^{-1} \times \int_{-1}^{1} \psi(a,\mu)\mu\,d\mu \quad . \tag{32}$$

For $A = 1$ and $B = 0$, Δ is the ratio of the average flux in region 1 to the flux at infinity, whereas for $A = 0$ and $B = 1$, Δ is the ratio of the average flux in region 1 to what the average flux there would be if $c_1 = c_2$. By writing[4]

$$\psi_1(x,\mu) = A(\nu_0)[\phi_1(\nu_0,\mu)\exp(-x/\nu_0) + \phi_1(-\nu_0,\mu)\exp(x/\nu_0)] + \int_0^1 A(\nu) \times [\phi_1(\nu,\mu)\exp(-x/\nu) + \phi_1(-\nu,\mu)\exp(x/\nu)]d\nu \tag{33}$$

and

$$\psi_2(x,\mu) = B(\eta_0)\phi_2(\eta_0,\mu)\exp(-x/\eta_0) + \int_0^1 B(\eta)\phi_2(\eta,\mu)\exp(-x/\eta)\,d\eta + \frac{A}{1-c_2} + B\phi_2(-\eta_0,\mu)\exp(x/\eta_0) \ , \quad x > a \ , \tag{34}$$

we note that

$$\int_{-1}^{1} \psi(a,\mu)\phi_1(\xi,\mu)\mu\,d\mu + \exp(-2a/\xi) \times \int_{-1}^{1} \psi(a,\mu)\phi_1(-\xi,\mu)\mu\,d\mu = 0 \ , \quad \xi \in P \ , \tag{35a}$$

$$\int_{-1}^{1} \psi(a,\mu)\phi_2(-\eta,\mu)\mu\,d\mu = -A\eta \ , \quad \eta \in (0,1) \ , \tag{35b}$$

and

$$\int_{-1}^{1} \psi(a,\mu)\phi_2(-\eta_0,\mu)\mu\,d\mu = -A\eta_0 - B\exp(a/\eta_0)N_2(\eta_0) \ , \tag{35c}$$

where

$$N_2(\eta_0) = \frac{1}{2}\, c_2\eta_0^3\left(\frac{c_2}{\eta_0^2 - 1} - \frac{1}{\eta_0^2}\right) \quad . \tag{36}$$

If we now substitute

$$\psi(a,\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha \ , \quad \mu > 0 \ , \tag{37a}$$

and

$$\psi(a,-\mu) = \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha \ , \quad \mu > 0 \ , \tag{37b}$$

into Eqs. (35), we find the F_N equations,

$$\sum_{\alpha=0}^{N} a_\alpha\,[B_\alpha^{(1)}(\nu_0) + \exp(-2a/\nu_0)A_\alpha(\nu_0)] - \sum_{\alpha=0}^{N} b_\alpha[A_\alpha(\nu_0) + \exp(-2a/\nu_0)B_\alpha^{(1)}(\nu_0)] = 0 \ , \tag{38a}$$

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha^{(2)}(\eta_0) - a_\alpha A_\alpha(\eta_0)] = \frac{2}{c_2}\left[A + \frac{B}{\eta_0} N_2(\eta_0)\exp(a/\eta_0)\right] \ , \tag{38b}$$

$$\sum_{\alpha=0}^{N} a_\alpha\,[B_\alpha^{(1)}(\nu_j) + \exp(-2a/\nu_j)A_\alpha(\nu_j)] - \sum_{\alpha=0}^{N} b_\alpha[A_\alpha(\nu_j) + \exp(-2a/\nu_j)B_\alpha^{(1)}(\nu_j)] = 0 \ , \quad \nu_j \in [0,1] \ , \tag{38c}$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha^{(2)}(\eta_j) - a_\alpha A_\alpha(\eta_j)] = \frac{2}{c_2} A \ , \quad \eta_j \in [0,1] \ . \tag{38d}$$

Note that we have added a superscript to $B_\alpha(\xi)$ to indicate the use of c_1 and c_2 in Eq. (10b). For our F_0 approximation, we use Eqs. (38a) and (38b) to find a_0 and b_0, while for the F_N approximation, we use Eqs. (38a) and (38b) plus $\nu_j = \eta_j \in [0,1]$, $j = 1, 2, 3, \ldots, N$, in Eqs. (38c) and (38d) to establish the required a_α and b_α. Finally, Δ can be computed from

$$\Delta = \left[\frac{2aA(1-c_1)}{1-c_2} + 2(1-c_1)B\eta_0 \sinh(a/\eta_0)\right]^{-1} \times \sum_{\alpha=0}^{N} \left[\frac{b_\alpha - a_\alpha}{\alpha + 2}\right] \ . \tag{39}$$

Typical results appear in Tables XII and XIII.

TABLE XII

Δ for Constant Source

c_1	c_2	a	F_0	F_1	F_2	F_3	F_4	F_5	Exact
0.1	0.8	1.0	0.1666	0.1638	0.1697	0.1713	0.1713	0.1713	0.1713
0.5	0.8	1.0	0.2684	0.2584	0.2663	0.2687	0.2687	0.2687	0.2688
0.9	0.8	1.0	0.6477	0.6344	0.6414	0.6436	0.6436	0.6436	0.6436
0.5	0.95	0.1	0.5716	0.7432	0.6837	0.6632	0.6650	0.6666	0.6678
0.5	0.95	0.5	0.2937	0.2919	0.2985	0.3028	0.3027	0.3027	0.3027
0.5	0.95	1.0	0.1773	0.1731	0.1763	0.1771	0.1771	0.1771	0.1771
0.1	0.95	0.3	0.2488	0.2749	0.2766	0.2837	0.2841	0.2843	0.2842
0.1	0.975	0.3	0.2048	0.2215	0.2228	0.2274	0.2276	0.2277	0.2277

TABLE XIII

Δ for Exponential Flux

c_1	c_2	a	F_0	F_1	F_2	F_3	F_4	F_5	Exact
0.1	0.8	1.0	0.3838	0.3613	0.3528	0.3581	0.3585	0.3587	0.3587
0.5	0.8	1.0	0.6187	0.5736	0.5492	0.5627	0.5635	0.5637	0.5638
0.9	0.8	1.0	1.4927	1.4133	1.3158	1.3506	1.3531	1.3538	1.3539
0.5	0.95	0.1	0.6193	0.7693	0.7152	0.6969	0.6985	0.7000	0.7010
0.5	0.95	0.5	0.3683	0.3616	0.3689	0.3727	0.3726	0.3726	0.3726
0.5	0.95	1.0	0.2640	0.2560	0.2595	0.2601	0.2601	0.2601	0.2601
0.1	0.95	0.3	0.2903	0.3134	0.3172	0.3236	0.3239	0.3241	0.3240
0.1	0.975	0.3	0.2266	0.2422	0.2446	0.2489	0.2491	0.2492	0.2492

V. CONCLUSIONS

As a result of our studies, we see clearly that the F_N approximation, although especially concise, yields results for $N \leq 5$ that are consistently accurate to three or four significant figures. We note that the computation time generally is very short, since the inversion of at most a 10×10 matrix, whose elements are very elementary functions, is required. We did encounter some difficulty with the convergence of the F_N method for very thin slabs, e.g., slabs of optical thickness <0.1; however, we believe this feature of the method can be improved by finding a more convenient representation of the equations to be solved for the thin slab limit. We note that the results we quote as exact are those deduced from the converged F_N solution and are confirmed when possible with the numerical results of Kavenoky.[11]

It is clear that the F_N method can readily be generalized to one-speed problems with more general scattering kernels and to multigroup theory. Although we are less certain, we are optimistic about the extension of the F_N method to spherical and cylindrical geometries.

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[11]A. KAVENOKY, "La méthode C_N de résolution de l'équation du transport," Doctoral Thesis, Paris (1973).