Technical Notes

Three Basic Neutron-Transport Problems in Spherical Geometry

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Received October 5, 1978
Accepted December 1, 1978

ABSTRACT

The method of elementary solutions and the F_N method are used to solve three basic problems concerning neutron diffusion in spheres.

THE CRITICAL SPHERE

As is well known,[1] the critical problem for a bare sphere can be solved by considering the pseudo-slab problem defined by

$$\mu \frac{\partial}{\partial r} \Phi(r,\mu) + \Phi(r,\mu) = \frac{c}{2} \int_{-1}^{1} \Phi(r,\mu')d\mu' \quad , \tag{1}$$

$$\Phi(r,\mu) = -\Phi(-r,-\mu) \quad , \tag{2a}$$

and

$$\Phi(R,-\mu) = 0 \quad , \quad \mu > 0 \quad . \tag{2b}$$

Of course, the solution to this slab problem can be established "exactly" by the method of elementary solutions, for example[2]; however, here we use the approximation

$$\Phi(R,\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha \quad , \quad \mu > 0 \quad , \tag{3}$$

and the F_N method[3,4] to deduce the very simple and concise equations

$$\sum_{\alpha=1}^{N} a_\alpha [B_\alpha(\nu_\beta) - \exp(-2R/\nu_\beta)A_\alpha(\nu_\beta)] = -[B_0(\nu_\beta) - \exp(-2R/\nu_\beta)A_0(\nu_\beta)] \tag{4a}$$

and

$$\sum_{\alpha=0}^{N} a_\alpha[B_\alpha(\nu_0) - \exp(-2R/\nu_0)A_\alpha(\nu_0)] = 0 \quad , \quad a_0 = 1 \quad , \tag{4b}$$

which we can use, for $c > 1$, to find the critical radius R. We note that ν_0 is the "positive" zero of

$$\Lambda(z) = 1 + \frac{c}{2} z \int_{-1}^{1} \frac{d\mu}{\mu - z} \quad , \tag{5}$$

$$B_0(\xi) = \frac{2}{c} - 1 - \xi \log\left(1 + \frac{1}{\xi}\right) \quad , \tag{6a}$$

$$B_\alpha(\xi) = \xi B_{\alpha-1}(\xi) - \frac{1}{\alpha + 1} \quad , \tag{6b}$$

$$A_0(\xi) = 1 - \xi \log\left(1 + \frac{1}{\xi}\right) \quad , \tag{7a}$$

and

$$A_\alpha(\xi) = -\xi A_{\alpha-1}(\xi) + \frac{1}{\alpha + 1} \quad . \tag{7b}$$

For the F_N method, we use N values of ν_β, spaced equally in the interval [0,1], and iterate between Eqs. (4a) and (4b) to find the critical radius R. We note that the calculation of R in this manner is particularly efficient because only polynomials and the log function are required as coefficients in the system of algebraic equations given by Eq. (4a). In Table I, we list

TABLE I

The Critical Radius

c	F_3	F_4	F_5	F_6	Exact
1.05	7.2769	7.2771	7.2771	7.2772	7.2772
1.07	6.0067	6.0068	6.0069	6.0069	6.0069
1.09	5.1869	5.1870	5.1871	5.1871	5.1871
1.1	4.8725	4.8726	4.8727	4.8727	4.8727
1.3	2.4247	2.4247	2.4248	2.4248	2.4248
1.5	1.6900	1.6901	1.6902	1.6902	1.6902
1.7	1.3120	1.3125	1.3126	1.3126	1.3127
1.9	1.0767	1.0778	1.0781	1.0780	1.0780

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[1] G. J. MITSIS, "Transport Solutions to the Monoenergetic Critical Problems," (Thesis), ANL-6787, Argonne National Laboratory (1963).

[2] K. M. CASE, *Ann. Phys.*, **9**, 1 (1960).

[3] C. E. SIEWERT and P. BENOIST, *Nucl. Sci. Eng.*, **69**, 156 (1979).

[4] P. GRANDJEAN and C. E. SIEWERT, *Nucl. Sci. Eng.*, **69**, 161 (1979).

our numerical results for the F_N method and the exact results obtained[5] by the Case method.[2]

THE ALBEDO PROBLEM

For a bare sphere of radius R, we consider

$$\mu \frac{\partial}{\partial r} \psi(r,\mu) + \frac{(1-\mu^2)}{r} \frac{\partial}{\partial \mu} \psi(r,\mu) + \psi(r,\mu) = \frac{c}{2} \int_{-1}^{1} \psi(r,\mu')d\mu' \tag{8}$$

and

$$\psi(R,-\mu) = 1 \quad , \quad \mu > 0 \quad , \tag{9}$$

and we wish to compute the albedo

$$A^* = 2 \int_0^1 \psi(R,\mu)\mu d\mu \tag{10a}$$

or

$$A^* = 1 - \frac{2}{R^2}(1-c) \int_0^R r^2 \rho(r) dr \quad . \tag{10b}$$

As discussed by Wu and Siewert,[6] the density

$$\rho(r) = \int_{-1}^{1} \psi(r,\mu)d\mu \tag{11}$$

is given by

$$\rho(r) = \frac{1}{r} \Phi_0(r) \quad , \tag{12}$$

where, in general,

$$\Phi_\alpha(r) = \int_{-1}^{1} \Phi(r,\mu)\mu^\alpha d\mu \quad , \tag{13}$$

and where $\Phi(r,\mu)$ is the solution of the pseudo-slab problem defined by

$$\mu \frac{\partial}{\partial r} \Phi(r,\mu) + \Phi(r,\mu) = \frac{c}{2} \int_{-1}^{1} \Phi(r,\mu')d\mu' \quad , \tag{14}$$

$$\Phi(r,\mu) = -\Phi(-r,-\mu) \quad , \tag{15a}$$

and

$$\Phi(-R,\mu) = -(R+\mu) \quad , \quad \mu > 0 \quad . \tag{15b}$$

From Eq. (14), we note that

$$\frac{d}{dr} \Phi_1(r) + (1-c)\Phi_0(r) = 0 \tag{16a}$$

and

$$\frac{d}{dr} \Phi_2(r) + \Phi_1(r) = 0 \quad , \tag{16b}$$

which can be used to obtain

$$A^* = 1 + \frac{2}{R^2} [R\Phi_1(R) + \Phi_2(R)] \quad . \tag{17}$$

If we now introduce the approximation

$$\Phi(R,\mu) = -\sum_{\alpha=0}^{N} a_\alpha \mu^\alpha \quad , \quad \mu > 0 \quad , \tag{18}$$

then we can write

$$\Phi_1(R) = -\sum_{\alpha=0}^{N} \frac{a_\alpha}{\alpha+2} - \frac{1}{2}R - \frac{1}{3} \quad , \tag{19a}$$

$$\Phi_2(R) = -\sum_{\alpha=0}^{N} \frac{a_\alpha}{\alpha+3} + \frac{1}{3}R + \frac{1}{4} \quad , \tag{19b}$$

and

$$A^* = \frac{1}{2R^2} - \frac{2}{R^2} \sum_{\alpha=0}^{N} a_\alpha \left(\frac{R}{\alpha+2} + \frac{1}{\alpha+3} \right) \quad . \tag{20}$$

To complete the calculation of the albedo, we must solve the following F_N equations to find the coefficients a_α:

$$\sum_{\alpha=0}^{N} a_\alpha [B_\alpha(\xi_j) - \exp(-2R/\xi_j)A_\alpha(\xi_j)] = -RA_0(\xi_j) - A_1(\xi_j) + \exp(-2R/\xi_j) \times [RB_0(\xi_j) + B_1(\xi_j)] \quad . \tag{21}$$

For the F_N approximation, we use $\xi_0 = \nu_0$ and N additional values of ξ_j, spaced equally distant in the interval [0,1], in Eq. (21) to establish $N + 1$ linear algebraic equations, which can be readily solved to yield the required a_α. In Table II, we give our numerical results along with "exact" results deduced from our F_N calculations for large N and the results of Sanchez[7] and Mordant.[8]

THE POINT SOURCE IN A FINITE SPHERE

In a very recent publication, Dubi and Horowitz[9] reported numerical results for the flux distribution, in a finite sphere,

TABLE II

The Albedo for a Sphere

c	R	F_3	F_4	F_5	F_6	Exact
0.1	1.0	0.33726	0.33712	0.33681	0.33685	0.33687
0.1	3.0	0.08606	0.08597	0.08592	0.08591	0.08590
0.1	9.0	0.03108	0.03100	0.03094	0.03094	0.03093
0.3	1.0	0.42893	0.42921	0.42881	0.42884	0.42884
0.3	3.0	0.16232	0.16222	0.16209	0.16209	0.16206
0.3	9.0	0.09145	0.09134	0.09121	0.09120	0.09118
0.5	1.0	0.54229	0.54256	0.54225	0.54227	0.54226
0.5	3.0	0.26619	0.26612	0.26598	0.26598	0.26595
0.5	9.0	0.17393	0.17383	0.17369	0.17369	0.17366
0.7	1.0	0.68660	0.68672	0.68657	0.68658	0.68657
0.7	3.0	0.42259	0.42255	0.42246	0.42246	0.42244
0.7	9.0	0.30001	0.29993	0.29984	0.29984	0.29982
0.9	1.0	0.87801	0.87803	0.87801	0.87801	0.87801
0.9	3.0	0.71032	0.71031	0.71029	0.71029	0.71029
0.9	9.0	0.55476	0.55473	0.55470	0.55470	0.55469

[5] J. MAIORINO, Private Communication (1978).

[6] S. WU and C. E. SIEWERT, *J. Appl. Math. Phys. (ZAMP)*, **26**, 637 (1975).

[7] R. SANCHEZ, "Application de la méthode de Garlerkin à la résolution de l'équation intégrale du transport unidimensionnelle: probabilités de collisions généralisées tenant compte du gradient du flux et du choc linéairement anisotrope," Thèse, Université de Paris-Sud (1974); see also, CEA-N-1793, Centre d'Etudes Nucléaires de Saclay (1974).

[8] M. MORDANT, "La méthode C_N de résolution de l'équation du transport des neutrons en géométrie sphérique. Comparaison avec les résultats des expériences critiques Godiva et Jezebel," Thèse, Université Paris-Sud (1973); see also, CEA-N-1658, Centre d'Etudes Nucléaires de Saclay (1973).

[9] A. DUBI and Y. S. HOROWITZ, *Nucl. Sci. Eng.*, **66**, 118 (1978).

due to a point source located at the origin. To provide a benchmark result, we evaluate here the solution reported some years ago by Erdmann and Siewert.[10] For the special case of a point source, the solution given by Erdmann and Siewert[10] can be written as

$$\rho(r) = \rho_\infty(r) - \frac{1}{4\pi r}\left[E(\nu_0)\exp(-R/\nu_0)\sinh(r/\nu_0) + \int_0^1 E(\nu)\exp(-R/\nu)\sinh(r/\nu)d\nu\right] , \quad (22)$$

where

$$\rho_\infty(r) = \frac{1}{4\pi r}\left[\frac{1}{\nu_0 N(\nu_0)}\exp(-r/\nu_0) + \int_0^1 \frac{1}{\nu N(\nu)}\exp(-r/\nu)d\nu\right] \quad (23)$$

is the infinite-medium solution. Here, the expansion coefficient $E(\nu_0)$ is given by

$$E(\nu_0) = G(\nu_0) + \int_0^1 K(x)E(x)dx \quad , \quad (24)$$

where $E(\nu)$ satisfies the Fredholm equation,

$$E(\nu) = G(\nu) + \int_0^1 K(x \to \nu)E(x)dx \quad . \quad (25)$$

In addition,

$$N(\nu_0) = \frac{c}{2}\nu_0^3\left(\frac{c}{\nu_0^2 - 1} - \frac{1}{\nu_0^2}\right) \quad , \quad (26a)$$

$$N(\nu) = \nu\left[(1 - c\nu \tanh^{-1}\nu)^2 + \frac{1}{4}c^2\nu^2\pi^2\right] \quad , \quad (26b)$$

and the known functions $G(\nu_0)$, $G(\nu)$, $K(x)$, and $K(x \to \nu)$ are as follows:

$$\{1 - \exp[-2(R + z_0)/\nu_0]\}\, G(\nu_0) = \frac{2}{\nu_0 N(\nu_0)}\exp[-(R + 2z_0)/\nu_0] + \frac{c\nu_0}{H(\nu_0)N(\nu_0)}\int_0^1 \frac{\exp(-R/\nu)}{H(\nu)N(\nu)}\frac{d\nu}{\nu + \nu_0} \quad , \quad (27a)$$

$$\{1 - \exp[-2(R + z_0)/\nu_0]\}\, K(x) = \frac{cx\nu_0}{2(\nu_0 + x)H(\nu_0)N(\nu_0)H(x)} \times \exp(-2R/x) \quad , \quad (27b)$$

$$H(\nu)N(\nu)G(\nu) = \frac{c\nu\nu_0}{2(\nu + \nu_0)H(\nu_0)}\exp(-R/\nu_0) \times \left[G(\nu_0)\exp(-R/\nu_0) + \frac{2}{\nu_0 N(\nu_0)}\right] + c\nu\int_0^1 \frac{1}{(x + \nu)H(x)N(x)}\exp(-R/x)dx \quad , \quad (27c)$$

and

$$H(\nu)N(\nu)K(x \to \nu) = \frac{c\nu\nu_0}{2(\nu + \nu_0)H(\nu_0)}\exp(-2R/\nu_0)\, K(x) + \frac{c\nu x}{2(\nu + x)H(x)}\exp(-2R/x) \quad . \quad (27d)$$

In addition, $H(\xi)$ is the Chandrasekhar H function,[11] and the extrapolated endpoint (from the Milne problem) is

$$z_0 = \frac{1}{2}\nu_0 \ln\left[\frac{4N(\nu_0)H^2(\nu_0)}{c\nu_0}\right] \quad . \quad (28)$$

In Table III, we list our numerical results obtained by approximating all integrals by a Gaussian quadrature scheme and subsequently by solving Eq. (25) iteratively and using Eqs. (24) and (22). We note that the results of Dubi and Horowitz[9] for $c = 0.3$ are significantly different from ours, and we refer to recent[12] corrections to Ref. 9.

TABLE III

The Neutron Flux ($R = 1$)

	$c = 0.3$		$c = 0.9$	
r	$4\pi r^2\rho_\infty(r)$	$4\pi r^2\rho(r)$	$4\pi r^2\rho_\infty(r)$	$4\pi r^2\rho(r)$
0	1	1	1	1
0.1	0.96440	0.96417	1.1208	1.1147
0.2	0.91965	0.91871	1.2342	1.2099
0.3	0.87101	0.86885	1.3383	1.2832
0.4	0.82091	0.81692	1.4326	1.3336
0.5	0.77076	0.76421	1.5169	1.3602
0.6	0.72147	0.71143	1.5914	1.3620
0.7	0.67364	0.65883	1.6567	1.3372
0.8	0.62765	0.60620	1.7129	1.2828
0.9	0.58374	0.55247	1.7607	1.1914
1.0	0.54204	0.49154	1.8006	1.0258

ACKNOWLEDGMENTS

The authors wish to express their gratitude to W. Dunn and T. Duracz for communicating Monte Carlo and ANISN results concerning the point source problem.

One of the authors (C. E. S.) is grateful to the Centre d'Etudes Nucléaires de Saclay for kind hospitality and partial support of this work, which was also supported in part by the National Science Foundation.

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