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SOUND-WAVE PROPAGATION

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SUNTO. — Facendo uso delle soluzioni di un modello linearizzato dell'equazione di Boltzmann, si risolve il problema della lastra oscillante in un semispazio.

INTRODUCTION.

This work concerning sound-wave propagation is a contribution to the collaborative work of Siewert and Thomas [1] and Loyalka and Cheng [2].

In a recent paper [3], hereafter referred to as *SB*, the linearized and modeled Boltzmann equation

$$(1) \qquad \left(\frac{\partial}{\partial t} + c_x \frac{\partial}{\partial x} + 1\right) h(x, \mathbf{c}, t) = (\pi)^{-3/2} \int h(x, \mathbf{c}', t) \cdot \left[1 + 2\, \mathbf{c} \cdot \mathbf{c}' + \frac{2}{3}\left(c'^2 - \frac{3}{2}\right)\left(c^2 - \frac{3}{2}\right)\right] e^{-c'^2}\, d^3 c'$$

was investigated and various elementary solutions were reported. We now wish to use the established elementary solutions to solve the problem of sound-wave propagation in a semi-infinite medium. Here $h(x, \mathbf{c}, t)$ represents the perturbation of the distribution function from the Maxwellian distribution, $\mathbf{c}$, with components c_x, c_y and c_z and magnitude c, is the velocity, t is the time and x is the space variable.

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For the case of an oscillating plate located at $x = 0$, we seek a solution of Eq. (1) subject to

$$\lim_{x \to \infty} h(x, \mathbf{c}, t) < \infty \tag{2a}$$

and

$$h(0, \mathbf{c}, t) = \left(c_x + \frac{\sqrt{\pi}}{2}\right) e^{i\omega t} + \frac{2}{\pi} \int_{-\infty}^{\infty} dc'_y \int_{-\infty}^{\infty} dc'_z \int_0^{\infty} dc'_x \, h(0, -c'_x, c'_y, c'_z, t) \, e^{-c'^2} c'_x , \quad c_x > 0 , \tag{2b}$$

where u_0, the amplitude of the oscillation, has been taken equal to 1/2 and ω is the frequency of oscillation. We note that this boundary-value problem has been discussed by Buckner and Ferziger in a fundamental paper [4] published in 1966. Since Buckner and Ferziger did not have available the required half-range analysis [3], they developed only an approximate solution that was based on their full-range theory. Further, it seems that Buckner and Ferziger did not insist on particle conservation at the wall.

Here we wish to use the half-range analysis [3], to solve the given problem explicitly in terms of the $\mathbf{H}$ matrix [3].

Basic formulation.

Since we are concerned here with temperature-density effects, we can take « moments » of Eq. (1) to obtain equations dependent only on x and c_x. Thus we let

$$\psi_1(x, c_x, t) = (\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(c_y^2 + c_z^2)} \, h(x, \mathbf{c}, t) \, dc_y \, dc_z \tag{3a}$$

and

$$\psi_2(x, c_x, t) = (\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(c_y^2 + c_z^2)} \, h(x, \mathbf{c}, t)(c_y^2 + c_z^2 - 1) \, dc_y \, dc_z \tag{3b}$$

so that the density perturbation

$$\Delta N(x, t) = \pi^{-3/2} \int h(x, \mathbf{c}, t) \, e^{-c^2} d^3 c \tag{4}$$

and the temperature perturbation

$$\Delta T(x, t) = \frac{2}{3} \pi^{-3/2} \int h(x, \mathbf{c}, t) \left(c^2 - \frac{3}{2}\right) e^{-c^2} d^3 c \tag{5}$$

can be expressed as

$$\Delta N(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi_1(x, \mu, t)\, e^{-\mu^2} d\mu \tag{6}$$

and

$$\Delta T(x, t) = \frac{2}{3\pi} \int_{-\infty}^{\infty} \left[\left(\mu^2 - \frac{1}{2}\right) \psi_1(x, \mu, t) + \psi_2(x, \mu, t)\right] e^{-\mu^2} d\mu\,, \tag{7}$$

where we have used μ for c_x. If we now multiply Eq. (1) by $\exp(-c_y^2 - c_z^2)$ and integrate from $-\infty$ to ∞ over c_y and c_z, and then multiply Eq. (1) by $(c_y^2 + c_z^2 - 1)\exp(-c_y^2 - c_z^2)$ and integrate similarly, we find that the resulting two equations can be expressed as

$$\left(\frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x} + 1\right) \boldsymbol{\Psi}(x, \mu, t) = \tag{8}$$

$$= (\pi)^{-1/2} \int_{-\infty}^{\infty} [\mathbf{Q}(\mu)\, \mathbf{Q}^T(\mu') + 2\mu\mu'\, \mathbf{P}]\, \boldsymbol{\Psi}(x, \mu', t)\, e^{-\mu'^2} d\mu'$$

where

$$\mathbf{Q}(\mu) = \begin{vmatrix} \left(\frac{2}{3}\right)^{1/2} \left(\mu^2 - \frac{1}{2}\right) & 1 \\ \left(\frac{2}{3}\right)^{1/2} & 0 \end{vmatrix}, \quad \mathbf{P} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \tag{9}$$

and $\boldsymbol{\Psi}(x, \mu, t)$ is a vector with components $\psi_1(x, \mu, t)$ and $\psi_2(x, \mu, t)$.

The boundary conditions on $\boldsymbol{\Psi}(x, \mu, t)$ can readily be established by taking the appropriate moments of Eqs. (2) and using Eqs. (3). We thus seek a solution of Eq. (8) subject to

$$\lim_{x \to \infty} \boldsymbol{\Psi}(x, \mu, t) < \infty \tag{10a}$$

and

$$\boldsymbol{\Psi}(0, \mu, t) = \sqrt{\pi} \left(\mu + \frac{\sqrt{\pi}}{2}\right) \begin{vmatrix} 1 \\ 0 \end{vmatrix} e^{i\omega t} + \tag{10b}$$

$$+ 2 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \int_0^{\infty} \mu'\, \boldsymbol{\Psi}(0, -\mu', t)\, e^{-\mu'^2} d\mu'\,, \quad \mu > 0\,.$$

In order to construct the desired solution, we write

$$\Psi(x, \mu, t) = \sqrt{\pi}\ \Psi_1(x, \mu, t) + \sqrt{\pi}\ E\ \Psi_0(x, \mu, t)\,, \tag{11}$$

where $\Psi_0(x, \mu, t)$ and $\Psi_1(x, \mu, t)$ both are bounded, for $x > 0$, solutions of Eq. (8) constrained to satisfy the boundary conditions

$$\Psi_\beta(0, \mu, t) = \mu^\beta e^{i\omega t} \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \quad \mu > 0, \quad \beta = 0 \text{ and } 1\,. \tag{12}$$

We find, on substituting Eq. (11) into Eq. (10b), the required expression for the constant E:

$$E = \frac{4\, U_{1,1} + \sqrt{\pi}}{2 - 4\, U_{0,1}}\,, \tag{13}$$

where $U_{0,1}$ and $U_{1,1}$ are, respectively, the upper elements of

$$\mathbf{U}_\beta = e^{-i\omega t} \int_0^\infty \mu' e^{-\mu'^2}\, \Psi_\beta\, (0, -\mu', t)\, d\mu', \quad \beta = 0 \text{ and } 1. \tag{14}$$

Of course, once $\Psi(x, \mu, t)$ is established, such quantities as the xx component of the perturbed pressure tensor

$$\Delta P_{xx}(x, t) = \frac{1}{\pi^{3/2}} \int h(x, \mathbf{c}, t)\, c_x^2\, e^{-c^2}\, d^3 c \tag{15}$$

and the perturbed gas pressure [5]

$$\Delta P(x, t) = \frac{1}{3\pi^{3/2}} \int h(x, \mathbf{c}, t)\, c^2\, e^{-c^2}\, d^3 c \tag{16}$$

are readily available. We find

$$\Delta P_{xx}(x, t) = \frac{1}{\pi} \begin{vmatrix} 1 \\ 0 \end{vmatrix}^T \int_{-\infty}^{\infty} \Psi(x, \mu, t)\, e^{-\mu^2} \mu^2\, d\mu \tag{17}$$

and

$$\Delta P(x, t) = \frac{1}{3\pi} \int_{-\infty}^{\infty} \begin{vmatrix} 1 + \mu^2 \\ 1 \end{vmatrix}^T \Psi(x, \mu, t)\, e^{-\mu^2}\, d\mu\,. \tag{18}$$

In addition, Eqs. (6) and (7) give, respectively, the perturbed density and temperature distributions in terms of $\Psi(x, \mu, t)$.

SOLUTION.

Since the elementary solutions of Eq. (8) are given in *SB*, we can express the two vectors required in Eq. (11) as

$$\boldsymbol{\Psi}_\beta(x, \mu, t) = e^{i\omega t} \left\{ \sum_{\alpha=1}^{k(i\omega)} A_\beta(\nu_\alpha)\, \boldsymbol{\Phi}(\nu_\alpha, \mu; i\omega)\, e^{-(i\omega+1)x/\nu_\alpha} + \right. \tag{19}$$

$$\left. + \int_0^\infty \boldsymbol{\Phi}(\nu, \mu; i\omega)\, \mathbf{A}_\beta(\nu)\, e^{-(i\omega+1)x/\nu}\, d\nu \right\}, \quad \beta = 0 \text{ and } 1,$$

where

$$\boldsymbol{\Phi}(\nu, \mu; i\omega) = \theta\, \nu\, Pv \left(\frac{1}{\nu - \mu}\right) \mathbf{Q}(\mu)\, (\mathbf{I} + \gamma\, \nu\, \mu \mathbf{D}) + \tag{20}$$

$$+ \delta(\nu - \mu)\, e^{\nu^2}\, \mathbf{Q}^{-T}(\nu)\, \boldsymbol{\lambda}(\nu; i\omega),$$

and

$$\boldsymbol{\Phi}(\nu_\alpha, \mu; i\omega) = \theta\, \nu_\alpha \left(\frac{1}{\nu_\alpha - \mu}\right) \mathbf{Q}(\mu)\, (\mathbf{I} + \gamma\, \nu_\alpha\, \mu\, \mathbf{D})\, \mathbf{M}(\nu_\alpha), \tag{21}$$

with

$$\theta = \frac{1}{\sqrt{\pi}\,(i\omega + 1)} \quad \text{and} \quad \gamma = \frac{2\, i\omega}{i\omega + 1}. \tag{22}$$

Here the discrete eigenvalues ν_α are the « positive » zeros of $\Lambda(z; i\omega) = \det \boldsymbol{\Lambda}(z; i\omega)$, where

$$\boldsymbol{\Lambda}(z; i\omega) = \mathbf{I} + z \int_{-\infty}^{\infty} \boldsymbol{\Psi}(\mu)\, \frac{d\mu}{\mu - z}, \tag{23}$$

with the characteristic matrix given by

$$\boldsymbol{\Psi}(\mu) = \theta\, e^{-\mu^2}\, \mathbf{Q}^T(\mu)\, \mathbf{Q}(\mu)\, (\mathbf{I} + \gamma\, \mu^2\, \mathbf{D}). \tag{24}$$

In addition, $\mathbf{M}(\nu_\alpha)$ is a null vector of $\boldsymbol{\Lambda}(\nu_\alpha; i\omega)$,

$$\boldsymbol{\lambda}(\nu; i\omega) = \mathbf{I} + \nu\, P \int_{-\infty}^{\infty} \boldsymbol{\Psi}(\mu)\, \frac{d\mu}{\mu - \nu} \tag{25}$$

and

$$\mathbf{D} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}. \tag{26}$$

We use $k(i\omega)$ to denote the number of $\pm$ pairs of discrete eigenvalues, and we use the convention that ν_α (rather than $-\nu_\alpha$) is such that $\boldsymbol{\Psi}_0(x, \mu, t)$ and $\boldsymbol{\Psi}_1(x, \mu, t)$, as given by Eq. (19), remain bounded as x approaches infinity. In a previous paper [6] we reported explicit expressions for the discrete eigenvalues and, as did Buckner and Ferziger [4], concluded that $k(i\omega)$ could be 0, 1, or 2; *i.e.*, $k(i\omega) = 2$ for $0 < \omega < 0.646453\ldots$, $k(i\omega) = 1$ for $0.646453\ldots < \omega < 2.14517\ldots$, and $k(i\omega) = 0$ for $\omega > 2.14517\ldots$.

The expansion coefficients appearing in Eq. (19) can be determined by constraining the solutions to meet the boundary conditions given by Eq. (12). Thus we must solve

$$\mu^\beta \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \sum_{\alpha=1}^{k(i\omega)} A_\beta(\nu_\alpha)\, \boldsymbol{\Phi}(\nu_\alpha, \mu; i\omega) + \int_0^\infty \boldsymbol{\Phi}(\nu, \mu; i\omega)\, \mathbf{A}_\beta(\nu)\, d\nu, \quad \mu > 0. \tag{27}$$

In *SB* a half-range completeness proof was given for the case $k(i\omega) = 0$, which ensures that Eq. (27) has a solution for $k(i\omega) = 0$. In addition, Siewert and Kriese [7] have deduced the half-range adjoint functions so that Eq. (27) can formally be solved for general index $k(i\omega)$ in terms of the $\mathbf{H}$ matrix [3]. We find that we can write the continuum coefficients as

$$\mathbf{A}_0(\nu) = \left(\frac{2z_1}{\theta}\right)\left(\frac{1}{z_1+\nu}\right) \mathbf{L}^{-1}(\nu)\, \boldsymbol{\pi}^{-1}(\nu)\, \mathbf{H}^{-T}(\nu) \cdot [\mathbf{H}(z_1) + \mathbf{D}\,\mathbf{H}(-z_1)]^{-1} \begin{vmatrix} 0 \\ 1 \end{vmatrix} \tag{28}$$

and

$$\mathbf{A}_1(\nu) = \left(\frac{z_1}{\theta}\right)\left(\frac{1}{z_1+\nu}\right) \mathbf{L}^{-1}(\nu)\, \boldsymbol{\pi}^{-1}(\nu)\, \mathbf{H}^{-T}(\nu) \cdot [(\nu + z_1)\,\mathbf{I} - 2z_1\,\mathbf{K}]\,(\mathbf{I} - \mathbf{H}_0^T) \begin{vmatrix} 0 \\ 1 \end{vmatrix}, \tag{29}$$

where

$$\mathbf{L}(\nu) = \boldsymbol{\lambda}(\nu; i\omega)\, \boldsymbol{\Psi}^{-1}(\nu)\, \boldsymbol{\lambda}(\nu; i\omega) + \pi^2 \nu^2\, \boldsymbol{\Psi}(\nu), \tag{30}$$

$$\mathbf{H}_0^T = \int_0^\infty \mathbf{H}^T(\mu)\, \widehat{\boldsymbol{\Psi}}(\mu)\, d\mu, \quad \widehat{\boldsymbol{\Psi}}(\mu) = \boldsymbol{\pi}(\mu)\, \boldsymbol{\Psi}(\mu)\, \boldsymbol{\pi}^{-1}(\mu), \tag{31}$$

$$\pi(z) = \begin{vmatrix} 1 & 0 \\ 0 & \dfrac{z_1 - z}{z_1} \end{vmatrix} \quad \text{and} \quad \gamma^{1/2} z_1 = i. \tag{32}$$

In addition, the $\mathbf{K}$ matrix is given by

$$\mathbf{K} = [\mathbf{H}(z_1) + \mathbf{D}\,\mathbf{H}(-z_1)]^{-1}\,\mathbf{D}\,\mathbf{H}(-z_1), \tag{33}$$

and the $\mathbf{H}$ matrix is a generalization of Chandrasekhar's H function [8].

We can also use the orthogonality relations discussed by Siewert and Kriese [7] to find the discrete coefficients $A_0(\nu_\alpha)$ and $A_1(\nu_\alpha)$. We find, for $\alpha = 1, 2, \ldots, k(i\omega)$,

$$A_0(\nu_\alpha) = \left(\frac{2z_1}{\theta}\right)\left(\frac{1}{z_1 + \nu_\alpha}\right)\frac{\nu_\alpha}{N(\nu_\alpha)}\,\mathbf{M}^T(\nu_\alpha)\,\boldsymbol{\pi}(-\nu_\alpha)\,\mathbf{H}^{-T}(\nu_\alpha)\cdot \cdot\,[\mathbf{H}(z_1) + \mathbf{D}\,\mathbf{H}(-z_1)]^{-1}\begin{vmatrix} 0 \\ 1 \end{vmatrix} \tag{34}$$

and

$$A_1(\nu_\alpha) = \left(\frac{z_1}{\theta}\right)\left(\frac{1}{z_1 + \nu_\alpha}\right)\frac{\nu_\alpha}{N(\nu_\alpha)}\,\mathbf{M}^T(\nu_\alpha)\,\boldsymbol{\pi}(-\nu_\alpha)\,\mathbf{H}^{-T}(\nu_\alpha)\cdot \cdot\,[(\nu_\alpha + z_1)\,\mathbf{I} - 2\,z_1\,\mathbf{K}]\,(\mathbf{I} - \mathbf{H}_0^T)\begin{vmatrix} 0 \\ 1 \end{vmatrix}, \tag{35}$$

where the normalization integral $N(\nu_\alpha)$ is given by

$$N(\nu_\alpha) = \nu_\alpha^2\,\mathbf{M}^T(\nu_\alpha)\,\boldsymbol{\pi}(\nu_\alpha)\,\boldsymbol{\pi}(-\nu_\alpha)\,\boldsymbol{\Lambda}'(\nu_\alpha)\,\mathbf{M}(\nu_\alpha). \tag{36}$$

Since Eqs. (28), (29), (34) and (35) explicitly express the desired expansion coefficients in terms of the $\mathbf{H}$ matrix, the function $\boldsymbol{\Psi}(x, \mu, t)$ is readily available and can be used, for example, in Eqs. (6), (7), (17) and (18) to give, respectively, the perturbed density, temperature, xx component of pressure and the pressure. On substituting Eq. (11) into Eqs. (6), (7), (17) and (18) and carrying out the indicated integration, we find we can write

$$\Delta N(x, t) = \frac{1}{\sqrt{\pi}}\,e^{i\omega t}\begin{vmatrix} 0 \\ 1 \end{vmatrix}^T \left[\sum_{\alpha=1}^{k(i\omega)} A(\nu_\alpha)\,\mathbf{M}(\nu_\alpha)\,e^{-(i\omega+1)x/\nu_\alpha} + \int_0^\infty \mathbf{A}(\nu)\,e^{-(i\omega+1)x/\nu}\,d\nu\right], \tag{37}$$

$$\Delta T(x,t) = \sqrt{\frac{2}{3\pi}}\, e^{i\omega t} \begin{vmatrix} 1 \\ 0 \end{vmatrix}^T \left[\sum_{\alpha=1}^{k(i\omega)} A(\nu_\alpha)\, \mathbf{M}(\nu_\alpha)\, e^{-(i\omega+1)x/\nu_\alpha} + \right.$$
$$\left. + \int_0^\infty \mathbf{A}(\nu)\, e^{-(i\omega+1)x/\nu}\, d\nu \right], \tag{38}$$

$$\Delta P_{xx}(x,t) = \frac{1}{\sqrt{\pi}}\, e^{i\omega t} \left(\frac{i\omega}{i\omega+1}\right)^2 \begin{vmatrix} 0 \\ 1 \end{vmatrix}^T \left[\sum_{\alpha=1}^{k(i\omega)} \nu_\alpha^2 A(\nu_\alpha)\, \mathbf{M}(\nu_\alpha)\, e^{-(i\omega+1)x/\nu_\alpha} + \right.$$
$$\left. + \int_0^\infty \nu^2 \mathbf{A}(\nu)\, e^{-(i\omega+1)x/\nu}\, d\nu \right], \tag{39}$$

and

$$\Delta P(x,t) = \frac{1}{\sqrt{6\pi}}\, e^{i\omega t} \begin{vmatrix} 1 \\ \sqrt{3/2} \end{vmatrix}^T \left[\sum_{\alpha=1}^{k(i\omega)} A(\nu_\alpha)\, \mathbf{M}(\nu_\alpha)\, e^{-(i\omega+1)x/\nu_\alpha} + \right.$$
$$\left. + \int_0^\infty \mathbf{A}(\nu)\, e^{-(i\omega+1)x/\nu}\, d\nu \right]. \tag{40}$$

Here

$$\mathbf{A}(\nu) = \mathbf{A}_1(\nu) + E\,\mathbf{A}_0(\nu) \quad \text{and} \quad A(\nu_\alpha) = A_1(\nu_\alpha) + E\,A_0(\nu_\alpha). \tag{41}$$

Although the constant E can be computed from Eqs. (13) and (14), we prefer to use the $\mathbf{R}$ matrix of Siewert and Kriese [7] to obtain

$$E = \frac{z_1 \begin{vmatrix} 0 \\ 1 \end{vmatrix}^T \mathbf{W}_1 \begin{vmatrix} 0 \\ 1 \end{vmatrix}}{\begin{vmatrix} 0 \\ 1 \end{vmatrix}^T \mathbf{W}_0 \begin{vmatrix} 0 \\ 1 \end{vmatrix}}, \tag{42}$$

where

$$\mathbf{W}_0 = [\mathbf{H}^{-T}(z_1) - \mathbf{H}^{-T}(-z_1)]\,[\mathbf{H}(z_1) + \mathbf{D}\mathbf{H}(-z_1)]^{-1} \tag{43}$$

and

$$\mathbf{W}_1 = [(\mathbf{I} - \mathbf{H}_0) - \mathbf{H}^{-T}(z_1) +$$
$$+ \{\mathbf{H}^{-T}(z_1) - \mathbf{H}^{-T}(-z_1)\}\mathbf{K}]\,(\mathbf{I} - \mathbf{H}_0^T). \tag{44}$$

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ABSTRACT. — The elementary solutions of a linearized and modeled Boltzmann equation are used to solve the oscillating-plate problem for a semi-infinite half space.

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