On the solution of a $\tan(\xi - k\pi) + \tanh\xi = 0$

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ABSTRACT

Explicit solutions of the equation mentioned in the title are developed.

1. INTRODUCTION

In a recent paper [1], the zeros of

$$F_k(z)=\mathrm{Log}\,(z+1)-\mathrm{Log}(z-1)-\frac{i|a|}{a}[\log(z+i|a|)-\log(z-i|a|)] + 2k\pi i \quad (1)$$

were used, for $k = 0, 1, 2, \ldots$, to deduce explicit solutions of a transcendental equation,

$$a \tan\xi + \tanh\xi = 0, \quad a \in (-\infty, \infty), \quad (2)$$

basic to the study of vibrating plates [2,3]. In this note, we wish to show how these same zeros of $F_k(z)$ can, for different values of k, yield solutions of the more general transcendental equation

$$b(\tan\xi \tanh\xi - a) + a\tan\xi + \tanh\xi = 0 \quad (3a)$$

or

$$a \tan(\xi - k\pi) + \tanh\xi = 0, \quad (3b)$$

where

$$b = \tan k\pi. \quad (4)$$

It is clear that equation (2) is a special case of equations (3) corresponding to integer values of k. On the other hand, for $k = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$, equations (3) yield

$$\tan\xi - a\coth\xi = 0. \quad (5)$$

In general, we will use all of the zeros of $\{F_k(z)\}$, for which k yields a given value of b, to develop solutions of equations (3). Note that we need only consider $b > 0$ since $\xi(-b,a) = -\xi(b,a)$.

2. ANALYSIS

We note that the log functions appearing in equation (1) are such that

$$\mathrm{Log}(\zeta)=\ln|\zeta| + i\arg(\zeta), \quad \arg(\zeta) \in (-\pi,\pi), \quad (6a)$$

and

$$\log(\zeta)=\ln|\zeta| + i\arg(\zeta), \quad \arg(\zeta) \in (-\tfrac{1}{2}\pi, \tfrac{3}{2}\pi). \quad (6b)$$

It thus follows that, as previously discussed [1], $F_k(z)$ is analytic in the complex plane cut from -1 to 1 along the real axis and from $-i|a|$ to $i|a|$ along the imaginary axis. Also

$$F_k(z) \to 2k\pi i + \frac{2(1+a)}{z} + O\left(\frac{1}{z^3}\right), \quad \text{as } |z| \to \infty. \quad (7)$$

If, in equation (1), we consider k to be a general complex number, but $k \neq 0$, then we can use the argument principle to deduce, for $a > 0$, that $F_k(z)$ has one zero for all non-zero k contained in the region denoted as R_1 in figure 1 and no zeros for $k \notin R_1$. For $a < 0$, $F_k(z)$ has three zeros for all non-zero $k \in R_3$, one zero for $k \in R_1$, as shown in figure 2, and no zeros for other non-zero values of k. Since $F_{-k}(-z) = -F_k(z)$ and $-\overline{F_{\bar{k}}(-\bar{z})} = F_k(z)$, it is clear that to establish the zeros of $F_k(z)$, for all k, we need consider only the first quadrant of the k plane. However, since we wish to utilize the explicit results developed in our previous paper [1], we consider here that $k \in (-\infty, 0) \cup (0, \infty)$. We note that

$$F_k(iy) = i\left[-\frac{2|a|}{a}\tanh^{-1}\left(\frac{|a|}{y}\right) + 2\tan^{-1}(y) + (2k - \mathrm{sgn}(y))\pi\right],$$

$$y \in (-\infty, -|a|) \cup (|a|, \infty), \quad (8)$$

and thus for $\{a,k\} \in \mathcal{R}_1$, where $\mathcal{R}_1 \Rightarrow k \in (-\infty, 0) \cup (0, \infty)$ for $a > 0$ and $\mathcal{R}_1 \Rightarrow k \in (-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, \infty)$ for $a < 0$, $F_k(z)$ has one imaginary zero, $i\,\mathrm{sgn}(k)\mathrm{sgn}(a)|y_{|k|}|$, where $|y_{|k|}| > |a|$ for k finite. It therefore follows that

$$\xi_k = \left(k - \frac{1}{2}\frac{|ak|}{ak}\right)\pi + \frac{|ak|}{ak}\tan^{-1}(|y_k|), \quad \{a,k\} \in \mathcal{R}_1, \quad (9)$$

establishes real solutions of equations (3) for all those values of k such that $b = \tan k\pi$. We note that explicit expressions for y_k are given as equations (36) and (37) in our earlier paper [1].

In order to establish a convenient labeling of our solutions here, we let $k_\ell = k_0 + \ell$, $\ell = 0, \pm 1, \pm 2, \ldots$,

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where $\pi k_0 = \tan^{-1} b$, $k_0 \neq 0$, and use ξ_ℓ to denote the solution computed from $k = k_\ell$. Thus

$$\xi_\ell = \operatorname{sgn}[a(k_0+\ell)]\,[(\tfrac{|a|}{a}|k_0+\ell| - \tfrac{1}{2})\pi + \tan^{-1}|y_{|k_0+\ell|}|],$$
$$\{a, k_0+\ell\} \in \mathcal{R}_1. \tag{10}$$

We now wish to consider $\{a, k_0\} \in \mathcal{R}_3$; *i.e.*, $a < 0$ and $k_0 \in (0, \frac{1}{2})$. The case $k = 0$ has already been discussed [1] and special attention will be given to the case $a < 0$ and $k_0 = \frac{1}{2}$ since, as can be seen from figure 2, $F_{k_0}(z)$ vanishes on the cut. Since for $\{a, k_0\} \in \mathcal{R}_3$, $F_{k_0}(z)$ has three zeros (one of which is always imaginary) in the finite cut plane, we must modify the results given in our earlier paper [1] to obtain

$$F_{k_0}(z) = (z-z_{0,1})(z-z_{0,2})(z-iy_{k_0})2\pi i\, k_0 X_{k_0}(z),$$
$$\{a, k_0\} \in \mathcal{R}_3, \tag{11}$$

where

$$X_{k_0}(z) = \frac{1}{(z^2-1)(z-ia)} \exp\{\frac{1}{2\pi i}\int_0^1 [z\ell n M_{k_0}(x) + 2ix\theta_{k_0}(x)]\frac{dx}{x^2-z^2} + \frac{1}{\pi}\int_{-|a|}^{|a|} \phi_{k_0}(y)\frac{dy}{y+iz}\}. \tag{12}$$

We note that $M_{k_0}(x)$ and $\phi_{k_0}(y)$ are defined by equations (33) and (35) of [1]; however here

$$\theta_{k_0}(x) = \tan^{-1}[\frac{-2\pi R(x)}{R^2(x) + \pi^2(4k_0^2-1)}], \tag{13}$$

with

$$R(x) = 2\tanh^{-1}(x) + \frac{|a|}{a}[\operatorname{sgn}(x)\pi - 2\tan^{-1}(\frac{x}{|a|})], \tag{14}$$

is defined to vary continuously from $\tan^{-1}(\frac{1}{2k_0^2})$ at $x = 0^+$ to 2π at $x = 1$. Also we have written the three zeros of $F_{k_0}(z)$, for $a < 0$, as iy_{k_0}, $z_{0,1}$ and $z_{0,2}$. Since equation (11) is valid for all z, that equation can be evaluated at three different values of z, and subsequently the resulting three equations can be solved simultaneously to yield the three zeros iy_{k_0}, $z_{0,1}$ and $z_{0,2}$. On the other hand, we can investigate equation (11) as $|z| \to \infty$ to find that the three zeros iy_{k_0}, $z_{0,1}$ and $z_{0,2}$ are the solutions of

$$z^3 + pz^2 + qz + r = 0, \tag{15}$$

where

$$ip = \frac{1+a}{k_0\pi} + a + L_1, \tag{16a}$$

$$q = -\frac{a(1+a)}{k_0\pi} - 1 - [a + \frac{(1+a)}{k_0\pi}]L_1 + L_2 - \frac{1}{2}L_1^2, \tag{16b}$$

and

$$ir = \frac{(1-a^3)}{3k_0\pi} - [a + \frac{(1+a)}{k_0\pi}][1 - L_2 + \frac{1}{2}L_1^2] - \frac{a(1+a)L_1}{k_0\pi} + L_3 - L_1 + L_1L_2 - \frac{L_1^3}{6}. \tag{16c}$$

Here

$$L_1 = \frac{1}{2\pi}\int_0^1 \ell n M_{k_0}(x)dx - \frac{1}{\pi}\int_{-|a|}^{|a|}\phi_{k_0}(y)dy, \tag{17a}$$

$$L_2 = \frac{1}{\pi}\int_0^1 x\theta_{k_0}(x)dx - \frac{1}{\pi}\int_{-|a|}^{|a|} y\phi_{k_0}(y)dy, \tag{17b}$$

and

$$L_3 = \frac{1}{2\pi}\int_0^1 x^2 \ell n M_{k_0}(x)dx + \frac{1}{\pi}\int_{-|a|}^{|a|} y^2\phi_{k_0}(y)dy. \tag{17c}$$

We note that for all $a < 0$ the additional real solution of equations (3) is

$$\xi_{0,1} = (k_0 + \tfrac{1}{2})\pi - \tan^{-1}|y_{k_0}|, \quad k_0 \in (0, 1/2). \tag{18}$$

For $a \leqslant -1$, $z_{0,1}$ and $z_{0,2}$ are complex, with $\bar{z}_{0,1} = -z_{0,2}$; these zeros are not required here. For $a \in (-1,0)$, $z_{0,1}$ and $z_{0,2}$ can both be imaginary, or both complex; when they are imaginary, they are of the form $z_{0,1} = i|z_{0,1}|$ and $z_{0,2} = i|z_{0,2}|$ and they can be used in

$$\xi_{0,\alpha} = (k_0 - \tfrac{1}{2})\pi + \tan^{-1}|z_{0,\alpha-1}|, \quad \alpha = 2 \text{ and } 3,$$
$$k_0 \in (0, 1/2), \tag{19}$$

to yield two additional real solutions of equations (3). For the case $k_0 = \frac{1}{2}$ and $a < 0$, we note that special attention is required because $F^+_{1/2}(x)$ and $F^-_{-1/2}(x)$ both have zeros at $\pm x_0 \in (-1,1)$. We thus find it convenient here to investigate the function

$$\Lambda(z) = F_{1/2}(z)F_{-1/2}(z). \tag{20}$$

If we consider the Riemann-Hilbert problem defined by

$$X^+(\tau) = G(\tau)X^-(\tau), \quad \tau \in [-1,1] \cup [-i|a|, i|a|], \tag{21}$$

where

$$G(\tau) = \frac{\Lambda^+(\tau)}{\Lambda^-(\tau)}, \tag{22}$$

we find we can express $\Lambda(z)$ in terms of a canonical solution of equation (21) :

$$\frac{\Lambda(z)}{(z^2-x_0^2)(z^2+y_{1/2}^2)}$$

$$= \frac{\pi^2}{z^2(z^2+a^2)} \exp\left[\frac{2}{\pi}\int_0^1 x\theta_{1/2}(x)\frac{dx}{x^2-z^2}\right.$$

$$\left. + \frac{2}{\pi}\int_{-|a|}^{|a|} y\phi_{1/2}(y)\frac{dy}{y^2+z^2}\right]. \tag{23}$$

Here $\pm iy_{1/2}$ are the two imaginary zeros of $\Lambda(z)$,

$$\theta_{1/2}(x) = \tan^{-1}\left(\frac{-2\pi}{R(x)}\right), \tag{24}$$

with

$$R(x) = 2\tanh^{-1}(x) - \mathrm{sgn}(x)\pi + 2\tan^{-1}\left(\frac{x}{|a|}\right), \tag{25}$$

and

$$\phi_{1/2}(y) = \tan^{-1}\left(\frac{-\pi}{I_{1/2}(y)}\right), \tag{26}$$

with

$$I_{1/2}(y) = 2\tanh^{-1}\left(\frac{y}{|a|}\right)+2\tan^{-1}(y)+[1-\mathrm{sgn}(y)]\pi. \tag{27}$$

We note that the angle $\theta_{1/2}(x)$ appearing in equation (23) is continuous for $x \in [0,1]$ with $\theta_{1/2}(0) = -\pi + \tan^{-1}(2)$; the angle $\phi_{1/2}(y)$ varies from $-\pi$ to 0, as y varies from $-|a|$ to $|a|$, and is discontinuous at $y=0$.

If we now observe equation (23) as $|z| \to \infty$, we can deduce that x_0^2 and $y_{1/2}^2$ are the two zeros of the polynomial

$$\zeta^2 + b\zeta + c = 0 \tag{28}$$

where

$$b = a^2 + \frac{4(1+a)^2}{\pi^2} + M_1 \tag{29}$$

and

$$c = \frac{4(1+a)^2a^2}{\pi^2} + \frac{8}{3\pi^2}(1-a^3)(1+a)$$

$$+\left[\frac{4(1+a)^2}{\pi^2} + a^2\right]M_1 + \frac{1}{2}M_1^2 + M_2. \tag{30}$$

Here

$$M_1 = \frac{2}{\pi}\int_0^1 x\theta_{1/2}(x)dx - \frac{2}{\pi}\int_{-|a|}^{|a|} y\phi_{1/2}(y)dy \tag{31}$$

and

$$M_2 = \frac{2}{\pi}\int_0^1 x^3\theta_{1/2}(x)dx + \frac{2}{\pi}\int_{-|a|}^{|a|} y^3\phi_{1/2}(y)dy. \tag{32}$$

We can solve equation (28) to obtain x_0 and $y_{1/2}$; the latter result can be used in

$$\pm\,\xi_0 = \pi - \tan^{-1}|y_{1/2}| \tag{33}$$

to yield the additional required solutions for $k_0 = \frac{1}{2}$ and $a < 0$.

It is clear that a solution by iteration of equations (3) can be established; however, to deduce by iteration all of the solutions of interest here would require first approximations of sufficient accuracy so that the iteration schemes would, in fact, converge to each of the required results. To our knowledge this has not been done. For selected data cases, we have used a simple quadrature scheme to evaluate our "exact" solutions to get results for first approximations required in an iterative solution of equations (3).

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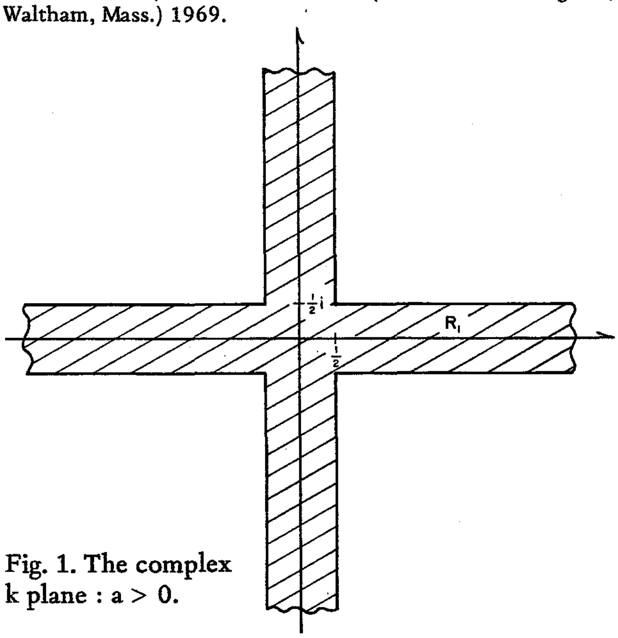

Fig. 1. The complex k plane : a > 0.

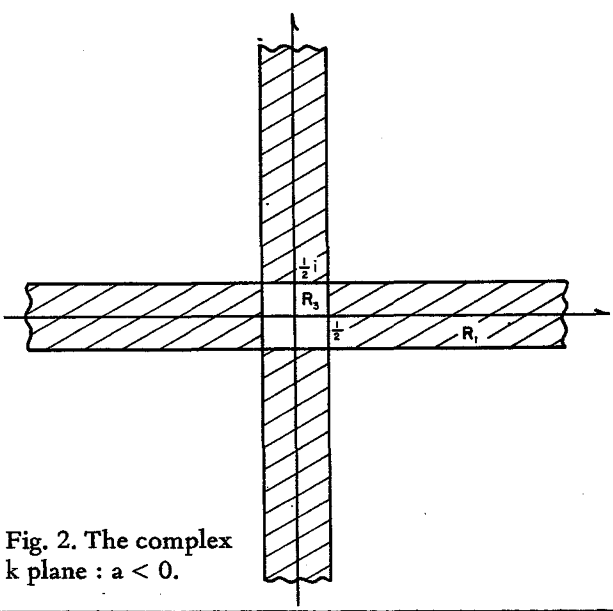

Fig. 2. The complex k plane : a < 0.