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THE EFFECT OF FORWARD AND BACKWARD SCATTERING ON THE SPHERICAL ALBEDO FOR A FINITE PLANE-PARALLEL ATMOSPHERE

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Abstract—A synthetic kernel is used to study the effect of forward and backward scattering on the spherical albedo and the transmission factor for a finite atmosphere.

INTRODUCTION

IN ORDER to extend a recent work[1] and thus to study the effect of forward and backward scattering in a finite plane-parallel medium, we consider the equation of transfer

$$\mu \frac{\partial}{\partial z} \psi(z, \mu) + \rho(\sigma + \kappa)\psi(z, \mu) = \rho\sigma \int_{-1}^{1} \psi(z, \mu')f(\mu' \to \mu)\, d\mu', \tag{1}$$

where

$$f(\mu' \to \mu) = l\delta(\mu' + \mu) + m\delta(\mu' - \mu) + \frac{1}{2} n(1 + \gamma\mu'\mu) \tag{2}$$

and $l + m + n = 1$. For the spherical albedo problem, we seek a solution of Eq. (1) for $z \in [0, d]$ such that

$$\psi(0, \mu) = 1, \quad \mu > 0, \tag{3a}$$

and

$$\psi(d, -\mu) = 0, \quad \mu > 0. \tag{3b}$$

We wish to compute the spherical albedo A^* as defined by AMBARTSUMYAN[2]

$$A^* = 2 \int_0^1 \psi(0, -\mu)\mu\, d\mu \tag{4a}$$

and the transmission factor

$$B^* = 2 \int_0^1 \psi(d, \mu)\mu\, d\mu. \tag{4b}$$

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If we now use the changes of variables introduced by İNÖNÜ[3] and exhibited explicitly in our earlier paper,[1] we find that we can write

$$A^* = -R + \frac{4B}{B+1}\int_0^1 \Psi(0,-\mu)\mu \, d\mu \tag{5a}$$

and

$$B^* = \frac{4B}{B+1}\int_0^1 \Psi(a,\mu)\mu \, d\mu, \tag{5b}$$

where $\Psi(y,\mu)$ satisfies

$$\mu \frac{\partial}{\partial y}\Psi(y,\mu) + \Psi(y,\mu) = \frac{\omega}{2}\int_{-1}^1 \Psi(y,\mu')(1+b\mu\mu')\, d\mu' \tag{6a}$$

subject to

$$\Psi(0,\mu) = \frac{2}{B+1} + R\Psi(0,-\mu), \quad \mu > 0, \tag{6b}$$

and

$$\Psi(a,-\mu) = R\Psi(a,\mu), \quad \mu > 0. \tag{6c}$$

Here

$$\omega_0 = \frac{\sigma}{\sigma+\kappa}, \quad \alpha = \frac{l\omega_0}{1-m\omega_0}, \quad \beta = \frac{n\omega_0}{1-m\omega_0}, \tag{7}$$

$$y = \sqrt{(1-\alpha^2)}\rho(\sigma+\kappa)(1-m\omega_0)z, \tag{8}$$

$$\omega = \frac{\beta}{1-\alpha}, \quad b = \left(\frac{1-\alpha}{1+\alpha}\right)\gamma, \quad B = \sqrt{\left(\frac{1-\alpha}{1+\alpha}\right)}, \tag{9}$$

$$R = \frac{B-1}{B+1}, \quad a = \sqrt{(1-\alpha^2)}(1-m\omega_0)\tau_0 \quad \text{and} \quad \tau_0 = \rho(\sigma+\kappa)d. \tag{10}$$

ANALYSIS

Using the notation of CASE and ZWEIFEL,[4] we can write $\Psi(y,\mu)$ in terms of the elementary solutions of Eq. (6a) as

$$\Psi(y,\mu) = A(\nu_0)\phi(\nu_0,\mu)\,e^{-y/\nu_0} + A(-\nu_0)\phi(-\nu_0,\mu)\,e^{y/\nu_0} + \int_0^1 A(\nu)\phi(\nu,\mu)\,e^{-y/\nu}\,d\nu + \int_0^1 A(-\nu)\phi(-\nu,\mu)\,e^{y/\nu}\,d\nu, \tag{11}$$

where $A(\pm\nu_0)$ and $A(\pm\nu)$ are expansion coefficients to be determined by Eqs. (6b) and (6c). If we now substitute Eq. (11) into Eqs. (6b) and (6c), then the resulting two equations can be regularized by using the half-range orthogonality relations given by MCCORMICK and KUŠČER.[5] Since the final equations for the expansion coefficients are special cases of the equations developed by BEACH, ÖZIŞIK, and SIEWERT,[6] we simply list the results here:

$$\mathbf{M}\mathbf{A}(\nu_0) = \mathbf{G} + \int_0^1 K_0(\nu')\mathbf{U}(\nu')\mathbf{A}(\nu')\,d\nu' \tag{12}$$

and

$$\mathbf{M}(\nu)\mathbf{A}(\nu) = \mathbf{G}(\nu) + K_1(\nu)\mathbf{U}(\nu_0)\mathbf{A}(\nu_0) + \int_0^1 K(\nu'\to\nu)\mathbf{U}(\nu')\mathbf{A}(\nu')\,d\nu', \quad \nu\in(0,1). \tag{13}$$

Here we have written the unknowns as vectors, i.e.

$$\mathbf{A}(\xi) = \begin{bmatrix} A(\xi) \\ A(-\xi) \end{bmatrix}, \quad \xi = \nu_o \quad \text{or} \quad \xi = \nu \in (0, 1). \tag{14}$$

In addition, the other quantities, all known explicitly, required in Eqs. (12) and (13) are

$$\mathbf{M} = \mathbf{M}(\nu_0) - \mathbf{U}(\nu_0)\, e^{-2z_0/\nu_0}, \tag{15a}$$

$$\mathbf{U}(\xi) = \begin{bmatrix} R & -1 \\ -e^{-a/\xi} & R\, e^{a/\xi} \end{bmatrix}, \tag{15b}$$

$$\mathbf{M}(\xi) = \begin{bmatrix} 1 & -R \\ -R\, e^{-a/\xi} & e^{a/\xi} \end{bmatrix}, \tag{15c}$$

$$\mathbf{G} = \left(\frac{2}{B+1}\right) \frac{\nu_0}{N(\nu_0)H(\nu_0)}\, q \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{15d}$$

$$\mathbf{G}(\nu) = \left(\frac{2}{B+1}\right) \frac{\nu}{N(\nu)H(\nu)}\, q \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{15e}$$

$$K_0(\nu) = \frac{\omega\nu\nu_0[1 - r\nu\nu_0 + c(\nu + \nu_0)]}{2(\nu + \nu_0)H(\nu)H(\nu_0)N(\nu_0)} \tag{15f}$$

$$K_1(\nu) = \frac{\omega\nu\nu_0[1 - r\nu\nu_0 + c(\nu + \nu_0)]}{2(\nu + \nu_0)H(\nu_0)H(\nu)N(\nu)} \tag{15g}$$

and

$$K(\nu' \to \nu) = \frac{\omega\nu\nu'[1 - r\nu\nu' + c(\nu + \nu')]}{2(\nu + \nu')H(\nu')H(\nu)N(\nu)}. \tag{15h}$$

In addition, $r = b(1 - \omega)$, z_0 is the extrapolated end-point for the Milne problem,

$$z_0 = \frac{\nu_0}{2} \ln \left[\frac{4N(\nu_0)H^2(\nu_0)}{\omega\nu_0[1 - r\nu_0^2 + 2\nu_0 c]} \right], \tag{16a}$$

$$q = \frac{2(1 - \omega)}{2 - \omega H_0}, \tag{16b}$$

$$c = \frac{\omega r H_1}{2 - \omega H_0}, \tag{16c}$$

and

$$H_\alpha = \int_0^1 \mu^\alpha H(\mu)\, d\mu, \tag{16d}$$

where $H(\mu)$ is Chandrasekhar's H function.[7] The normalization factors appearing in Eqs. (15) are

$$N(\nu) = \nu[(1 + r\nu^2)(1 - \omega\nu \tanh^{-1} \nu) - r(1 - \omega)\nu^2]^2 + \frac{1}{4}\omega^2\nu^3\pi^2(1 + r\nu^2)^2 \tag{17a}$$

and

$$N(\nu_0) = \frac{\omega\nu_0^2}{2}(1 + r\nu_0^2) \left[\frac{\omega(1 + r\nu_0^2)}{\nu_0(\nu_0^2 - 1)} - \frac{(1 - \omega)(1 + 3r\nu_0^2)}{\nu_0(1 + r\nu_0^2)} \right]. \tag{17b}$$

It is clear that we can substitute Eq. (12) into Eq. (13) to obtain a system of two Fredholm equations for the expansion coefficients $A(\nu)$ and $A(-\nu)$:

$$\mathbf{A}(\nu) = \mathbf{F}(\nu) + \int_0^1 \mathbf{K}(\nu' \to \nu)\mathbf{A}(\nu')\, d\nu', \quad \nu \in (0, 1), \tag{18}$$

Table 1. The spherical albedo; the indices l, m and n in $A^*(l, m, n)$ correspond to the quantities given in Eq. (2).

ω_0	γ	τ_0	$A^*(0,1,0)$	$A^*(0,\frac{2}{3},\frac{1}{3})$	$A^*(0,\frac{1}{3},\frac{2}{3})$	$A^*(0,0,1)$	$A^*(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$A^*(\frac{1}{3},0,\frac{2}{3})$	$A^*(\frac{2}{3},0,\frac{1}{3})$	$A^*(1,0,0)$
0.9	0.0	1.0	0.0	0.16495	0.27371	0.35271	0.35633	0.41551	0.46631	0.51297
0.9	0.0	3.0	0.0	0.26256	0.38803	0.46406	0.47160	0.51989	0.56411	0.61580
0.9	0.0	5.0	0.0	0.28636	0.40604	0.47634	0.48542	0.52874	0.57115	0.62540
0.9	0.0	10.0	0.0	0.29490	0.40978	0.47802	0.48758	0.52966	0.57177	0.62678
0.95	0.0	1.0	0.0	0.19130	0.31127	0.39640	0.39887	0.46222	0.51418	0.55928
0.95	0.0	3.0	0.0	0.33592	0.47785	0.55876	0.56243	0.61343	0.65375	0.69820
0.95	0.0	5.0	0.0	0.38627	0.51828	0.58859	0.59327	0.63578	0.67130	0.71855
0.95	0.0	10.0	0.0	0.41551	0.53322	0.59649	0.60190	0.64041	0.67437	0.72380
0.999	0.0	1.0	0.0	0.22240	0.35409	0.44552	0.44767	0.51499	0.56896	0.61245
0.999	0.0	3.0	0.0	0.44384	0.60668	0.69487	0.69615	0.75103	0.78961	0.81811
0.999	0.0	5.0	0.0	0.56250	0.71379	0.78636	0.78703	0.82930	0.85753	0.87804
0.999	0.0	10.0	0.0	0.70836	0.82318	0.87104	0.87126	0.89741	0.91408	0.92699
0.9	0.1	3.0	0.0	0.25683	0.38157	0.45772	0.46954	0.51691	0.56299	0.61580
0.9	0.3	3.0	0.0	0.24496	0.36798	0.44428	0.46533	0.51077	0.56074	0.61580
0.9	0.5	3.0	0.0	0.23252	0.35342	0.42971	0.46101	0.50437	0.55845	0.61580
0.9	0.7	3.0	0.0	0.21947	0.33778	0.41385	0.45658	0.49769	0.55612	0.61580
0.9	0.9	3.0	0.0	0.20574	0.32091	0.39651	0.45203	0.49071	0.55375	0.61580

Table 2. The transmission factor; the indices l, m and n in $B^*(l, m, n)$ correspond to the quantities given in Eq. (2).

ω_0	γ	τ_0	$B^*(0,1,0)$	$B^*(0,\frac{2}{3},\frac{1}{3})$	$B^*(0,\frac{1}{3},\frac{2}{3})$	$B^*(0,0,1)$	$B^*(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$B^*(\frac{1}{3},0,\frac{2}{3})$	$B^*(\frac{2}{3},0,\frac{1}{3})$	$B^*(1,0,0)$
0.9	0.0	1.0	0.83258	0.66301	0.55359	0.47475	0.47544	0.41488	0.36865	0.33597
0.9	0.0	3.0	0.60008	0.33222	0.21755	0.15417	0.15968	0.11728	0.09467	0.08992
0.9	0.0	5.0	0.44321	0.17388	0.09053	0.05342	0.05838	0.03583	0.02682	0.02937
0.9	0.0	10.0	0.21938	0.03599	0.01042	0.00386	0.00495	0.00189	0.00119	0.00222
0.95	0.0	1.0	0.90984	0.71628	0.59606	0.51098	0.50993	0.44608	0.39565	0.35787
0.95	0.0	3.0	0.76455	0.42210	0.28427	0.20849	0.21055	0.16178	0.13124	0.11972
0.95	0.0	5.0	0.64937	0.26257	0.14814	0.09501	0.09800	0.06664	0.05025	0.05055
0.95	0.0	10.0	0.44321	0.08496	0.03098	0.01415	0.01557	0.00768	0.00483	0.00735
0.999	0.0	1.0	0.99801	0.77561	0.64391	0.55249	0.55034	0.48302	0.42905	0.38559
0.999	0.0	3.0	0.99406	0.55019	0.38736	0.29916	0.29789	0.24301	0.20444	0.17625
0.999	0.0	5.0	0.99016	0.42759	0.27631	0.20376	0.20310	0.16084	0.13264	0.11293
0.999	0.0	10.0	0.98055	0.27203	0.15737	0.10965	0.10945	0.08344	0.06694	0.05665
0.9	0.1	3.0	0.60008	0.33737	0.22292	0.15902	0.16129	0.11938	0.09541	0.08992
0.9	0.3	3.0	0.60008	0.34806	0.23432	0.16943	0.16458	0.12376	0.09691	0.08992
0.9	0.5	3.0	0.60008	0.35932	0.24666	0.18090	0.16797	0.12837	0.09844	0.08992
0.9	0.7	3.0	0.60008	0.37119	0.26005	0.19360	0.17147	0.13325	0.10001	0.08992
0.9	0.9	3.0	0.60008	0.38371	0.27462	0.20772	0.17508	0.13840	0.10162	0.08992

where the known vector is

$$\mathbf{F}(\nu) = \mathbf{M}^{-1}(\nu)[\mathbf{G}(\nu) + K_1(\nu)\mathbf{U}(\nu_0)\mathbf{M}^{-1}\mathbf{G}] \tag{19}$$

and the matrix kernel is

$$\mathbf{K}(\nu' \to \nu) = \mathbf{M}^{-1}(\nu)[K_1(\nu)\mathbf{U}(\nu_0)\mathbf{M}^{-1}K_0(\nu') + K(\nu' \to \nu)\mathbf{U}(\nu')]. \tag{20}$$

In order to find the spherical albedo and the transmission factor, we can substitute Eq. (11) into Eqs. (5) to obtain, after using Eq. (12),

$$A^* = -R + \frac{4B}{B+1}\left(\mathbf{W}^T(\nu_0)\mathbf{M}^{-1}\mathbf{G} + \int_0^1 [\mathbf{W}^T(\nu_0)\mathbf{M}^{-1}K_0(\nu)\mathbf{U}(\nu) + \mathbf{W}^T(\nu)]\mathbf{A}(\nu)\,\mathrm{d}\nu\right) \tag{21a}$$

and

[redacted]

where

$$\mathbf{W}(\xi)=\frac{\omega\xi}{2}\begin{bmatrix}1-r\xi\left(\frac{1}{2}-\xi\right)-\xi(1+r\xi^2)\log\left(1+\frac{1}{\xi}\right)\\ \frac{2}{\omega}-1-r\xi\left(\frac{1}{2}-\xi\right)-\xi(1+r\xi^2)\log\left(1+\frac{1}{\xi}\right)\end{bmatrix} \tag{22}$$

and

$$\mathbf{E}(\xi)=\begin{bmatrix}0 & e^{a/\xi}\\ e^{-a/\xi} & 0\end{bmatrix}. \tag{23}$$

Numerical results which show the effect of l, m and n on the spherical albedo and the transmission factor are given in Tables 1 and 2.

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