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The Half-Space Green's Function for an Atmosphere with a Polarized Radiation Field*

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This paper presents a rigorous solution for the half-space Green's function of the matrix transport equation that describes the flow of polarized radiation in a free-electron atmosphere. The singular normal modes expansion technique is used to construct the solution in such a manner that the expansion coefficients can be determined by applying the proper boundary conditions. The necessary completeness and orthogonality theorems are proved, and thus all expansion coefficients are found by simply taking scalar products. In addition, the albedo problem for a semi-infinite half space is solved explicitly.

I. INTRODUCTION

IN a recent paper, hereafter referred to as **I**, Siewer and Fraley found the set of normal modes to the homogeneous matrix transport equation for the radiative-transfer problem in a free-electron atmosphere.[1] Also in **I**: the half-range completeness theorem was proved, the half-range adjoint functions were presented, the half-range normalization integrals were calculated, and the classical Milne problem was solved. The technique used by Siewert and Fraley was based upon Case's method of singular eigenfunctions that was developed for problems in one-speed neutron transport theory.[2,3] The formulation of the matrix transport equation, as given by Chandrasekhar, was reviewed briefly in **I**.[4,5] In addition, Kuščer and Ribarič have extended the matrix formulation in the theory of diffusion of light to include more generalized scattering laws.[6] The polarized light problem has been investigated recently by several authors.[7–10]

In this paper we extend the procedures discussed in **I** and solve explicitly for the half-space Green's function. In addition, the complete solution to the half-space albedo problem is obtained. Our procedure for finding the half-space Green's function is to develop a pseudoinfinite-medium Green's function from which subtractions can be made in order to meet the boundary condition of zero re-entrant radiation. The infinite-medium solution that we use is nonphysical, since it is allowed to diverge at an optical distance of minus infinity. The fact that this infinite-medium solution is nonphysical is a consequence of the conservative nature of our system and causes no real concern, since it is used only as an intermediate

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step in the construction of the half-space solution. This same procedure has been used for other physical models.[11,12]

In Sec. II we briefly review the cogent results formulated in I. In Sec. III, the necessary full-range completeness theorem is proved, since it is needed for the determination of the pseudo infinite-medium Green's function. Section IV is devoted to the full-range orthogonality theorem and the calculation of the full-range normalization integrals. Finally, in Sec. V the half-space Green's function and the albedo problem are solved explicitly.

II. THE NORMAL MODES OF THE BASIC EQUATION

The equation that mathematically describes the scattering of radiation in a free-electron atmosphere may be written in matrix notation as[1,4]:

$$\mu \frac{\partial}{\partial x} \boldsymbol{\Psi}(x,\mu) + \boldsymbol{\Psi}(x,\mu) = \frac{1}{2}\int_{-1}^{1} \boldsymbol{\varkappa}(\mu,\mu')\boldsymbol{\Psi}(x,\mu')\, d\mu', \tag{1}$$

where $\boldsymbol{\Psi}$ is a vector whose two components represent the parallel and perpendicular states of the polarized radiation field. The transfer matrix is

$$\boldsymbol{\varkappa}(\mu,\mu') = \tfrac{3}{4}\begin{bmatrix} 2(1-\mu'^2)(1-\mu^2) + \mu'^2\mu^2 & \mu^2 \\ \mu'^2 & 1 \end{bmatrix}; \tag{2}$$

x is the optical distance measured in units of the Thomson scattering coefficient, and μ is the direction cosine measured from the inward normal to the free surface.[13]

In order to separate the variables in Eq. (1), we seek solutions of the form

$$\boldsymbol{\Psi}(x,\mu) = e^{-x/\eta}\boldsymbol{\Phi}(\eta,\mu). \tag{3}$$

This *ansatz* leads to the following equation for the determination of $\boldsymbol{\Phi}(\eta,\mu)$:

$$(\eta - \mu)\boldsymbol{\Phi}(\eta,\mu) = \frac{\eta}{2}\int_{-1}^{1} \boldsymbol{\varkappa}(\mu,\mu')\boldsymbol{\Phi}(\eta,\mu')\, d\mu'. \tag{4}$$

Siewert and Fraley found the solutions of Eq. (4) to be[1]:

$$\boldsymbol{\Phi}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{5a}$$

$$\boldsymbol{\Phi}_1(\eta,\mu) = \begin{bmatrix} \dfrac{3\eta}{2}(1-\mu^2)\dfrac{P}{\eta-\mu} + \lambda_1(\eta)\delta(\eta-\mu) \\ 0 \end{bmatrix}, \quad \eta \in [-1,1], \tag{5b}$$

and

$$\boldsymbol{\Phi}_2(\eta,\mu) = \begin{bmatrix} -\dfrac{3\eta}{2}(\eta+\mu) \\ \dfrac{3\eta}{2}(1-\eta^2)\dfrac{P}{\eta-\mu} + \lambda_2(\eta)\delta(\eta-\mu) \end{bmatrix}, \quad \eta \in [-1,1]. \tag{5c}$$

In addition, they found a solution to Eq. (1) of the form

$$\boldsymbol{\Psi}_-(x,\mu) = (x-\mu)\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{6}$$

In Eqs. (5) and throughout this work, the symbol P denotes that integrals involving these functions are to be carried out in the Cauchy principal value sense. Also $\delta(x)$ is the Dirac delta function,

$$\lambda_1(\eta) = -1 + 3(1-\eta^2)[1 - \eta T(\eta)], \tag{7a}$$

$$\lambda_2(\eta) = 1 + 3(1-\eta^2)[1 - \eta T(\eta)], \tag{7b}$$

and $T(x)$ denotes $\tanh^{-1}x$.

Note that $\boldsymbol{\Phi}_+$, $\boldsymbol{\Phi}_1(\eta,\mu)$, and $\boldsymbol{\Phi}_2(\eta,\mu)$ are solutions of Eq. (4); whereas, $\boldsymbol{\Psi}_-(x,\mu)$ is a solution only of Eq. (1). The complete solution to Eq. (1) can thus be written as

$$\boldsymbol{\Psi}(x,\mu) = A_+\boldsymbol{\Phi}_+ + A_-\boldsymbol{\Psi}_-(x,\mu) + \int_{-1}^{1} \alpha(\eta)\boldsymbol{\Phi}_1(\eta,\mu)\, e^{-x/\eta}\, d\eta + \int_{-1}^{1} \beta(\eta) e^{-x/\eta}\boldsymbol{\Phi}_1(\eta,\mu)\, d\eta, \tag{8}$$

where A_+, A_-, $\alpha(\eta)$, and $\beta(\eta)$ are arbitrary expansion coefficients.

Since in later sections we will need it, we state the theorem proved in I regarding the half-range completeness of the elementary solutions, Eqs. (5).

Theorem I: The eigensolutions $\boldsymbol{\Phi}_+$, $\boldsymbol{\Phi}_1(\eta,\mu)$, and $\boldsymbol{\Phi}_2(\eta,\mu)$ are complete on the half range, $\mu \in [0,1]$, in the sense that an arbitrary two-component vector $\boldsymbol{\Psi}(\mu)$ defined for $0 \le \mu \le 1$ can be expanded in the form

$$\boldsymbol{\Psi}(\mu) = A_+\boldsymbol{\Phi}_+ + \int_0^1 \alpha(\eta)\boldsymbol{\Phi}_1(\eta,\mu)\, d\eta + \int_0^1 \beta(\eta)\boldsymbol{\Phi}_2(\eta,\mu)\, d\eta. \tag{9}$$

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[11] C. E. Siewert and P. F. Zweifel, Ann. Phys. (N.Y.) **36**, 61 (1966).
[12] N. J. McCormick and I. Kuščer, J. Math. Phys. **6**, 1939 (1965).
[13] We choose to measure the velocity vector from the inward normal rather than from the outward one in order that Case's method of normal modes may be directly applicable.

Also in **I**: the half-range adjoint functions were found to be

$$\mathbf{\Phi}_+^\dagger = \begin{bmatrix} \gamma_1(\mu) \\ -\gamma_2(\mu)(a + b\mu) \end{bmatrix}, \tag{10a}$$

$$\mathbf{\Phi}_1^\dagger(\eta, \mu) = \begin{bmatrix} \gamma_1(\mu)[\frac{3}{2}\eta(1 - \eta^2)\dfrac{P}{\eta - \mu} + \lambda_1(\eta)\delta(\eta - \mu) + \frac{3}{2}\eta(c + \eta)] \\ (15\eta/2b)\gamma_2(\mu) \end{bmatrix}, \tag{10b}$$

and

$$\mathbf{\Phi}_2^\dagger(\eta, \mu) = \begin{bmatrix} (3\eta/2b)\gamma_1(\mu) \\ \gamma_2(\mu)[\frac{3}{2}\eta(1 - \mu^2)\dfrac{P}{\eta - \mu} + \lambda_2(\eta)\delta(\eta - \mu) - \frac{3}{2}\eta(c + \mu)] \end{bmatrix}, \tag{10c}$$

where

$$\gamma_1(\mu) = \mu[X^+(\mu)/\Omega^+(\mu)], \tag{11a}$$

$$\gamma_2(\mu) = \mu[Y^+(\mu)/\Lambda^+(\mu)], \tag{11b}$$

$$\Omega(z) = -1 + 3(1 - z^2)[1 - zT(1/z)], \tag{11c}$$

and

$$\Lambda(z) = 1 + 3(1 - z^2)[1 - zT(1/z)]. \tag{11d}$$

The auxiliary functions $X(z)$ and $Y(z)$ are given by

$$X(z) = \frac{1}{z - 1}\exp\left[\frac{1}{\pi}\int_0^1 \arg\Omega^+(\mu)\frac{d\mu}{\mu - z}\right] \tag{12a}$$

and

$$Y(z) = \exp\left[\frac{1}{\pi}\int_0^1 \arg\Lambda^+(\mu)\frac{d\mu}{\mu - z}\right]. \tag{12b}$$

Also

$$a = X(1)Y(-1) + X(-1)Y(1), \tag{13a}$$

$$b = X(1)Y(-1) - X(-1)Y(1), \tag{13b}$$

and

$$c = \frac{X(1)Y(-1) + X(-1)Y(1)}{X(1)Y(-1) - X(-1)Y(1)}. \tag{13c}$$

The half-range orthogonality theorem is stated.

Theorem II: The eigensolutions $\mathbf{\Phi}_+$, $\mathbf{\Phi}_1(\eta, \mu)$, and $\mathbf{\Phi}_2(\eta, \mu)$ have corresponding half-range adjoint solutions, $\mathbf{\Phi}_+^\dagger$, $\mathbf{\Phi}_1^\dagger(\eta, \mu)$, and $\mathbf{\Phi}_2^\dagger(\eta, \mu)$, such that

$$\int_0^1 \tilde{\mathbf{\Phi}}^\dagger(\eta', \mu)\mathbf{\Phi}(\eta, \mu)\, d\mu = 0, \quad \eta \neq \eta', \quad \eta \text{ and } \eta' \geq 0. \tag{14}$$

Here the superscript tilde denotes the transpose operation. Using the notation

$$\langle i \mid j\rangle = \int_0^1 \mathbf{\Phi}_i^\dagger(\eta', \mu)\mathbf{\Phi}_j(\eta, \mu)\, d\mu, \quad \text{for } i, j = +, 1, 2, \tag{15}$$

we list the results found previously[1]:

$$\langle i \mid j\rangle = 0 \quad \text{for} \quad i \neq j, \tag{16a}$$

$$\langle + \mid +\rangle = N_+, \tag{16b}$$

$$\langle 1 \mid 1\rangle = \mathcal{S}_1(\eta)\delta(\eta - \eta'), \tag{16c}$$

$$\langle 2 \mid 2\rangle = \mathcal{S}_2(\eta)\delta(\eta - \eta'), \tag{16d}$$

where

$$N_+ = -\tfrac{2}{3}b \tag{17a}$$

and

$$\mathcal{S}_i(\eta) = \gamma_i(\eta)[\lambda_i^2(\eta) + \tfrac{9}{4}\pi^2\eta^2(1 - \eta^2)^2]. \tag{17b}$$

These functions and their relationships will be useful in solving the problems described in a later section.

III. FULL-RANGE COMPLETENESS

We wish to prove the necessary

Theorem III: The eigensolutions $\mathbf{\Phi}_+$, $\mathbf{\Phi}_1(\eta, \mu)$, $\mathbf{\Phi}_2(\eta, \mu)$, and $\mathbf{\Psi}_-(0, \mu)$ are complete on the full-range, $\mu \in [-1, 1]$, in the sense that an arbitrary two-component vector $\mathbf{\Psi}(\mu)$ defined for $-1 \leq \mu \leq 1$ can be expanded in the form

$$\mathbf{\Psi}(\mu) = A_+\mathbf{\Phi}_+ + A_-\mathbf{\Psi}_-(0, \mu) + \int_{-1}^1 \alpha(\eta)\mathbf{\Phi}_1(\eta, \mu)\, d\eta + \int_{-1}^1 \beta(\eta)\mathbf{\Phi}_2(\eta, \mu)\, d\eta. \tag{18}$$

We first investigate the feasibility of expanding an arbitrary function in terms only of the continuum solutions $\mathbf{\Phi}_1(\eta, \mu)$ and $\mathbf{\Phi}_2(\eta, \mu)$. For the theorem to be true, this procedure should lead to restrictions which can be removed only by adding to the expansion the discrete solutions $\mathbf{\Phi}_+$ and $\mathbf{\Psi}_-(0, \mu)$.[14] Thus, we propose

$$\mathbf{\Psi}'(\mu) = \int_{-1}^1 \alpha(\eta)\mathbf{\Phi}_1(\eta, \mu)\, d\eta + \int_{-1}^1 \beta(\eta)\mathbf{\Phi}_2(\eta, \mu)\, d\eta, \tag{19}$$

[14] Simmons has also developed full-range completeness and orthogonality theorems; however, in his formulation the eigenfunctions were not explicitly available, and thus the determination of the expansion coefficients does not follow as readily as here.

or, expanding in terms of the components, we find

$$\Psi_1'(\mu) = \int_{-1}^{1} \alpha(\eta)\left[\tfrac{3}{2}\eta(1-\mu^2)\frac{P}{\eta-\mu} + \lambda_1(\eta)\delta(\eta-\mu)\right] d\eta + \int_{-1}^{1} \beta(\eta)[-\tfrac{3}{2}\eta(\eta+\mu)]\, d\eta \quad (20)$$

and

$$\Psi_2'(\mu) = \int_{-1}^{1} \beta(\eta)\left[\tfrac{3}{2}\eta(1-\eta^2)\frac{P}{\eta-\mu} + \lambda_2(\eta)\delta(\eta-\mu)\right] d\eta. \quad (21)$$

The procedure will be to solve for $\beta(\eta)$ from Eq. (21). Thus

$$\Psi_2'(\mu) = P\int_{-1}^{1} \tfrac{3}{2}\eta(1-\eta^2)\frac{\beta(\eta)}{\eta-\mu}\, d\eta + \lambda_2(\mu)\beta(\mu). \quad (22)$$

Noting the form of $\lambda_2(\mu)$ in Eq. (7b), we introduce the function

$$\Lambda(z) = 1 + 3(1-z^2)\left[1 - \frac{z}{2}\int_{-1}^{1}\frac{d\eta}{z-\eta}\right] \quad (23)$$

and note that it is analytic in the entire complex plane cut on the real line from -1 to 1. Other important properties of $\Lambda(z)$ that will be used later are

$$\Lambda(1) = \Lambda(-1) = 1,$$
$$\lim_{z\to\infty} \Lambda(z) = 2,$$

and $\Lambda(z)$ has no zeros in the complex plane cut from -1 to 1. Defining the boundary values, $\Lambda^+(z)$ and $\Lambda^-(z)$, to be the values of $\Lambda(z)$ as z approaches the real line on the cut from above and below, respectively, we obtain from Cauchy's theorem

$$\Lambda^{\pm}(\eta) = 1 + 3(1-\eta^2)[1 - \eta T(\eta) \pm \pi i \tfrac{1}{2}\eta]. \quad (24)$$

Thus,

$$\tfrac{1}{2}[\Lambda^+(\eta) + \Lambda^-(\eta)] = \lambda_2(\eta) \quad (25a)$$

and

$$\Lambda^+(\eta) - \Lambda^-(\eta) = 3\pi i\eta(1-\eta^2). \quad (25b)$$

We also define

$$N_2(z) \triangleq \frac{1}{2\pi i}\int_{-1}^{1}\eta(1-\eta^2)\frac{\beta(\eta)}{\eta - z}\, d\eta, \quad (26)$$

and observe that

(a) $N_2(z)$ is analytic in the complex plane cut from -1 to 1, and

(b) $N_2(z) \to -z^{-1}$ as $z \to \infty$.

The boundary values of $N_2(z)$ are

$$N_2^{\pm}(\mu) = \frac{1}{2\pi i} P\int_{-1}^{1}\eta(1-\eta^2)\frac{\beta(\eta)}{\eta-\mu}\, d\eta \pm \tfrac{1}{2}[\mu(1-\mu^2)\beta(\mu)]. \quad (27)$$

Thus,

$$N_2^+(\mu) - N_2^-(\mu) = \mu(1-\mu^2)\beta(\mu), \quad (28a)$$

and

$$N_2^+(\mu) + N_2^-(\mu) = \frac{1}{\pi i} P\int_{-1}^{1}\eta\beta(\eta)(1-\eta^2)\frac{d\eta}{\eta-\mu}. \quad (28b)$$

Substituting Eqs. (25) and (28) into Eq. (22), we arrive at the recognizable inhomogeneous Hilbert problem[15]:

$$N_2^+(\mu)\Lambda^+(\mu) - N_2^-(\mu)\Lambda^-(\mu) = \mu(1-\mu^2)\Psi_2'(\mu), \quad (29)$$

which has the solution

$$N_2(z)\Lambda(z) = \frac{1}{2\pi i}\int_{-1}^{1}\mu(1-\mu^2)\frac{\Psi_2'(\mu)}{\mu - z}\, d\mu + P_k(z), \quad (30)$$

where we have determined $N_2(z)\Lambda(z)$ to within an arbitrary polynomial $P_k(z)$. Noting the properties previously described for $\Lambda(z)$, we find that the function

$$N_2(z) = \frac{1}{2\pi i\Lambda(z)}\int_{-1}^{1}\mu(1-\mu^2)\frac{\Psi_2'(\mu)}{\mu-z}\, d\mu \quad (31)$$

does indeed have the properties derived from its definition in Eq. (26). Thus the polynomial of z required to meet our restrictions on $N_2(z)$ is simply zero.

Having determined $N_2(z)$ in terms of the arbitrary function, $\Psi_2'(\mu)$, we can evaluate $\beta(\eta)$ from Eq. (28a).

Now that $\beta(\eta)$ is known, an attempt is made to determine $\alpha(\eta)$ from Eq. (20). Thus

$$\Psi_1'(\mu) = \int_{-1}^{1}\alpha(\eta)\tfrac{3}{2}\eta(1-\mu^2)\frac{P}{\eta-\mu}\, d\eta + \lambda_1(\mu)\alpha(\mu) - g(\mu), \quad (32)$$

where

$$g(\mu) = \frac{3}{2}\int_{-1}^{1}\eta(\eta+\mu)\beta(\eta)\, d\eta. \quad (33)$$

The dispersion function

$$\Omega(z) = -1 + 3(1-z^2)\left[1 - \frac{z}{2}\int_{-1}^{1}\frac{d\eta}{z-\eta}\right] \quad (34)$$

has the properties:

(a) $\Omega(z)$ is analytic in the complex plane cut from -1 to 1, and

(b) $\Omega(z) \to z^{-2}$ as $z \to \infty$.

The boundary values are given by

$$\Omega^{\pm}(\mu) = -1 + 3(1-\mu^2)\left[1 - \frac{\mu}{2} P\int_{-1}^{1}\frac{d\eta}{\mu-\eta} \pm \pi i\frac{\mu}{2}\right]. \quad (35)$$

Thus,

$$\tfrac{1}{2}[\Omega^+(\mu) + \Omega^-(\mu)] = \lambda_1(\mu), \quad (36a)$$

and

$$\Omega^+(\mu) - \Omega^-(\mu) = 3\pi i\mu(1-\mu^2). \quad (36b)$$

[15] N. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953).

In the same manner used to solve for $\beta(\eta)$, we define

$$N_1(z) = \frac{1}{2\pi i}\int_{-1}^{1} \alpha(\eta)\frac{\eta}{\eta - z}\, d\eta \tag{37}$$

and note that it is analytic in the complex plane cut from -1 to 1 and vanishes as $1/z$ as $z \to \infty$. Also, the boundary values satisfy

$$N_1^+(\mu) - N_1^-(\mu) = \mu\alpha(\mu) \tag{38a}$$

and

$$N_1^+(\mu) + N_1^-(\mu) = \frac{1}{\pi i} P\int_{-1}^{1} \alpha(\eta)\eta \frac{d\eta}{\eta - \mu}. \tag{38b}$$

Inserting Eqs. (36) and (38) into Eq. (32), we obtain another inhomogeneous Hilbert problem

$$N_1^+(\mu)\Omega^+(\mu) - N_1^-(\mu)\Omega^-(\mu) = \mu[\Psi_1'(\mu) + g(\mu)], \tag{39}$$

which has as its solutions, to within an arbitrary polynomial in z,

$$N_1(z) = \frac{1}{2\pi i \Omega(z)}\int_{-1}^{1} \mu[\Psi_1'(\mu) + g(\mu)]\frac{d\mu}{\mu - z}. \tag{40}$$

In the limit as $z \to \infty$,

$$N_1(z) \to -\frac{z}{2\pi i}\int_{-1}^{1} \tfrac{3}{2}\mu[\Psi_1'(\mu) + g(\mu)] \times \left[1 + \frac{\mu}{z} + \left(\frac{\mu}{z}\right)^2 + \cdots\right] d\mu, \tag{41}$$

so that for $N_1(z)$ to vanish as $1/z$, which it must by its definition, the following must apply:

$$\int_{-1}^{1} \mu[\Psi_1'(\mu) + g(\mu)]\, d\mu = 0 \tag{42a}$$

and

$$\int_{-1}^{1} \mu^2[\Psi_1'(\mu) + g(\mu)]\, d\mu = 0. \tag{42b}$$

Thus we cannot find $\alpha(\eta)$ from the arbitrary function $\Psi_1'(\mu)$ without the restrictions on $\Psi_1'(\mu)$ that are indicated by Eqs. (42). To circumvent these restrictions, we assume that the arbitrary vector $\mathbf{\Psi}(\mu)$ can be expanded in terms of the continuum plus both discrete modes; i.e.,

$$\mathbf{\Psi}(\mu) = \mathbf{\Psi}'(\mu) + A_+\mathbf{\Phi}_+ + A_-\mathbf{\Psi}_-(0, \mu). \tag{43}$$

Writing Eq. (43) explicitly in terms of the components, we have

$$\Psi_i(\mu) = \Psi_i'(\mu) + A_+ - A_-\mu, \quad i = 1, 2. \tag{44}$$

We can thus make use of Eqs. (42) to evaluate A_+ and A_-. The coefficients are found to be

$$A_+ = \frac{3}{2}\int_{-1}^{1} \mu^2[\Psi_1(\mu) + g(\mu)]\, d\mu \tag{45a}$$

and

$$A_- = -\frac{3}{2}\int_{-1}^{1} \mu[\Psi_1(\mu) + g(\mu)]\, d\mu. \tag{45b}$$

The expansion coefficient $\alpha(\eta)$ may be determined from Eq. (38a) to complete the determination of the expansion coefficients for any given $\mathbf{\Psi}(\mu)$.[16] The theorem is therefore proved.

IV. FULL-RANGE ORTHOGONALITY AND NORMALIZATION

The full-range orthogonality is stated as

Theorem IV: The eigenvectors $\mathbf{\Phi}_+$, $\mathbf{\Phi}_1(\eta, \mu)$, and $\mathbf{\Phi}_2(\eta, \mu)$ are orthogonal on the full range with respect to the weight function μ, i.e.,

$$\int_{-1}^{1} \mu\tilde{\mathbf{\Phi}}(\eta', \mu)\mathbf{\Phi}(\eta, \mu)\, d\mu = 0, \quad \eta \neq \eta'. \tag{46}$$

The proof of this theorem follows in the usual manner; i.e., we write the eigenvalue equation for η and η' as[17]

$$\left\{1 - \frac{\mu}{\eta}\right\}\mathbf{\Phi}(\eta, \mu) = \frac{1}{2}\int_{-1}^{1} \boldsymbol{\varkappa}(\mu, \mu')\mathbf{\Phi}(\eta, \mu)\, d\mu \tag{47a}$$

and

$$\left\{1 - \frac{\mu}{\eta'}\right\}\mathbf{\Phi}(\eta', \mu) = \frac{1}{2}\int_{-1}^{1} \boldsymbol{\varkappa}(\mu, \mu')\mathbf{\Phi}(\eta', \mu)\, d\mu. \tag{47b}$$

Multiplying Eq. (47a) from the left by $\tilde{\mathbf{\Phi}}(\eta', \mu)$, transposing Eq. (47b) and then multiplying it by $\mathbf{\Phi}(\eta, \mu)$ from the right, integrating over μ from -1 to 1 and subtracting the two equations proves the theorem immediately; i.e.,

$$\left\{\frac{1}{\eta} - \frac{1}{\eta'}\right\}\int_{-1}^{1} \mu\tilde{\mathbf{\Phi}}(\eta', \mu)\mathbf{\Phi}(\eta, \mu)\, d\mu = 0. \tag{48}$$

Here we have made use of the fact that

$$\tilde{\boldsymbol{\varkappa}}(\mu, \mu') = \boldsymbol{\varkappa}(\mu', \mu).$$

In this orthogonality theorem, we might expect that there would be a minor complication introduced by the fact that $\mathbf{\Phi}_1(\eta, \mu)$ and $\mathbf{\Phi}_2(\eta, \mu)$ are degenerate in the sense that they have the same eigenvalue spectrum. It turns out, however, that this is not the case because

$$\int_{-1}^{1} \mu\tilde{\mathbf{\Phi}}_1(\eta', \mu)\mathbf{\Phi}_2(\eta, \mu)\, d\mu = 0, \tag{49}$$

as can be easily verified. The adjoint vectors for the full range are thus

$$\mathbf{\Phi}^\dagger(\eta, \mu) = \mu\mathbf{\Phi}(\eta, \mu). \tag{50}$$

We choose to present only the results for the

[16] It is obvious that in order to determine $\alpha(\eta)$ and $\beta(\eta)$ in terms of $\mathbf{\Psi}(\mu)$ rather than $\mathbf{\Psi}'(\mu)$, one must make the proper substitutions as indicated in Eq. (43).

[17] The vector $\mathbf{\Psi}_-(0, \mu)$ is not included in the set of orthogonal functions since it is not a solution of Eqs. (47).

various normalization integrals, since these calculations, although they are very straightforward, are quite tedious. The Poincaré–Bertrand formula as given by Kuščer, McCormick, and Summerfield has been used to specify the technique for handling the double principal value integrals that were encountered.[18] Defining the full-range scalar product[19]

$$\langle i \mid j \rangle \triangleq \int_{-1}^{1} \tilde{\mathbf{\Phi}}_i^{\dagger}(\eta', \mu)\mathbf{\Phi}_j(\eta, \mu)\, d\mu, \quad i, j = +, 1, 2, \quad (51)$$

we find

$$\langle i \mid j \rangle = 0;\ i \neq j = +, 1, \text{ or } 2, \quad (52a)$$

$$\langle + \mid + \rangle = 0, \quad (52b)$$

$$\langle 1 \mid 1 \rangle = \eta\Omega^+(\eta)\Omega^-(\eta)\delta(\eta - \eta'), \quad (52c)$$

and

$$\langle 2 \mid 2 \rangle = \eta\Lambda^+(\eta)\Lambda^-(\eta)\delta(\eta - \eta'). \quad (52d)$$

In addition to the above results, we will need, in the next section, integrals of the form

$$\langle i \mid - \rangle = \int_{-1}^{1} \tilde{\mathbf{\Phi}}_i^{\dagger}(\eta, \mu)\mathbf{\Psi}_-(0, \mu)\, d\mu. \quad (53)$$

We find

$$\langle + \mid - \rangle = -\tfrac{4}{3} \quad (54a)$$

and

$$\langle i \mid - \rangle = 0, \quad i = -, 1, 2. \quad (54b)$$

With all of the necessary formalism and theorems now established, we proceed to solve in the next section the two problems of interest.

V. THE HALF-SPACE GREEN'S FUNCTION AND THE ALBEDO PROBLEM

Now that the completeness and orthogonality theorems have been established for both the full range and the half range, the solutions for the two problems of interest can be constructed with a minimum of manipulation. The solutions to the homogeneous transport equation have already been found, so that the only remaining task is to find the expansion coefficients such that the boundary conditions are satisfied.

We consider the half-space Green's function, where the transport equation takes the form

$$\mu \frac{\partial}{\partial x} \mathbf{\Psi}(x, \mu) + \mathbf{\Psi}(x, \mu) = \frac{1}{2}\int_{-1}^{1} \mathbf{\varkappa}(\mu, \mu')\mathbf{\Psi}(x, \mu')\, d\mu' + \mathbf{Q}. \quad (55)$$

Here the source term is given by

$$\mathbf{Q} = \delta(x - x_0)\begin{bmatrix} q_0\delta(\mu - \mu_0) \\ q_1\delta(\mu - \mu_1) \end{bmatrix}. \quad (56)$$

In order to have complete generality, one might consider that there are two Green's functions: one corresponding to $q_1 = 0$ and $x_0 = x_0$, and the other to $q_0 = 0$ and $x_0 = x_1$. Thus the solution of the transport equation for any given source term or inhomogenity could be constructed from these two Green's functions. However, the Green's function developed here includes these two cases.

We therefore seek a solution to Eq. (1) subject to the following boundary conditions[3,20]:

(a) The $\lim_{x \to \infty} \mathbf{\Psi}_g(x_0, \mu_0, \mu_1 \to x, \mu)$ is to be bounded,

(b)

$$\mu\{\mathbf{\Psi}_g(x_0^+, \mu_0, \mu_1 \to x, \mu) - \mathbf{\Psi}_g(x_0^-, \mu_0, \mu_1 \to x, \mu)\} = \begin{bmatrix} q_0\delta(\mu - \mu_0) \\ q_1\delta(\mu - \mu_1) \end{bmatrix},$$

and

(c) $\mathbf{\Psi}_g(x_0, \mu_0, \mu_1 \to 0, \mu) = 0$ for $\mu > 0$.

We construct a pseudoinfinite-medium Green's function in the forms

$$\mathbf{\chi}(x_0, \mu_0, \mu_1 \to x, \mu) = B_+\mathbf{\Phi}_+ + \int_0^1 B_1(\eta)e^{-(x-x_0)/\eta}\mathbf{\Phi}_1(\eta, \mu)\, d\eta + \int_0^1 B_2(\eta)e^{-(x-x_0)/\eta}\mathbf{\Phi}_2(\eta, \mu)\, d\eta, \quad x > x_0; \quad (57a)$$

and

$$\mathbf{\chi}(x_0, \mu_0, \mu_1 \to x, \mu) = -B_-\mathbf{\Psi}_-(x - x_0, \mu) - \int_{-1}^{0} B_1(\eta)e^{-(x-x_0)/\eta}\mathbf{\Phi}_1(\eta, \mu)\, d\eta - \int_{-1}^{0} B_2(\eta)e^{-(x-x_0)/\eta}\mathbf{\Phi}_2(\eta, \mu)\, d\eta, \quad x < x_0. \quad (57b)$$

We note that $\mathbf{\chi}(x_0, \mu_0, \mu_1 \to x, \mu)$ satisfies boundary condition (a); its divergence as x approaches $-\infty$ is of no concern since we will consider only $x \geq 0$. Applying boundary condition (b), we find that the expansion coefficients are to be determined from

$$\mathbf{\Psi}(\mu) \triangleq \frac{1}{\mu}\begin{bmatrix} q_0\delta(\mu - \mu_0) \\ q_1\delta(\mu - \mu_1) \end{bmatrix} = B_+\mathbf{\Phi}_+ + B_-\mathbf{\Psi}_-(0, \mu) + \int_{-1}^{1} B_1(\eta)\mathbf{\Phi}_1(\eta, \mu)\, d\eta + \int_{-1}^{1} B_2(\eta)\mathbf{\Phi}_2(\eta, \mu)\, d\eta, \quad \mu \in [-1, 1]. \quad (58)$$

This is simply a full-range expansion in terms of the normal modes. Theorem III therefore is applicable,

[18] I. Kuščer, N. J. McCormick, and G. C. Summerfield, Ann. Phys. (N.Y.) **30**, 411 (1964).

[19] Note that we have used the same symbols here for the full-range adjoint functions and scalar products as were used in Sec. II for the half range.

[20] Note that we replace the inhomogenity introduced by **Q** by the equivalent "jump" condition (b).

and the expansion coefficients can be found by taking full-range scalar products, e.g.,

$$B_1(\eta) = \left[\int_{-1}^{1} \tilde{\mathbf{\Phi}}_1^\dagger(\eta, \mu)\mathbf{\Psi}(\mu)\, d\mu\right] \Big/ [\eta\Omega^+(\eta)\Omega^-(\eta)]. \quad (59)$$

We find

$$B_1(\eta) = \frac{q_0\{\frac{3}{2}\eta(1-\mu_0^2)[P/(\eta-\mu_0)] + \lambda_1(\eta)\delta(\eta-\mu_0)\}}{\eta\Omega^+(\eta)\Omega^-(\eta)}, \quad (60a)$$

$$B_2(\eta) = \frac{(-q_0\frac{3}{2}\eta(\eta+\mu_0) + q_1\{\frac{3}{2}\eta(1-\eta^2)[P/(\eta-\mu_1)] + \lambda_2(\eta)\delta(\eta-\mu_1)\})}{\eta\Lambda^+(\eta)\Lambda^-(\eta)}, \quad (60b)$$

$$B_+ = \tfrac{3}{4}[q_0\mu_0 + q_1\mu_1] \quad (60c)$$

and

$$B_- = -\tfrac{3}{4}(q_0 + q_1). \quad (60d)$$

With these expressions for the expansion coefficients, $\mathbf{\chi}(x_0, \mu_0, \mu_1 \to x, \mu)$, as given by Eqs. (57), satisfies the first two boundary conditions. We propose that the half-space Green's function can be written in the form

$$\mathbf{\Psi}_g(x_0, \mu_0, \mu_1 \to x, \mu) = \mathbf{\chi}(x_0, \mu_0, \mu_1 \to x, \mu) - A_+\mathbf{\Phi}_+ - \int_0^1 A_1(\eta)e^{-x/\eta}\mathbf{\Phi}_1(\eta, \mu)\, d\eta - \int_0^1 A_2(\eta)e^{-x/\eta}\mathbf{\Phi}_2(\eta, \mu)\, d\eta. \quad (61)$$

Stipulating the condition of zero re-entrant radiation, boundary condition (c), we obtain the half-range expansion

$$\mathbf{\chi}(x_0, \mu_0, \mu_1 \to 0, \mu) = A_+\mathbf{\Phi}_+ + \int_0^1 A_1(\eta)\mathbf{\Phi}_1(\eta, \mu)\, d\eta + \int_0^1 A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)\, d\eta, \quad \mu \in [0, 1]. \quad (62)$$

Since $\mathbf{\chi}(x_0, \mu_0, \mu_1 \to 0, \mu)$ is known and the half-range completeness and orthogonality have been established through Theorems I and II, the coefficients A_+, $A_1(\eta)$, and $A_2(\eta)$ are found by taking, this time, *half-range* scalar products. Thus

$$A_+ = \frac{1}{N_+}\int_0^1 \tilde{\mathbf{\Phi}}_+^\dagger \mathbf{\chi}(x_0, \mu_0, \mu_1 \to 0, \mu)\, d\mu, \quad (63a)$$

$$A_1(\eta) = \frac{1}{\mathcal{S}_1(\eta)}\int_0^1 \tilde{\mathbf{\Phi}}_1^\dagger(\eta, \mu)\mathbf{\chi}(x_0, \mu_0, \mu_1 \to 0, \mu)\, d\mu, \quad (63b)$$

and

$$A_2(\eta) = \frac{1}{\mathcal{S}_2(\eta)}\int_0^1 \tilde{\mathbf{\Phi}}_2^\dagger(\eta, \mu)\mathbf{\chi}(x_0, \mu_0, \mu_1 \to 0, \mu)\, d\mu. \quad (63c)$$

Although the explicit evaluation of the scalar products indicated above is a tedious task, it is a straightforward one. We illustrate the procedure by further developing the expression for A_+.

Inspection of Eq. (57b) shows that for $x = 0$ it can be written as

$$\mathbf{\chi}(x_0, \mu_0, \mu_1 \to 0, \mu) = x_0 B_-\mathbf{\Phi}_+ - B_-\mathbf{\Psi}_-(0, \mu) - \int_0^1 B_1(-\eta)e^{-x_0/\eta}\mathbf{\Phi}_1(-\eta, \mu)\, d\eta - \int_0^1 B_2(-\eta)e^{-x_0/\eta}\mathbf{\Phi}_2(-\eta, \mu)\, d\eta. \quad (64)$$

It is at once apparent that upon taking the scalar product of Eq. (64) with $\mathbf{\Phi}_+^\dagger$, we encounter integrals similar to those that we have already discussed in the section on half-range normalization. Therefore, in order to proceed, we must evaluate integrals of the type

$$M_{ij}(\eta', \eta) \triangleq \int_0^1 \tilde{\mathbf{\Phi}}_i^\dagger(\eta', \mu)\mathbf{\Phi}_j(-\eta, \mu)\, d\mu, \quad \eta, \eta' \in [0, 1], \quad i = +1, 2 \text{ and } j = 1, 2. \quad (65)$$

We note that for η and $\mu \in [0, 1]$, $\mathbf{\Phi}_j(-\eta, \mu)$ is not singular; the complexity of the above integrals is thereby greatly reduced. We find

$$M_{+1}(\eta', \eta) = \eta X(-\eta), \quad (66a)$$

$$M_{+2}(\eta', \eta) = \eta(b\eta - a)Y(-\eta), \quad (66b)$$

$$M_{11}(\eta', \eta) = \tfrac{3}{2}\eta\eta'[X(-\eta)/(\eta + \eta')] \times [\eta\eta' + c(\eta + \eta') + 1], \quad (66c)$$

$$M_{12}(\eta', \eta) = (15/2b)\eta\eta' Y(-\eta), \quad (66d)$$

$$M_{21}(\eta', \eta) = (3/2b)\eta\eta' X(-\eta), \quad (66e)$$

and

$$M_{22}(\eta', \eta) = \tfrac{3}{2}\eta\eta'[Y(-\eta)/(\eta + \eta')] \times [\eta\eta' - c(\eta + \eta') + 1]. \quad (66f)$$

In addition, the determination of A_+, $A_1(\eta)$, and $A_2(\eta)$ necessitates the evaluation of the three integrals

$$M_{i-}(\eta') \triangleq \int_0^1 \tilde{\mathbf{\Phi}}_i^\dagger(\eta', \mu)\mathbf{\Psi}_-(0, \mu)\, d\mu, \quad \eta' \in [0, 1] \quad \text{and} \quad i = +, 1, 2. \quad (67)$$

These same integrals were encountered in the solution to the Milne problem in I. It was found there that

$$M_{+-}(\eta) = -N_+ z_0 = -N_+\left\{c + \tfrac{1}{2}[Y(1) - Y(-1)] + \frac{3}{4}\int_0^1 \frac{\mu^3}{Y(-\mu)}\, d\mu\right\}, \quad (68a)$$

$$M_{1-}(\eta) = -5\eta/b, \quad (68b)$$

and

$$M_{2-}(\eta) = -\eta(\eta - c). \tag{68c}$$

Here z_0 is the Milne problem extrapolation distance.

The solution for the expansion coefficient A_+ can now be expressed in terms of known functions; we find

$$A_+ = B_-(x_0 + z_0) + \frac{3}{2b}\int_0^1 B_1(-\eta)e^{-x_0/\eta}\eta X(-\eta)\,d\eta + \frac{3}{2b}\int_0^1 B_2(-\eta)e^{-x_0/\eta}\eta(b\eta - a)Y(-\eta)\,d\eta. \tag{69}$$

The coefficients $A_1(\eta)$ and $A_2(\eta)$ can be determined in a completely analogous manner. For the sake of brevity we present only the final results. We find

$$\begin{aligned}\mathbf{\Psi}_g(x_0, \mu_0, \mu_1 \to x, \mu) &= T_+\mathbf{\Phi}_+ + \int_0^1 T_1(\eta)e^{-x/\eta}\mathbf{\Phi}_1(\eta, \mu)\,d\eta \\ &\quad + \int_0^1 T_2(\eta)e^{-x/\eta}\mathbf{\Phi}_2(\eta, \mu)\,d\eta, \quad x > x_0, \end{aligned} \tag{70a}$$

where

$$\begin{aligned}T_+ &= \tfrac{3}{4}[q_0\mu_0 + q_1\mu_1 + (q_0 + q_1)(x_0 + z_0)] \\ &\quad - \frac{3}{2b}\int_0^1 B_1(-\eta)e^{-x_0/\eta}\eta X(-\eta)\,d\eta \\ &\quad - \frac{3}{2b}\int_0^1 B_2(-\eta)e^{-x_0/\eta}\eta(b\eta - a)Y(-\eta)\,d\eta, \end{aligned} \tag{70b}$$

$$T_1(\eta) = B_1(\eta)e^{x_0/\eta} - A_1(\eta), \tag{70c}$$

and

$$T_2(\eta) = B_2(\eta)e^{x_0/\eta} - A_2(\eta). \tag{70d}$$

For $x < x_0$, we obtain

$$\begin{aligned}\mathbf{\Psi}_g(x_0, \mu_0, \mu_1 \to x, \mu) &= Q_+\mathbf{\Phi}_+ - \int_0^1 [B_1(-\eta)e^{-(x_0-x)/\eta}\mathbf{\Phi}_1(-\eta, \mu) \\ &\quad + A_1(\eta)e^{-x/\eta}\mathbf{\Phi}_1(\eta, \mu)]\,d\eta \\ &\quad - \int_0^1 [B_2(-\eta)e^{-(x_0-x)/\eta}\mathbf{\Phi}_2(-\eta, \mu) \\ &\quad + A_2(\eta)e^{-x/\eta}\mathbf{\Phi}_2(\eta, \mu)]\,d\eta, \quad x < x_0, \end{aligned} \tag{71a}$$

where

$$\begin{aligned}Q_+ &= \tfrac{3}{4}(q_0 + q_1)(x + z_0 - \mu) \\ &\quad - \frac{3}{2b}\int_0^1 B_1(-\eta)e^{-x_0/\eta}\eta X(-\eta)\,d\eta \\ &\quad - \frac{3}{2b}\int_0^1 B_2(-\eta)e^{-x_0/\eta}\eta(b\eta - a)Y(-\eta)\,d\eta, \end{aligned} \tag{71b}$$

$$\begin{aligned}A_1(\eta) = -\frac{1}{\mathcal{S}_1(\eta)}\Big[&\frac{15\eta}{4b}(q_0 + q_1) \\ &+ \int_0^1 B_1(-t)e^{-x_0/t}M_{11}(\eta, t)\,dt \\ &+ \int_0^1 B_2(-t)e^{-x_0/t}M_{12}(\eta, t)\,dt\Big], \end{aligned} \tag{71c}$$

and

$$\begin{aligned}A_2(\eta) = -\frac{1}{\mathcal{S}_2(\eta)}\Big[&\tfrac{3}{4}(q_0 + q_1)\eta(\eta - c) \\ &+ \int_0^1 B_1(-t)e^{-x_0/t}M_{21}(\eta, t)\,dt \\ &+ \int_0^1 B_2(-t)e^{-x_0/t}M_{22}(\eta, t)\,dt\Big]. \end{aligned} \tag{71d}$$

The solution to the half-space Green's function corresponding to the source $\mathbf{Q}$ is thus complete.

The fact that the solution is complicated might have been anticipated, since the class of problems for which it could be used to generate solutions is a very broad one.

Let us now consider the somewhat simpler albedo problem.[21] Here we seek a solution to Eq. (1) subject to the following boundary conditions:

(i) $\mathbf{\Psi}_a(0, \mu) = \begin{bmatrix} s_0\delta(\mu - \mu_0) \\ s_1\delta(\mu - \mu_1) \end{bmatrix}$ for μ, μ_0, and $\mu_1 \geq 0$

and

(ii) $\mathbf{\Psi}_a(x, \mu)$ must remain finite as x increases without bound.

We immediately write the normal modes that satisfy condition (ii); thus

$$\mathbf{\Psi}_a(x, \mu) = A_+\mathbf{\Phi}_+ + \int_0^1 \alpha(\eta)e^{-x/\eta}\mathbf{\Phi}_1(\eta, \mu)\,d\eta + \int_0^1 \beta(\eta)e^{-x/\eta}\mathbf{\Phi}_2(\eta, \mu)\,d\eta. \tag{72}$$

In order to determine the expansion coefficients, we apply condition (i) to obtain

$$\begin{bmatrix} s_0\delta(\mu - \mu_0) \\ s_1\delta(\mu - \mu_1) \end{bmatrix} = A_+\mathbf{\Phi}_+ + \int_0^1 \alpha(\eta)\mathbf{\Phi}_1(\eta, \mu)\,d\eta + \int_0^1 \beta(\eta)\mathbf{\Phi}_2(\eta, \mu)\,d\eta, \quad \mu > 0. \tag{73}$$

Taking half-range scalar products of Eq. (73), we find the following results for the expansion coefficients:

$$A_+ = \frac{3}{2b}\,[s_1\gamma_2(\mu_1)(a + b\mu_1) - s_0\gamma_1(\mu_0)], \tag{74a}$$

$$\begin{aligned}\alpha(\eta) = [\mathcal{S}_1(\eta)]^{-1}\{&s_0\gamma_1(\mu_0)[\tfrac{3}{2}\eta(1 - \eta^2)[P/(\eta - \mu_0)] \\ &+ \lambda_1(\eta)\delta(\eta - \mu_0) + \tfrac{3}{2}\eta(c + \eta)] \\ &+ s_1(15\eta/2b)\gamma_2(\mu_1)\}, \end{aligned} \tag{74b}$$

and

$$\begin{aligned}\beta(\eta) = [\mathcal{S}_2(\eta)]^{-1}\{&s_0(3\eta/2b)\gamma_1(\mu_0) + s_1\gamma_2(\mu_1) \\ &\times [\tfrac{3}{2}\eta(1 - \mu_1^2)[P/(\eta - \mu_1)] + \lambda_2(\eta)\delta(\eta - \mu_1) \\ &- \tfrac{3}{2}\eta(c + \mu_1)]\}. \end{aligned} \tag{74c}$$

With the expansion coefficients thus determined, the solution is completed.

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[21] Although the albedo problem was sketched in I, the explicit results were not given there.