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On Establishing a Two-Term Scattering Law in the Theory of Radiative Transfer

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Introduction

In a recent work [1] the inverse problem for an isotropically scattering medium was solved in terms of the angular distribution of radiation entering and leaving a half space. Here we extend the previous result to include a linearly anisotropically scattering model.

Analysis

In terms of the problem in radiative transfer defined by

$$\mu \frac{\partial}{\partial x} \psi(x, \mu) + \psi(x, \mu) = \frac{\omega}{2} \int_{-1}^{1} (1 + b\mu\mu')\psi(x, \mu')\, d\mu', \tag{1}$$

$$\psi(0, \mu) = f(\mu), \quad \mu > 0, \tag{2a}$$

and

$$\psi(a, -\mu) = 0, \quad \mu > 0, \tag{2b}$$

we seek to find the single scattering albedo ω and the coefficient b in the scattering law in terms of $f(\mu)$ and the presumed measurable $\psi(0, -\mu)$, $\mu > 0$, and $\psi(a, \mu)$, $\mu > 0$. From Chandrasekhar's monumental work [2] on radiative transfer we note that

$$\psi(0, -\mu) = \frac{\omega}{2\mu} \int_{0}^{1} S(\mu, \mu') f(\mu')\, d\mu', \quad \mu > 0, \tag{3a}$$

and

$$\psi(a, \mu) = f(\mu) e^{-a/\mu} + \frac{\omega}{2\mu} \int_{0}^{1} T(\mu, \mu') f(\mu')\, d\mu', \quad \mu > 0, \tag{3b}$$

where

$$\begin{aligned}\left(\frac{1}{\mu'} + \frac{1}{\mu}\right) S(\mu, \mu') = {} & X(\mu)X(\mu')[1 - c_1(\mu + \mu') - r\mu\mu'] \\ & - Y(\mu)Y(\mu')[1 + c_1(\mu + \mu') - r\mu\mu'] \\ & - c_2(\mu + \mu')[X(\mu)Y(\mu') + Y(\mu)X(\mu')]\end{aligned} \tag{4a}$$

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and

$$\left(\frac{1}{\mu'} - \frac{1}{\mu}\right)T(\mu, \mu') = Y(\mu)X(\mu')[1 + c_1(\mu - \mu') + r\mu\mu'] - X(\mu)Y(\mu')[1 - c_1(\mu - \mu') + r\mu\mu'] + c_2(\mu - \mu')[X(\mu)X(\mu') + Y(\mu)Y(\mu')]. \tag{4b}$$

Here $r = b(1 - \omega)$ and $X(\mu)$ and $Y(\mu)$ are the functions associated with the characteristic function

$$\Psi(\mu) = \tfrac{1}{2}\omega(1 + r\mu^2), \tag{5}$$

used by Chandrasekhar [2] for the linearly anisotropically scattering model. In addition, the constants c_1 and c_2 can be expressed in terms of moments of the X and Y functions. We note that Chandrasekhar [2] has also deduced that

$$\rho(0) = \int_0^1 f(\mu')[(1 - c_1\mu')X(\mu') - c_2\mu' Y(\mu')]\, d\mu', \tag{6a}$$

$$\rho(a) = \int_0^1 f(\mu')[(1 + c_1\mu')Y(\mu') + c_2\mu' X(\mu')]\, d\mu', \tag{6b}$$

$$j(0) = \int_0^1 f(\mu')[q_1 X(\mu') + p_1 Y(\mu')]\mu'\, d\mu' \tag{6c}$$

and

$$j(a) = \int_0^1 f(\mu')[q_1 Y(\mu') + p_1 X(\mu')]\mu'\, d\mu' \tag{6d}$$

where

$$\rho(x) = \int_{-1}^1 \psi(x, \mu)\, d\mu, \qquad j(x) = \int_{-1}^1 \psi(x, \mu)\mu\, d\mu \tag{7}$$

and q_1 and p_1 are additional constants that can be expressed in terms of moments of the X and Y functions. Multiplying Eqns. (3) by μ^2 and integrating over μ, we find that we can write

$$rK(0) = \int_0^1 f(\mu')([c_1(1 - x_0) - c_2 y_0 + r(1 - x_0)\mu']X(\mu') + [c_2(1 - x_0) - c_1 y_0 + r y_0 \mu']Y(\mu'))\mu'\, d\mu' \tag{8a}$$

and

$$rK(a) = \int_0^1 f(\mu')([c_2 y_0 - c_1(1 - x_0) + r(1 - x_0)\mu']Y(\mu') + [c_1 y_0 - c_2(1 - x_0) + r y_0 \mu']X(\mu'))\mu'\, d\mu' \tag{8b}$$

where

$$K(x) = \int_{-1}^1 \psi(x, \mu)\mu^2\, d\mu, \tag{9}$$

$$x_0 = \int_0^1 X(\mu)\Psi(\mu)\, d\mu \quad \text{and} \quad y_0 = \int_0^1 Y(\mu)\Psi(\mu)\, d\mu. \tag{10}$$

If we now multiply Eqn. (3a) by $f(\mu)$ and integrate we find, after considerable algebra, that

$$S_0 = \int_0^1 f(\mu)\psi(0, -\mu)\, d\mu \tag{11}$$

can be expressed as

$$4S_0 = \omega(\rho^2(0) - \rho^2(a) - b[j^2(0) - j^2(a)]), \tag{12}$$

after using Eqns. (6). Clearly we could carry out two experiments, say $f(\mu) = f_1(\mu)$ and $f(\mu) = f_2(\mu)$, and thus generate two versions of Eqn. (12) that could be solved for ω and b. On the other hand, results for ω and b can be obtained from only one experiment if we consider

$$S_2 = \int_0^1 f(\mu)\psi(0, -\mu)\mu^2\, d\mu. \tag{13}$$

On multiplying Eqn. (3a) by $f(\mu)\mu^2$ and integrating, we find we can write

$$4S_2 = \omega\left(\frac{[j^2(0) - j^2(a)]}{1 - \omega} - \frac{b[K^2(0) - K^2(a)]}{1 - \omega b/3}\right). \tag{14}$$

We can now solve Eqns. (12) and (14) to find our final results:

$$\omega = \frac{-B - [B^2 - 4AC]^{1/2}}{2A} \tag{15a}$$

and

$$b = \left[\rho^2(0) - \rho^2(a) - \frac{4}{\omega} S_0\right][j^2(0) - j^2(a)]^{-1} \tag{15b}$$

where

$$A = -[\rho^2(0) - \rho^2(a)][j^2(0) - j^2(a) + 4S_2 - 3K^2(0) + 3K^2(a)], \tag{16a}$$

$$\begin{aligned} B = {} & 4S_0[j^2(0) - j^2(a) + 4S_2 - 3K^2(0) + 3K^2(a)] \\ & + [\rho^2(0) - \rho^2(a)][4S_2 - 3K^2(0) + 3K^2(a)] \\ & + 3[j^2(0) - j^2(a)][4S_2 + j^2(0) - j^2(a)] \end{aligned} \tag{16b}$$

and

$$C = -4S_0[4S_2 - 3K^2(0) + 3K^2(a)] - 12S_2[j^2(0) - j^2(a)]. \tag{16c}$$

It is evident that the solutions given by Eqns. (15) are in terms only of the surface quantities $\psi(0, \mu)$ and $\psi(a, \mu)$.

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References

[1] C. E. Siewert, J. Math. Phys. **19**, 1587 (1978).

[2] S. Chandrasekhar, *Radiative Transfer*, Oxford University Press, London (1950).

Abstract

The inverse problem for a linearly anisotropically scattering model in the theory of radiative transfer or neutron transport is solved, in terms only of surface quantities, for a finite slab.

Zusammenfassung

Das inverse Problem für ein linear anisotropes Streumodell in der Theorie der Strahlungsübertragung oder der Neutronen-Transporttheorie wird gelöst; für eine endliche Schicht wird die Lösung mit Hilfe von Oberflächengrössen ausgedrückt.

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