# ON MULTI-REGION PROBLEMS IN RADIATIVE TRANSFER

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**Abstract.** The $F_N$ method is used to solve radiative transfer problems, based on the general anisotropically scattering model, in multi-layer atmospheres.

## 1. Introduction

The recently developed (cf. Siewert and Benoist, 1979) $F_N$ method for solving problems in neutron-transport theory (Grandjean and Siewert, 1979) and radiative transfer (Siewert, 1978, 1979), has proved to be efficient and economical to use, from the point-of-view of numerical calculations, and thus we wish to report here our analysis of a typical many-layer problem. We consider that the atmosphere is stratified into $K$ homogeneous regions, and thus we take the following equation of transfer to be applicable in each of the $K$ regions, $\tau \in [\tau_{i-1}, \tau_i]$, $i = 1, 2, 3, \ldots, K$:

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \omega_i \sum_{l=0}^{L_i} \left(\frac{2l+1}{2}\right) f_{i,l} P_l(\mu) \int_{-1}^{1} P_l(\mu') I(\tau, \mu')\, \mathrm{d}\mu', \tag{1}$$

where $\tau$ is the optical variable, $\mu$ is the direction cosine of the propagating radiation (as measured from the positive $\tau$ axis), $\omega_i$ is the single-scattering albedo for region $i$, and the constants $f_{i,l}$, $l = 0, 1, 2, \ldots, L_i$, are the coefficients in a Legendre expansion of the phase function for region $i$. We assume that the radiation entering the system of layers is specified, and thus we write

$$I(0, \mu) = f_0(\mu), \qquad \mu > 0, \tag{2a}$$

and

$$I(\tau_K, -\mu) = f_K(\mu), \qquad \mu > 0. \tag{2b}$$

We assume that $\tau_0 = 0$, and consider that $f_0(\mu)$ and $f_K(\mu)$ are given. In addition to the

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boundary conditions given by Equations (2), we use the fact that the angular distribution $I(\tau, \mu)$ should be continuous (except perhaps for $\mu = 0$) at each interface $\tau_j$, $j = 1, 2, 3, \ldots, K-1$. Although the complete solution for $I(\tau, \mu)$ will be established, we are concerned primarily with computing the exit quantities

$$A^* = \frac{1}{\Delta}\int_0^1 I(0, -\mu)\mu \, d\mu \tag{3a}$$

and

$$B^* = \frac{1}{\Delta}\int_0^1 I(\tau_K, \mu)\mu \, d\mu, \tag{3b}$$

where

$$\Delta = \int_0^1 [f_0(\mu) + f_K(\mu)]\mu \, d\mu. \tag{4}$$

## 2. Analysis

We begin our analysis of the considered multi-region problem by expressing $I(\tau, \mu)$ in terms of the well-known elementary solutions (cf. McCormick and Kuščer, 1973)

$$I(\tau, \mu) = \sum_{\beta=0}^{\kappa_i - 1} [A(\nu_{i,\beta})\phi_i(\nu_{i,\beta}, \mu) \, e^{-\tau/\nu_{i,\beta}} + A(-\nu_{i,\beta})\phi_i(-\nu_{i,\beta}, \mu) \, e^{\tau/\nu_{i,\beta}}] + \\ + \int_{-1}^1 A_i(\nu)\phi_i(\nu, \mu) \, e^{-\tau/\nu} \, d\nu, \tag{5}$$

for $\tau \in [\tau_{i-1}, \tau_i]$, $i = 1, 2, 3, \ldots, K$; where

$$\phi_i(\nu, \mu) = \tfrac{1}{2}\omega_i \nu g_i(\nu, \mu) Pv\left(\frac{1}{\nu - \mu}\right) + \lambda_i(\nu)\delta(\nu - \mu), \tag{6a}$$

$$g_i(\nu, \mu) = \sum_{l=0}^{L_i} (2l + 1) f_{i,l} g_{i,l}(\nu) P_l(\mu), \tag{6b}$$

$$\lambda_i(\nu) = 1 + \nu P \int_{-1}^1 \psi_i(x) \frac{dx}{x - \nu}, \tag{6c}$$

$$\psi_i(x) = \tfrac{1}{2}\omega_i g_i(x, x) \tag{6d}$$

and the polynomials $g_{i,l}(\nu)$ can be generated from

$$\nu h_{i,l} g_{i,l}(\nu) = (l + 1) g_{i,l+1}(\nu) + l g_{i,l-1}(\nu) \tag{7a}$$

with

$$g_{i,0}(\nu) = 1 \quad \text{and} \quad g_{i,1}(\nu) = \nu(1 - \omega_i). \tag{7b, c}$$

In addition,

$$h_{i,l} = (2l + 1)(1 - \omega_i f_{i,l}), \tag{7d}$$

$$\phi_i(\nu_{i,\beta}, \mu) = \tfrac{1}{2}\omega_i \nu_{i,\beta} g_i(\nu_{i,\beta}, \mu)\left(\frac{1}{\nu_{i,\beta} - \mu}\right) \tag{8}$$

and $\nu_{i,\beta}$, $\beta = 0, 1, 2, \ldots, \kappa_i - 1$, denote the 'positive' zeros of

$$\Lambda_i(z) = 1 + z \int_{-1}^1 \psi_i(x) \frac{dx}{x - z} \tag{9}$$

in the complex plane cut from $-1$ to 1 along the real axis. The expansion coefficients $A(\pm\nu_{i,\beta})$ and $A_i(\nu)$, $\nu \in (-1, 1)$, appearing in Equation (5) are to be determined by the boundary conditions. We note that $I(\tau, \mu)$, as given by Equation (5), satisfies exactly the appropriate equations of transfer.

As discussed by Grandjean and Siewert (1979), the full-range orthogonality relations concerning the functions $\phi_i(\xi, \mu)$ can be used to develop a system of singular integral equations and constraints for the distribution of radiation leaving a single medium (the same singular integral equations and constraints were also reported by Bowden *et al.* 1968). Rather straightforwardly we can generalize these singular integral equations and constraints to our multi-region problem. Thus we consider the following system of 'exact' equations to be solved for the exit distributions $I(0, -\mu)$, $\mu > 0$, and $I(\tau_K, \mu)$, $\mu > 0$:

$$\int_0^1 \mu\phi_1(\xi, \mu)I(0, -\mu)\,\mathrm{d}\mu + \\ + \mathrm{e}^{-\Delta_1/\xi}\int_0^1 \mu[\phi_1(-\xi, \mu)I(\tau_1, \mu) - \phi_1(\xi, \mu)I(\tau_1, -\mu)]\,\mathrm{d}\mu = R_0(\xi), \\ \xi \in P_1, \tag{10a}$$

$$\int_0^1 \mu[\phi_1(\xi, \mu)I(\tau_1, \mu) - \phi_1(-\xi, \mu)I(\tau_1, -\mu)]\,\mathrm{d}\mu + \\ + \mathrm{e}^{-\Delta_1/\xi}\int_0^1 \mu\phi_1(-\xi, \mu)I(0, -\mu)\,\mathrm{d}\mu = S_0(\xi), \qquad \xi \in P_1, \tag{10b}$$

$$\int_0^1 \mu[\phi_i(\xi, \mu)I(\tau_{i-1}, -\mu) - \phi_i(-\xi, \mu)I(\tau_{i-1}, \mu)]\,\mathrm{d}\mu + \\ + \mathrm{e}^{-\Delta_i/\xi}\int_0^1 \mu[\phi_i(-\xi, \mu)I(\tau_i, \mu) - \phi_i(\xi, \mu)I(\tau_i, -\mu)]\,\mathrm{d}\mu = 0, \\ \xi \in P_i, \tag{10c}$$

$$\int_0^1 \mu[\phi_i(\xi, \mu)I(\tau_i, \mu) - \phi_i(-\xi, \mu)I(\tau_i, -\mu)]\,\mathrm{d}\mu + \\ + \mathrm{e}^{-\Delta_i/\xi}\int_0^1 \mu[\phi_i(-\xi, \mu)I(\tau_{i-1}, -\mu) - \phi_i(\xi, \mu)I(\tau_{i-1}, \mu)]\,\mathrm{d}\mu = 0, \\ \xi \in P_i, \tag{10d}$$

for $i = 2, 3, 4, \ldots, K - 1$, and

$$\int_0^1 \mu[\phi_K(\xi, \mu)I(\tau_{K-1}, -\mu) - \phi_K(-\xi, \mu)I(\tau_{K-1}, \mu)]\,\mathrm{d}\mu + \\ + \mathrm{e}^{-\Delta_K/\xi}\int_0^1 \mu\phi_K(-\xi, \mu)I(\tau_K, \mu)\,\mathrm{d}\mu = R_K(\xi), \qquad \xi \in P_K, \tag{10e}$$

$$\int_0^1 \mu\phi_K(\xi, \mu)I(\tau_K, \mu)\,\mathrm{d}\mu + \\ + \mathrm{e}^{-\Delta_K/\xi}\int_0^1 \mu[\phi_K(-\xi, \mu)I(\tau_{K-1}, -\mu) - \phi_K(\xi, \mu)I(\tau_{K-1}, \mu)]\,\mathrm{d}\mu = S_K(\xi), \\ \xi \in P_K. \tag{10f}$$

The known functions are

$$R_0(\xi) = \int_0^1 \mu\phi_1(-\xi, \mu) f_0(\mu)\, d\mu, \tag{11a}$$

$$S_0(\xi) = e^{-\Delta_1/\xi} \int_0^1 \mu\phi_1(\xi, \mu) f_0(\mu)\, d\mu, \tag{11b}$$

$$R_K(\xi) = e^{-\Delta_K/\xi} \int_0^1 \mu\phi_K(\xi, \mu) f_K(\mu)\, d\mu, \tag{11c}$$

and

$$S_K(\xi) = \int_0^1 \mu\phi_K(-\xi, \mu) f_K(\mu)\, d\mu. \tag{11d}$$

In addition,

$$\Delta_i = \tau_i - \tau_{i-1} \tag{12}$$

and

$$\xi \in P_i \Rightarrow \xi \in \{\nu_{i,\beta}\} \cup (0, 1). \tag{13}$$

In order to establish the $F_N$ equations, we now introduce the approximations

$$I(\tau_i, -\mu) = \sum_{\alpha=0}^{N} a_{i,\alpha}\mu^\alpha, \qquad \mu > 0, \qquad i = 0, 1, 2, 3, \ldots, K-1, \tag{14a}$$

and

$$I(\tau_i, \mu) = \sum_{\alpha=0}^{N} b_{i,\alpha}\mu^\alpha, \qquad \mu > 0, \qquad i = 1, 2, 3, \ldots, K, \tag{14b}$$

which can be substituted into Equations (10) to obtain

$$\sum_{\alpha=0}^{N} [a_{0,\alpha}B_\alpha^{(1)}(\xi) + e^{-\Delta_1/\xi}[b_{1,\alpha}A_\alpha^{(1)}(\xi) - a_{1,\alpha}B_\alpha^{(1)}(\xi)]] = \frac{2}{\omega_1\xi} R_0(\xi), \qquad \xi \in P_1, \tag{15a}$$

$$\sum_{\alpha=0}^{N} [b_{1,\alpha}B_\alpha^{(1)}(\xi) - a_{1,\alpha}A_\alpha^{(1)}(\xi) + e^{-\Delta_1/\xi}a_{0,\alpha}A_\alpha^{(1)}(\xi)] = \frac{2}{\omega_1\xi} S_0(\xi), \qquad \xi \in P_1, \tag{15b}$$

$$\sum_{\alpha=0}^{N} [a_{i-1,\alpha}B_\alpha^{(i)}(\xi) - b_{i-1,\alpha}A_\alpha^{(i)}(\xi) + e^{-\Delta_i/\xi}[b_{i,\alpha}A_\alpha^{(i)}(\xi) - a_{i,\alpha}B_\alpha^{(i)}(\xi)]] = 0, \qquad \xi \in P_i, \tag{15c}$$

$$\sum_{\alpha=0}^{N} [b_{i,\alpha}B_\alpha^{(i)}(\xi) - a_{i,\alpha}A_\alpha^{(i)}(\xi) + e^{-\Delta_i/\xi}[a_{i-1,\alpha}A_\alpha^{(i)}(\xi) - b_{i-1,\alpha}B_\alpha^{(i)}(\xi)]] = 0, \qquad \xi \in P_i, \tag{15d}$$

for $i = 2, 3, 4, \ldots, K-1$, and

$$\sum_{\alpha=0}^{N} [a_{K-1,\alpha}B_\alpha^{(K)}(\xi) - b_{K-1,\alpha}A_\alpha^{(K)}(\xi) + e^{-\Delta_K/\xi}b_{K,\alpha}A_\alpha^{(K)}(\xi)] = \frac{2}{\omega_K\xi} R_K(\xi), \qquad \xi \in P_K, \tag{15e}$$

$$\sum_{\alpha=0}^{N} [b_{K,\alpha}B_\alpha^{(K)}(\xi) + e^{-\Delta_K/\xi}[a_{K-1,\alpha}A_\alpha^{(K)}(\xi) - b_{K-1,\alpha}B_\alpha^{(K)}(\xi)]] = \frac{2}{\omega_K\xi} S_K(\xi), \qquad \xi \in P_K. \tag{15f}$$

The known functions required here have been reported by Siewert (1978a) – *viz.*,

$$A_0^{(i)}(\xi) = 1 - \frac{2}{\omega_i}\xi\psi_i(\xi)\log\left(1 + \frac{1}{\xi}\right) + \sum_{l=1}^{L_i}(2l+1)f_{i,l}g_{i,l}(\xi)\Pi_l(\xi),$$
$$i = 1, 2, 3, \ldots, K, \qquad (16a)$$

$$B_0^{(i)}(\xi) = \frac{2}{\omega_i} - 2 + A_0^{(i)}(\xi), \qquad i = 1, 2, 3, \ldots, K, \qquad (16b)$$

$$A_{\alpha+1}^{(i)}(\xi) = -\xi A_\alpha^{(i)}(\xi) + \sum_{l=0}^{L_i}(2l+1)f_{i,l}(-1)^l g_{i,l}(\xi)\Delta_{\alpha,l}, \qquad (17a)$$

and

$$B_{\alpha+1}^{(i)}(\xi) = \xi B_\alpha^{(i)}(\xi) - \sum_{l=0}^{L_i}(2l+1)f_{i,l}g_{i,l}(\xi)\Delta_{\alpha,l}. \qquad (17b)$$

In addition, the $\Pi$ polynomials can be generated (for $l > 0$) from

$$(2l+1)\xi\Pi_l(\xi) = (-1)^l(2l+1)\Delta_{0,l} + (l+1)\Pi_{l+1}(\xi) + l\Pi_{l-1}(\xi), \qquad (18)$$

with

$$\Pi_0(\xi) = 1, \qquad \Pi_1(\xi) = \xi - \tfrac{1}{2} \quad \text{and} \quad \Pi_2(\xi) = \tfrac{3}{2}\xi(\xi - \tfrac{1}{2}). \qquad (19a, b, c)$$

It is apparent that we can now use $2(N+1)K$ values of $\xi$ in Equations (15) to generate $2(N+1)K$ linear algebraic equations that can readily be solved to yield the $2(N+1)K$ desired constants $a_{i,\alpha}$ and $b_{i,\alpha}$. We note that the coefficient matrix for this system of equations is not dense. We see also that the elements of the coefficient matrix have been expressed in terms of polynomials and the log function and thus can be readily computed numerically. In the next section we discuss the manner in which we select the values of $\xi$ to be used in Equations (15) and give some typical numerical results.

We note that the $F_N$ method yields first of all the angular intensity at each of the two surfaces and at all of the interfaces. To establish the solution for all $\tau$ we can use the full-range orthogonality properties of the functions $\phi_i(\xi, \mu)$ to find the expansion coefficients required in Equation (5):

$$A(\xi)N_1(\xi)\,\mathrm{e}^{-\tau_1/\xi} = \tfrac{1}{2}\omega_1\xi\sum_{\alpha=0}^{N}[b_{1,\alpha}B_\alpha^{(1)}(\xi) - a_{1,\alpha}A_\alpha^{(1)}(\xi)], \qquad \xi \in P_1, \qquad (20a)$$

$$A(-\xi)N_1(-\xi) = \int_0^1 \mu\phi_1(-\xi, \mu)f_0(\mu)\,\mathrm{d}\mu - \tfrac{1}{2}\omega_1\xi\sum_{\alpha=0}^{N}a_{0,\alpha}B_\alpha^{(1)}(\xi), \qquad \xi \in P_1,$$
$$(20b)$$

$$A(\xi)N_i(\xi)\,\mathrm{e}^{-\tau_i/\xi} = \tfrac{1}{2}\omega_i\xi\sum_{\alpha=0}^{N}[b_{i,\alpha}B_\alpha^{(i)}(\xi) - a_{i,\alpha}A_\alpha^{(i)}(\xi)], \qquad \xi \in P_i,$$
$$i = 2, 3, 4, \ldots, K-1, \qquad (21a)$$

$$A(-\xi)N_i(-\xi)\,\mathrm{e}^{\tau_{i-1}/\xi} = \tfrac{1}{2}\omega_i\xi\sum_{\alpha=0}^{N}[b_{i-1,\alpha}A_\alpha^{(i)}(\xi) - a_{i-1,\alpha}B_\alpha^{(i)}(\xi)], \qquad \xi \in P_i,$$
$$i = 2, 3, 4, \ldots, K-1, \qquad (21b)$$

$$A(\xi)N_K(\xi)\,e^{-\tau_K/\xi} = \tfrac{1}{2}\omega_K\xi \sum_{\alpha=0}^{N} b_{K,\alpha}B_\alpha^{(K)}(\xi) - \int_0^1 \mu\phi_K(-\xi,\mu)f_K(\mu)\,d\mu, \quad \xi \in P_K, \tag{22a}$$

$$A(-\xi)N_K(-\xi)\,e^{\tau_{K-1}/\xi} = \tfrac{1}{2}\omega_K\xi \sum_{\alpha=0}^{N} [b_{K-1,\alpha}A_\alpha^{(K)}(\xi) - a_{K-1,\alpha}B_\alpha^{(K)}(\xi)], \quad \xi \in P_K. \tag{22b}$$

Here

$$N_i(\pm\nu_{i,\beta}) = \pm\tfrac{1}{2}\omega_i\nu_{i,\beta}^2 g_i(\nu_{i,\beta},\nu_{i,\beta})\Lambda_i'(\nu_{i,\beta}) \tag{23}$$

and

$$N_i(\pm\nu) = \pm\nu[\lambda_i^2(\nu) + \pi^2\nu^2\psi_i^2(\nu)]. \tag{24}$$

## 3. Numerical Results

In order to demonstrate the numerical accuracy of the $F_N$ method for anisotropic scattering we first consider a single slab that is described by the scattering data given in Table I. In Table II we list $\nu_0$ and $\nu_1$ (where appropriate) as a function of the single-scattering albedo $\omega$ for the scattering law listed in Table I. Table III contains the results of the calculation of the albedo and transmission factor (as a function of $\omega$ and the slab thickness $\tau_1$) for the $L = 8$ scattering model. Our scheme for selecting the points $\xi_\beta$, $\beta = 0, 1, 2, \ldots, N$, at which to evaluate and solve Equations (15) for the desired constants $a_{0,\alpha}$ and $b_{1,\alpha}$, $\alpha = 0, 1, 2, \ldots, N$, was to use $\xi_0 = \nu_0$, $\xi_1 = 0$, $\xi_2 = 1$ and the remaining $\xi_\beta$ spaced equally distant in the interval [0, 1] for the case $\kappa_0 = 1$. For the cases ($\omega \geqslant 0.9$) when $\kappa_0 = 2$ we used $\xi_0 = \nu_0$, $\xi_1 = \nu_1$, $\xi_2 = 0$, $\xi_3 = 1$ and the remaining $\xi_\beta$ spaced equally in [0, 1]. Note that some care must be taken when deducing the correct forms of Equations (15) for the case of a single slab.

For a multi-region application of the method we consider an atmosphere stratified

TABLE I

Scattering Law

| $l$ | $(2l + 1)f_l$ |
|---|---|
| 0 | 1 |
| 1 | 2.00916 |
| 2 | 1.56339 |
| 3 | 0.67407 |
| 4 | 0.22215 |
| 5 | 0.04725 |
| 6 | 0.00671 |
| 7 | 0.00068 |
| 8 | 0.00005 |

TABLE II

Discrete eigenvalues

| $\omega$ | $\nu_0$ | $\nu_1 - 1$ |
|---|---|---|
| 0.65 | 1.5481091387 | |
| 0.70 | 1.6848084756 | |
| 0.75 | 1.8625916761 | |
| 0.80 | 2.1052210741 | |
| 0.85 | 2.4621183578 | |
| 0.90 | 3.0606033561 | 1.95 (−12) |
| 0.99 | 10.0034767275 | 7.5177669527 (−4) |
| 0.999 | 31.7549147640 | 1.0924521440 (−3) |
| 0.9999 | 100.4568325323 | 1.1310738139 (−3) |

TABLE III

Albedo due to isotropic incidence

| $\omega$ | $\tau_1$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | 'Exact' |
|---|---|---|---|---|---|---|
| 0.9 | 1.0 | 0.17204 | 0.17205 | 0.17198 | 0.17194 | 0.17192 |
| 0.9 | 10.0 | 0.29084 | 0.29075 | 0.29072 | 0.29071 | 0.29070 |
| 0.99 | 1.0 | 0.22671 | 0.22674 | 0.22668 | 0.22664 | 0.22662 |
| 0.99 | 10.0 | 0.62209 | 0.62207 | 0.62207 | 0.62206 | 0.62206 |
| 0.999 | 1.0 | 0.23319 | 0.23322 | 0.23316 | 0.23313 | 0.23310 |
| 0.999 | 10.0 | 0.70697 | 0.70695 | 0.70695 | 0.70695 | 0.70694 |
| 0.9999 | 1.0 | 0.23385 | 0.23388 | 0.23382 | 0.23379 | 0.23376 |
| 0.9999 | 10.0 | 0.71694 | 0.71692 | 0.71692 | 0.71692 | 0.71691 |

TABLE IV

Transmission due to isotropic incidence

| $\omega$ | $\tau_1$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | 'Exact' |
|---|---|---|---|---|---|---|
| 0.9 | 1.0 | 0.65417 | 0.65413 | 0.65420 | 0.65425 | 0.65427 |
| 0.9 | 10.0 | 0.032929 | 0.032933 | 0.032935 | 0.032935 | 0.032936 |
| 0.99 | 1.0 | 0.75359 | 0.75356 | 0.75362 | 0.75365 | 0.75368 |
| 0.99 | 10.0 | 0.21076 | 0.21077 | 0.21078 | 0.21078 | 0.21078 |
| 0.999 | 1.0 | 0.76481 | 0.76478 | 0.76484 | 0.76487 | 0.76490 |
| 0.999 | 10.0 | 0.27342 | 0.27343 | 0.27344 | 0.27344 | 0.27344 |
| 0.9999 | 1.0 | 0.76595 | 0.76592 | 0.76598 | 0.76601 | 0.76604 |
| 0.9999 | 10.0 | 0.8107 | 0.8108 | 0.28109 | 0.28109 | 0.28109 |

TABLE V

Data for multi-region problem

| $l$ | $\omega_l$ | $\tau_l$ |
|---|---|---|
| 1 | 0.65 | 1.0 |
| 2 | 0.70 | 3.0 |
| 3 | 0.75 | 6.0 |
| 4 | 0.80 | 10.0 |
| 5 | 0.85 | 15.0 |
| 6 | 0.90 | 21.0 |

in six layers each with the same scattering law; but we allow $\omega$ to be different in each layer. In Table V we list the single-scattering albedo $\omega_l$ and the optical thickness $\tau_l$ for each of the six layers. In Tables VI and VII we list our results for the albedo and transmission factor for two cases: $L = 0$ – *i.e.*, isotropic scattering – and the $L = 8$ model of the scattering law listed in Table I. We note that the results given in Tables VI and VII correspond to incident radiation of the form

$$I(0, \mu) = \mu^{\beta}, \qquad \mu > 0, \tag{25a}$$

and

$$I(\tau_6, -\mu) = 0, \qquad \mu > 0. \tag{25b}$$

To have an idea of the speed of the method, we note that a typical $L = 8$ calculation by the $F_3$, $F_4$ and $F_5$ approximations of the albedo and transmission factor for the six-layer atmosphere required, for $\beta = 0$, approximately 2, 4, and 6 s respectively of computation time on the I.B.M. 370/155 machine.

In order to compare our results with those obtained by a modern particle-transport code, we have listed in Tables VI and VII ANISN results established and communicated by Clark 1978; (private communication). The ANISN results are based on using six mesh points per mean-free path and 32 angular quadrature points.

TABLE VI

Albedo for multi-region problem

| $\beta$ | $L$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | 'Exact' | ANISN |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.2320 | 0.2282 | 0.2281 | 0.2280 | 0.2280 | 0.2280 |
| 1 | 0 | 0.2154 | 0.2147 | 0.2148 | 0.2148 | 0.2148 | 0.2148 |
| 2 | 0 | 0.2074 | 0.2077 | 0.2078 | 0.2078 | 0.2079 | 0.2079 |
| 0 | 8 | 0.1054 | 0.1003 | 0.1002 | 0.1001 | 0.1001 | 0.1001 |
| 1 | 8 | 0.08170 | 0.08049 | 0.08054 | 0.08057 | 0.08058 | 0.08059 |
| 2 | 8 | 0.07064 | 0.07042 | 0.07047 | 0.07051 | 0.07052 | 0.07051 |

TABLE VII

Transmission Factor for multi-region problem

| $\beta$ | $L$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | 'Exact' | ANISN |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.6053 (−6) | 0.6161 (−6) | 0.6169 (−6) | 0.6171 (−6) | 0.6171 (−6) | 0.6099 (−6) |
| 1 | 0 | 0.6992 (−6) | 0.7166 (−6) | 0.7178 (−6) | 0.7181 (−6) | 0.7180 (−6) | 0.7098 (−6) |
| 2 | 0 | 0.7597 (−6) | 0.7879 (−6) | 0.7900 (−6) | 0.7905 (−6) | 0.7906 (−6) | 0.7815 (−6) |
| 0 | 8 | 0.7380 (−4) | 0.7416 (−4) | 0.7418 (−4) | 0.7419 (−4) | 0.7419 (−4) | 0.7391 (−4) |
| 1 | 8 | 0.8484 (−4) | 0.8540 (−4) | 0.8542 (−4) | 0.8543 (−4) | 0.8543 (−4) | 0.8512 (−4) |
| 2 | 8 | 0.9208 (−4) | 0.9303 (−4) | 0.9306 (−4) | 0.9307 (−4) | 0.9307 (−4) | 0.9274 (−4) |

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