J. Quant. Spectrosc. Radiat. Transfer Vol. 22, pp. 441–446
Pergamon Press Ltd., 1979. Printed in Great Britain

ON THE INVERSE PROBLEM FOR A THREE-TERM PHASE FUNCTION

C. E. SIEWERT
Nuclear Engineering Department, North Carolina State University Raleigh, NC 27650, U.S.A.

(Received 2 February 1979)

Abstract—Elementary considerations are used to establish the phase function for a three-term scattering law basic to the inverse problem for radiative transfer in a finite slab.

1. INTRODUCTION

In a recent paper[1] the X and Y functions of Chandrasekhar[2] were used to establish an exact solution of the inverse problem for a two-term scattering law. Here we would like to report an alternative development of the solution for a more general case and to discuss the extension to a three-term scattering model. This method of solution uses only the equation of transfer and the boundary conditions and thus does not require any detailed knowledge (say in terms of X and Y functions) of the intensity. First of all we consider the equation of transfer

$$\mu\frac{\partial}{\partial\tau}I(\tau, \mu)+I(\tau, \mu)=\frac{\omega}{2}\int_{-1}^{1}[1+b_1\mu\mu'+b_2P_2(\mu)P_2(\mu')]I(\tau, \mu')\mathrm{d}\mu', \quad \tau\in[L, R], \tag{1}$$

and the boundary conditions

$$I(L, \mu)=F_1(\mu), \quad \mu>0, \tag{2a}$$

and

$$I(R, -\mu)=F_2(\mu), \quad \mu>0. \tag{2b}$$

Here μ is the direction cosine of the propagating radiation, as measured from the positive τ axis, $\tau=L$ and $\tau=R$ correspond respectively to the left and right boundaries of the atmosphere and $F_1(\mu)$ and $F_2(\mu)$ are considered given incident distributions. We assume that we can measure the exit distributions $I(L, -\mu)$ and $I(R, \mu)$, $\mu>0$, and thus we wish to express ω, b_1, and b_2 in terms of $I(L, \mu)$ and $I(R, \mu)$, $\mu\in(-1, 1)$.

2. ANALYSIS

If we let

$$F(\tau, \mu)=\mu\frac{\partial}{\partial\tau}I(\tau, \mu) \tag{3}$$

then we can change μ to $-\mu$ and write Eq. (1) as

$$F(\tau, -\mu)+I(\tau, -\mu)=\frac{\omega}{2}[I_0(\tau)-b_1\mu I_1(\tau)+b_2P_2(\mu)I_2(\tau)], \tag{4}$$

where

$$I_\alpha(\tau)=\int_{-1}^{1}I(\tau, \mu)P_\alpha(\mu)\,\mathrm{d}\mu. \tag{5}$$

Now we can multiply Eq. (4) by $I(\tau, \mu)$ and integrate over μ from -1 to 1 to obtain

$$T_0(\tau) + 2\int_0^1 I(\tau, \mu)\, I(\tau, -\mu)\,\mathrm{d}\mu = \frac{\omega}{2}[I_0^2(\tau) - b_1 I_1^2(\tau) + b_2 I_2^2(\tau)], \tag{6}$$

where

$$T_0(\tau) = \int_{-1}^1 I(\tau, \mu) F(\tau, -\mu)\,\mathrm{d}\mu. \tag{7}$$

If we now differentiate Eq. (7) and use Eqs. (3) and (4) we can write

$$\frac{\mathrm{d}}{\mathrm{d}\tau}T_0(\tau) = \int_{-1}^1 I(\tau, \mu)\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\omega}{2}[I_0(\tau) - b_1\mu I_1(\tau) + b_2 P_2(\mu) I_2(\tau)] - I(\tau, -\mu)\right)\mathrm{d}\mu \tag{8}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\,T_0(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\omega}{4}[I_0^2(\tau) - b_1 I_1^2(\tau) + b_2 I_2^2(\tau)] - \int_0^1 I(\tau, \mu) I(\tau, -\mu)\,\mathrm{d}\mu\right). \tag{9}$$

Differentiating Eq. (6), we find

$$\frac{\mathrm{d}}{\mathrm{d}\tau}T_0(\tau) = 2\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\omega}{4}[I_0^2(\tau) - b_1 I_1^2(\tau) + b_2 I_2^2(\tau)] - \int_0^1 I(\tau, \mu) I(\tau, -\mu)\,\mathrm{d}\mu\right) \tag{10}$$

which can be used in Eq. (9) to yield

$$\frac{\mathrm{d}}{\mathrm{d}\tau}T_0(\tau) = 0 \tag{11}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\omega}{4}[I_0^2(\tau) - b_1 I_1^2(\tau) + b_2 I_2^2(\tau)] - \int_0^1 I(\tau, \mu) I(\tau, -\mu)\,\mathrm{d}\mu\right) = 0. \tag{12}$$

Finally we can integrate Eq. (12) from $\tau = L$ to $\tau = R$ to obtain

$$4S_0 = \omega[I_0^2(R) - I_0^2(L)] - 3\alpha[I_1^2(R) - I_1^2(L)] + \frac{5}{2}\beta[I_2^2(R) - I_2^2(L)] \tag{13}$$

where

$$S_0 = \int_0^1 I(R, \mu) F_2(\mu)\,\mathrm{d}\mu - \int_0^1 I(L, -\mu) F_1(\mu)\,\mathrm{d}\mu \tag{14}$$

and we have introduced

$$\alpha = \frac{1}{3}\omega b_1, \quad \beta = \frac{2}{5}\omega b_2. \tag{15a,b}$$

We observe that Eq. (13) contains only the known intensities at the two surfaces and the three unknowns ω, α, and β.

In a similar manner we can multiply Eq. (4) by $\mu^2 I(\tau, \mu)$ and integrate over μ from -1 to 1 to obtain

$$T_2(\tau) + 2\int_0^1 I(\tau, \mu) I(\tau, -\mu)\mu^2\,\mathrm{d}\mu = \frac{\omega}{2}\left(I_0(\tau)K(\tau) - b_1 I_1(\tau)L(\tau) + \frac{1}{2}b_2 I_2(\tau)[3M(\tau) - K(\tau)]\right), \tag{16}$$

where

$$K(\tau) = \int_{-1}^{1} I(\tau, \mu)\mu^2 \, d\mu, \tag{17a}$$

$$L(\tau) = \int_{-1}^{1} I(\tau, \mu)\mu^3 \, d\mu, \tag{17b}$$

$$M(\tau) = \int_{-1}^{1} I(\tau, \mu)\mu^4 \, d\mu, \tag{17c}$$

and

$$T_2(\tau) = \int_{-1}^{1} I(\tau, \mu)F(\tau, -\mu)\mu^2 \, d\mu. \tag{18}$$

Differentiating Eqs. (16) and (18) and eliminating between the two resulting equations, we deduce that

$$\frac{d}{d\tau}T_2(\tau) = \frac{\omega}{2}\Big[\frac{d}{d\tau}[I_0(\tau)K(\tau)] - 2I_0(\tau)\frac{d}{d\tau}K(\tau) - b_1\Big\{\frac{d}{d\tau}[I_1(\tau)L(\tau)] - 2I_1(\tau)\frac{d}{d\tau}L(\tau)\Big\} + \frac{1}{2}b_2\Big(\frac{d}{d\tau}\Big\{I_2(\tau)[3M(\tau) - K(\tau)]\Big\} - 2I_2(\tau)\frac{d}{d\tau}[3M(\tau) - K(\tau)]\Big)\Big]. \tag{19}$$

Equation (19) can be integrated from L to R and substituted into the difference between Eq. (16) evaluated at $\tau = R$ and Eq. (16) evaluated at $\tau = L$ to yield

$$\frac{2}{\omega}S_2 = \int_L^R I_0(\tau)\frac{d}{d\tau}K(\tau)\, d\tau - b_1\int_L^R I_1(\tau)\frac{d}{d\tau}L(\tau)\, d\tau + \frac{1}{2}b_2\int_L^R I_2(\tau)\frac{d}{d\tau}[3M(\tau) - K(\tau)]\, d\tau, \tag{20}$$

where

$$S_2 = \int_0^1 I(R, \mu)F_2(\mu)\mu^2 \, d\mu - \int_0^1 I(L, -\mu)F_1(\mu)\mu^2 \, d\mu. \tag{21}$$

Now if we multiply Eq. (1) by μ^k, $k = 0, 1, 2$, and 3, and integrate over μ, we find

$$\frac{d}{d\tau}I_1(\tau) + (1-\omega)I_0(\tau) = 0, \tag{22a}$$

$$\frac{d}{d\tau}K(\tau) + (1-\alpha)I_1(\tau) = 0, \tag{22b}$$

$$\frac{d}{d\tau}L(\tau) + K(\tau) = \frac{1}{3}[\omega I_0(\tau) + \beta I_2(\tau)], \tag{22c}$$

and

$$\frac{d}{d\tau}M(\tau) + L(\tau) = \frac{3}{5}\alpha I_1(\tau). \tag{22d}$$

Equations (22) can be used in Eq. (20) to obtain

$$4S_2 = \Big(\frac{\omega}{1-\omega}\Big)[I_1^2(R) - I_1^2(L)] - 3\Big(\frac{\alpha}{1-\alpha}\Big)[K^2(R) - K^2(L)] + 5\Big(\frac{\beta}{2-\beta}\Big)[M_2^2(R) - M_2^2(L)], \tag{23}$$

where

$$M_2(\tau) = \frac{3}{2}L(\tau) - \frac{1}{2}I_1(\tau). \tag{24}$$

It is clear that Eq. (23) contains only the known intensities at the surfaces and the three unknowns ω, α, and β. It is also clear that for the case $\beta = 0$ Eqs. (13) and (23) can be solved to yield the explicit results given previously[1] for $F_2(\mu) = 0$.

A third equation involving only the surface intensities and ω, α, and β can be established by multiplying Eq. (4) by $\mu^4 I(\tau, \mu)$, integrating over μ, and subsequently proceeding in a manner analogous to that used to deduce Eq. (23). This calculation involves a great deal of tedious manipulation and thus we quote here only the final result:

$$4S_4 = W(R) - W(L), \tag{25}$$

where

$$W(\tau) = \frac{1}{3}\left(\frac{1}{1-\alpha}\right)\left(\frac{\omega}{1-\omega}\right)^2 I_1^2(\tau) - \left(\frac{1}{1-\alpha}\right)^2\left(\frac{\omega}{1-\omega} + \frac{9}{5}\alpha^2\right)K^2(\tau) + \left(\frac{1}{1-\alpha}\right)$$
$$\times\left[\left(\frac{2\omega}{1-\omega}\right)I_1(\tau)L(\tau) - 6\alpha K(\tau)M(\tau) + 3\alpha L^2(\tau)\right] - 5\left(\frac{\beta}{2-\beta}\right)\left[2M_2(\tau)U(\tau)\right.$$
$$\left. + \frac{2}{3}\left(\gamma - \frac{9}{7}\right)\left(\frac{\beta}{2-\beta}\right)M_2^2(\tau) + \left[\frac{3}{2}M(\tau) - \gamma K(\tau)\right]^2\right]. \tag{26}$$

Here

$$S_4 = \int_0^1 I(R, \mu)F_2(\mu)\mu^4\,\mathrm{d}\mu - \int_0^1 I(L, -\mu)F_1(\mu)\mu^4\,\mathrm{d}\mu \tag{27}$$

and

$$U(\tau) = -\frac{2}{15}\left(\frac{\omega}{1-\omega}\right)\left(\frac{1}{1-\alpha}\right)I_1(\tau) - \frac{3}{2}\int_{-1}^1 I(\tau, \mu)\mu^5\,\mathrm{d}\mu + \gamma L(\tau), \tag{28}$$

with

$$\gamma = \frac{5-9\alpha}{10(1-\alpha)}. \tag{29}$$

It is apparent that Eqs. (13) and (23) can be solved to yield ω and α in terms of β. These results thus can be substituted into Eq. (25) to give a single equation to be solved for the third unknown β. Since this final equation for β will be complicated, a direct iterative solution of Eqs. (13), (23), and (25) may prove more expedient.

We note that though the results deduced here are, in principle, exact, it is not clear how the limited accuracy of the experimently measured intensities at the surfaces will affect the computed values of ω, b_1, and b_2. In addition, we may have some choice in the equations we use; for example if $b_1 = b_2 = 0$ then Eqs. (13), (23), and (25) can each be solved to yield, respectively,

$$\omega = 4S_0[I_0^2(R) - I_0^2(L)]^{-1}, \qquad b_1 = b_2 = 0, \tag{30}$$

$$\omega = 4S_2[I_1^2(R) - I_1^2(L) + 4S_2]^{-1}, \qquad b_1 = b_2 = 0, \tag{31}$$

and

$$\omega = \frac{-B - [B^2 + 16A\,S_4]^{1/2}}{2A}, \qquad b_1 = b_2 = 0. \tag{32}$$

Here

$$A = 4S_4 + \frac{1}{3}[I_1^2(R) - I_1^2(L)] - B \tag{33}$$

and

$$B = 8S_4 - [K^2(R) - K^2(L)] + 2[I_1(R)L(R) - I_1(L)L(L)]. \tag{34}$$

Finally, we observe that should $F_1(\mu) = F_2(\mu)$ an alternative procedure would be required since our results become indeterminate for this symmetric case.

3. ADDITIONAL REMARKS

First of all we wish to point out that in considering the azimuthally symmetric form of the equation of transfer, i.e. Eq. (1), we are not placing any such restriction on the incident radiation. It is well known[2] that a radiative transfer problem without azimuthal symmetry has as one component in the solution the considered symmetric problem. As discussed here the desired unknowns can be found from this component of the complete problem. We note that McCormick[3] has found it convenient to consider all of the components of the azimuthally dependent problem.

Secondly we wish to call attention to the fact that one of the attractive features of the developed solution for the unknowns ω, b_1, and b_2 is that the optical thickness of the considered medium is not required.

Let us now discuss the manner in which we can extend the foregoing analysis to include the effect of reflecting and non-transparent boundaries. If we wish to allow the boundaries (interfaces or walls) to introduce additional effects then we must distinguish between $\tau = L^{\pm}$ and $\tau = R^{\pm}$, i.e. just inside and just outside the slab. We thus write Eqs. (2) as

$$I(L^+, \mu) = 0_L(\mu)f_1(\mu) + \int_0^1 I(L^+, -\mu')\prod_L(\mu' \to \mu)\mu' \, d\mu', \quad \mu > 0, \tag{35a}$$

and

$$I(R^-, -\mu) = 0_R(\mu)f_2(\mu) + \int_0^1 I(R^-, \mu')\prod_R(\mu' \to \mu)\mu' \, d\mu', \quad \mu > 0, \tag{35b}$$

where the operator $0(\mu)$ is used to represent the effect of the boundary on the externally incident radiation $f(\mu)$, and the function $\prod(\mu' \to \mu)$ is used to characterize the internal reflection at the boundary. In addition, to express the effect of the boundaries on the exit radiation we write

$$I(L^-, -\mu) = M_L(\mu)I(L^+, -\mu), \quad \mu > 0, \tag{36a}$$

and

$$I(R^+, \mu) = M_R(\mu)I(R^-, \mu), \quad \mu > 0. \tag{36b}$$

It is clear that a detector placed outside the slab measures both the externally reflected radiation and the radiation escaping the slab, i.e.

$$I_m(L^-, -\mu) = M_L(\mu)I(L^+, -\mu) + N_L(\mu)f_1(\mu), \quad \mu > 0, \tag{37a}$$

and

$$I_m(R^+, \mu) = M_R(\mu)I(R^-, \mu) + N_R(\mu)f_2(\mu), \quad \mu > 0, \tag{37b}$$

where $N_L(\mu)$ and $N_R(\mu)$ are used to represent the effect of the external reflection of the incident radiation. It is apparent that if the boundary properties are known, to the extent that the operators $0(\mu)$ and $\prod(\mu' \to \mu)$ are known, and if we measure experimentally $I(L^+, -\mu)$ and

Table 1. The computed value of ω.

Result	F_0	F_1	F_2	F_3	F_4	F_5	Exact
Eq. (30)	0.7444	0.8245	0.8094	0.8021	0.8009	0.8005	0.8
Eq. (31)	0.7952	0.8053	0.8011	0.8000	0.8000	0.8000	0.8
Eq. (32)	0.7990	0.8014	0.8002	0.8000	0.8000	0.8000	0.8

$I(R^-, \mu)$, $\mu > 0$, then

$$F_1(\mu) = 0_L(\mu) f_1(\mu) + \int_0^1 I(L^+, -\mu') \prod_L (\mu' \to \mu) \mu' \, d\mu', \quad \mu > 0, \tag{38a}$$

and

$$F_2(\mu) = 0_R(\mu) f_2(\mu) + \int_0^1 I(R^-, \mu') \prod_R (\mu' \to \mu) \mu' \, d\mu', \quad \mu > 0, \tag{38b}$$

can be considered known, and thus the results of Section 2 are applicable here. On the other hand, if we can measure only $I_m(L^-, -\mu)$ and $I_m(R^+, \mu)$ then we need to specify the operators $M(\mu)$, $M^{-1}(\mu)$, and $N(\mu)$ and subsequently we can use Eqs. (37) and (38) to render the results of Section 2 appropriate here. Clearly if the boundary properties are unknown then the corresponding inverse problem is considerably more difficult and thus remains unsolved.

Finally we wish to report that a study of the effect of the accuracy of the surface intensities on the values of ω, b_1, and b_2 that can be found from Eqs. (13), (23), and (25) has been undertaken by Dunn and Maiorino and will be reported at a later date. In the meantime we quote in Table 1 some results communicated by Maiorino[4] of calculations based on the isotropically scattering model. In Table 1 various orders of the F_N approximation[5,6] were used to represent the effect of inaccuracy of the surface intensities for the considered case, $b_1 = b_2 = 0$, $R - L = 1$, $F_1(\mu) = 0$, and $\omega = 0.8$.

Acknowledgement—The author is grateful to J. R. Maiorino for communicating the numerical results shown in Table 1. This work was supported in part by the National Science Foundation through grant ENG. 7709405.

REFERENCES

1. C. E. Siewert, *ZAMP* **30**, 522 (1979).
2. S. Chandrasekhar, *Radiative Transfer.* Oxford University Press, London (1950).
3. N. J. McCormick, *J. Math. Phys.* **20**, 1504 (1979).
4. J. R. Maiorino, private communication (1979).
5. C. E. Siewert and P. Benoist, *Nucl. Sci. Engng* **69**, 156 (1979).
6. P. Grandjean and C. E. Siewert, *Nucl. Sci. Engng* **69**, 161 (1979).