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The H Matrix for Time-Dependent Problems in Rarefied Gas Dynamics

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Introduction

In several recent papers [1–3] the method of elementary solutions [4] was used to study basic problems based on the time-dependent *BGK* model in the kinetic theory of gases. We note [1–3] that solutions to typical half-space problems have been expressed in terms of well-known functions and the so-called **H** matrix. Here we wish to show that a system of singular integral equations and a set of integral constraints, which we consider to define $\mathbf{H}(\mu)$, $\mu \in [0, \infty)$, have a unique solution. In reference [1], to which we hereafter refer as SB, we introduced the matrix of sectionally analytic functions

$$\mathbf{\Omega}(z) = \mathbf{I} + z\int_{-\infty}^{\infty} \mathbf{\Psi}_*(x)\,\frac{dx}{x - z}, \tag{1}$$

where

$$\mathbf{\Psi}_*(x) = \theta\, e^{-x^2}\mathbf{\Pi}(x)\mathbf{Q}^T(x)\mathbf{Q}(x)\mathbf{\Pi}(-x), \tag{2}$$

$$\mathbf{Q}(x) = \begin{bmatrix} (\frac{2}{3})^{1/2}(x^2 - \frac{1}{2}) & 1 \\ (\frac{2}{3})^{1/2} & 0 \end{bmatrix}, \tag{3}$$

$$\mathbf{\Pi}(x) = \mathbf{I} - \left(\frac{x}{z_1}\right)\mathbf{D}, \tag{4}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5}$$

$\theta = (\pi)^{-1/2}(s + 1)^{-1}$, $\gamma = 2s(s + 1)^{-1}$, $(\gamma)^{1/2}z_1 = i$ and s is the Laplace transform variable. We note that a study of the zeros of $\Omega(z) = \det \mathbf{\Omega}(z)$ has been reported [5, 6]. There are in general κ pairs of $\pm$ zeros, $\pm\nu_\alpha$, of $\Omega(z)$ in the complex plane cut along the entire real axis. For $\operatorname{Re} s \geq 0$, i.e. the half plane relevant to the inversion integral the index κ can be 0, 1 or 2 depending [5, 6] on the exact value of s.

We consider $\mathbf{H}(\mu)$ to be the solution to

$$\mathbf{H}^T(\mu)\boldsymbol{\omega}(\mu) = \mathbf{I} + \mu P\int_0^\infty \mathbf{H}^T(x)\boldsymbol{\Psi}_*(x)\frac{dx}{x-\mu}, \qquad \mu \in [0, \infty), \tag{6a}$$

and

$$\left[\mathbf{I} + \nu_\alpha\int_0^\infty \mathbf{H}^T(x)\boldsymbol{\Psi}_*(x)\frac{dx}{x-\nu_\alpha}\right]\mathbf{W}(\nu_\alpha) = \mathbf{0}, \qquad \alpha = 1, 2, \ldots, \kappa. \tag{6b}$$

Here

$$\boldsymbol{\omega}(\mu) = \mathbf{I} + \mu P\int_{-\infty}^\infty \boldsymbol{\Psi}_*(x)\frac{dx}{x-\mu} \tag{7}$$

and $\mathbf{W}(\nu_\alpha)$ is a null vector of $\boldsymbol{\Omega}(\nu_\alpha)$:

$$\boldsymbol{\Omega}(\nu_\alpha)\mathbf{W}(\nu_\alpha) = \mathbf{0}. \tag{8}$$

Equation (6a) represents a system of singular integral equations and Eqn. (6b) is a system of κ integral constraints. We intend here to show that Eqns. (6) have a unique solution.

Analysis

If, following the ideas of Muskhelishvili [7], we introduce the matrix of sectionally analytic functions

$$\mathbf{N}(z) = \frac{1}{2\pi i}\int_0^\infty \mathbf{H}^T(x)\boldsymbol{\Psi}_*(x)\frac{dx}{x-z}, \tag{9}$$

then we can use the Plemelj formulae [7] to rewrite Eqn. (6a) as

$$[\mathbf{N}^+(t)]^T = \mathbf{G}(t)[\mathbf{N}^-(t)]^T + \boldsymbol{\Psi}_*^T(t)[\boldsymbol{\Omega}^-(t)]^{-T}, \quad t \in [0, \infty), \tag{10}$$

where

$$\mathbf{G}(t) = [\boldsymbol{\Omega}^+(t)]^T[\boldsymbol{\Omega}^-(t)]^{-T} \tag{11}$$

and

$$\boldsymbol{\Omega}^\pm(t) = \boldsymbol{\omega}(t) \pm \pi i t\boldsymbol{\Psi}_*(t). \tag{12}$$

We note that $-T$ is used for the transpose inverse operation, and that the $(\pm)$ superscripts are used to denote limiting values of sectionally analytic functions as branch cuts are approached from above $(+)$ and below $(-)$. As discussed in *SB*, there exists a canonical solution $\mathbf{X}(z)$, of ordered normal form at infinity, and with non-negative partial indices, of the homogeneous matrix Riemann problem defined by

$$\mathbf{X}^+(\mu) = \mathbf{G}(\mu)\mathbf{X}^-(\mu), \quad \mu \in [0, \infty), \tag{13}$$

and thus on using Eqn. (13) in Eqn. (12), we find

$$[\mathbf{X}^+(t)]^{-1}[\mathbf{N}^+(t)]^T - [\mathbf{X}^-(t)]^{-1}[\mathbf{N}^-(t)]^T = [\mathbf{X}^+(t)]^{-1}\mathbf{\Psi}_*^T(t)[\mathbf{\Omega}^-(t)]^{-T}. \tag{14}$$

Equation (14) can be solved to yield

$$\mathbf{N}^T(z) = \frac{1}{2\pi i}\mathbf{X}(z)\left[\int_0^\infty \mathbf{B}(t)\frac{dt}{t-z} + \mathbf{F}^T(z)\right], \tag{15}$$

where $\mathbf{F}(t)$ is a 2×2 matrix of (at this point) arbitrary polynomials and

$$\mathbf{B}(t) = [\mathbf{X}^+(t)]^{-1}\mathbf{\Psi}_*^T(t)[\mathbf{\Omega}^-(t)]^{-T}. \tag{16}$$

We can now use the analytic properties of $\mathbf{X}(z)$ to deduce that

$$\mathbf{X}^{-1}(z) = \mathbf{X}_{\text{asy}}^{-1}(z) - \mathbf{X}_{\text{asy}}^{-1}(0) + \mathbf{X}^{-1}(0) - z\int_0^\infty \mathbf{B}(t)\frac{dt}{t-z}, \tag{17}$$

where $\mathbf{X}_{\text{asy}}^{-1}(z)$ represents the principal part of $\mathbf{X}^{-1}(z)$ as $|z|$ tends to infinity. Thus Eqn. (15) can be written as

$$\mathbf{I} + 2\pi iz\mathbf{N}(z) = [\mathbf{X}_{\text{asy}}^{-T}(z) - \mathbf{X}_{\text{asy}}^{-T}(0) + \mathbf{X}^{-T}(0) + z\mathbf{F}(z)]\mathbf{X}^T(z). \tag{18}$$

Since

$$\lim_{|z|\to\infty} \mathbf{X}(z)\begin{bmatrix} z^{\kappa_1} & 0 \\ 0 & z^{\kappa_2} \end{bmatrix} = \mathbf{K}, \quad \det \mathbf{K} \neq 0, \tag{19}$$

where $\kappa_1 \leq \kappa_2$ and κ_2 are the partial indices and $\kappa_1 + \kappa_2 = \kappa$, we note from Eqn. (18) that we must impose the condition

$$\lim_{|z|\to\infty} z\mathbf{F}(z)\mathbf{X}^T(z) = \mathbf{A} < \infty; \tag{20}$$

if Eqn. (20) were not satisfied then Eqn. (18) would not yield an $\mathbf{N}(z)$ with the correct form at infinity. Thus

$$\mathbf{F}(z) \to \frac{1}{z}\mathbf{F}\begin{bmatrix} z^{\kappa_1} & 0 \\ 0 & z^{\kappa_2} \end{bmatrix}, \quad |z| \to \infty, \tag{21}$$

where $\mathbf{F}$ is a constant. It is now clear that Eqn. (6a) has a solution $\mathbf{H}(\mu)$, and thus we wish to show that Eqn. (6b) fixes the polynomial $\mathbf{F}(z)$ so that $\mathbf{N}(z)$ and thus $\mathbf{H}(\mu)$ will be unique.

In SB proof was given that the partial indices κ_1 and κ_2 are non-negative, and it was shown that the matrix $\mathbf{\Omega}(z)$ could be factored in the manner

$$\mathbf{\Omega}^T(z) = \mathbf{X}(z)\mathbf{P}(z)\mathbf{X}^T(-z), \tag{22}$$

where $\mathbf{P}(z)$ is a 2×2 matrix of polynomials with $P_{11}(0) \neq 0$. For the case $\kappa = 0$ there is no constraint, but since $\kappa_1 = \kappa_2 = 0$ it is clear that Eqn. (21) yields $\mathbf{F}(z) \equiv 0$. It follows that, for $\kappa = 0$, $\mathbf{N}(z)$ and consequently $\mathbf{H}(\mu)$ are uniquely determined. We consider now $\kappa \geq 1$ and write Eqn. (6b) as

$$[\mathbf{I} + 2\pi i\nu_\alpha \mathbf{N}(\nu_\alpha)]\mathbf{W}(\nu_\alpha) = 0, \quad \alpha = 1, 2, \ldots, \kappa. \tag{23}$$

For $\kappa = 1$, we note that Eqn. (21) yields

$$\mathbf{F}(z) = \begin{bmatrix} 0 & F_{12} \\ 0 & F_{22} \end{bmatrix}, \tag{24}$$

and thus on entering Eqn. (18) into Eqn. (23), we find

$$\begin{bmatrix} 0 & F_{12} \\ 0 & F_{22} \end{bmatrix} \mathbf{J}(\nu_1) = \mathbf{V}(\nu_1), \tag{25}$$

where

$$\mathbf{V}(z) = \frac{1}{z}\,[\mathbf{X}_{\text{asy}}^{-T}(0) - \mathbf{X}_{\text{asy}}^{-T}(z) - \mathbf{X}^{-T}(0)]\mathbf{X}^{T}(z)\mathbf{W}(z) \tag{26}$$

and

$$\mathbf{J}(z) = \mathbf{X}^{T}(z)\mathbf{W}(z). \tag{27}$$

Equation (25) clearly can be solved uniquely for F_{12} and F_{22} unless

$$\mathbf{J}(\nu_1) \propto \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{28}$$

which, after we use Eqns. (8) and (22), would imply

$$\mathbf{P}(-\nu_1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{0}. \tag{29}$$

We know Eqn. (29) to be false since, here, $P_{11}(z)$ is a constant and $P_{11}(0) \neq 0$. Thus $\mathbf{N}(z)$ and hence $\mathbf{H}(\mu)$ are uniquely determined for the case $\kappa = 1$. For $\kappa = 2$ we have two conditions

$$\mathbf{F}(\nu_\alpha)\mathbf{J}(\nu_\alpha) = \mathbf{V}(\nu_\alpha), \quad \alpha = 1 \text{ and } 2. \tag{30}$$

Considering firstly the possibility that $\kappa_1 = \kappa_2 = 1$, we note that Eqn. (21) would yield $\mathbf{F}(z) = \mathbf{F}$, and thus since $\mathbf{J}(\nu_1)$ and $\mathbf{J}(\nu_2)$ are linearly independent (recall that the eigenvalues ν_α are all different), Eqns. (30) could be solved to yield $\mathbf{F}$. If, on the other hand, $\kappa_1 = 0$ and $\kappa_2 = 2$, then $\mathbf{F}(z)$ would have zeros in the first column and would be linear in the second column. Thus, since again we know $\mathbf{J}(\nu_1)$ and $\mathbf{J}(\nu_2)$ are not of the form of Eqn. (28) and $\nu_1 \neq \nu_2$, we could solve Eqns. (30) uniquely to find $\mathbf{F}(z)$.

Therefore for $\kappa = 0$, 1 or 2, the polynomial $\mathbf{F}(z)$ is uniquely established, and thus we can use Eqns. (9) and (18) to find $\mathbf{H}(\mu)$:

$$\mathbf{H}(\mu) = \mathbf{X}^{-T}(-\mu)\mathbf{P}^{-T}(-\mu)\mathbf{P}_*^T(\mu), \quad \mu \in [0, \infty), \tag{31}$$

where we have used Eqn. (22), and

$$\mathbf{P}_*(z) = \mathbf{X}_{\text{asy}}^{-T}(z) - \mathbf{X}_{\text{asy}}^{-T}(0) + \mathbf{X}^{-T}(0) + z\mathbf{F}(z). \tag{32}$$

In order to develop some additional useful relationships, we first choose to define $\mathbf{H}(z)$ by extending the solution given by Eqn. (31). Thus

$$\mathbf{H}(z) = \mathbf{X}^{-T}(-z)\mathbf{P}^{-T}(-z)\mathbf{P}_*^T(z), \quad z \notin (-\infty, 0). \tag{33}$$

Since $\det \mathbf{P}(z)$ has zeros at $\pm\nu_\alpha$ and $\det \mathbf{P}_*(z)$ has zeros at ν_α, it is clear that $\det \mathbf{H}(z)$ has no zeros, and thus we can write

$$\mathbf{H}^{-1}(z) = \mathbf{P}_*^{-T}(z)\mathbf{P}^T(-z)\mathbf{X}^T(-z). \tag{34}$$

We note that $\mathbf{H}^{-1}(z)$ is analytic in the complex plane cut along the negative real axis. Equation (22) can be used in Eqn. (34) to obtain

$$\mathbf{H}^{-1}(z) = \mathbf{P}_*^{-T}(z)\mathbf{X}^{-1}(z)\mathbf{\Omega}^T(z). \tag{35}$$

We can now let $|z| \to \infty$ in Eqn. (35) to find

$$\mathbf{H}^{-1}(\infty) = \left[\mathbf{I} - \int_0^\infty \mathbf{H}^T(x)\mathbf{\Psi}_*(x)\,dx\right]^{-T} \mathbf{\Omega}^T(\infty). \tag{36}$$

Also, Eqns. (33) and (35) yield

$$\mathbf{\Omega}^T(z) = \mathbf{H}^{-T}(-z)\mathbf{R}(z)\mathbf{H}^{-1}(z), \tag{37}$$

where

$$\mathbf{R}(z) = \mathbf{P}_*(-z)\mathbf{P}^{-1}(z)\mathbf{P}_*^T(z). \tag{38}$$

It is clear from Eqn. (35) that $\mathbf{H}(0) = \mathbf{I}$, and thus since $\mathbf{R}(z)$ is bounded at infinity, then clearly $\mathbf{R}(z) = \mathbf{I}$, and Eqn. (37) yields the factorization

$$\mathbf{\Omega}^T(z) = \mathbf{H}^{-T}(-z)\mathbf{H}^{-1}(z). \tag{39}$$

Cauchy's integral theorem can now be used to write

$$\mathbf{H}^{-1}(z) = \mathbf{H}^{-1}(\infty) + \frac{1}{2\pi i}\int_{-\infty}^0 \{[\mathbf{H}^+(t)]^{-1} - [\mathbf{H}^-(t)]^{-1}\}\frac{dt}{t - z} \tag{40}$$

or (after we carry out some elementary manipulations)

$$\mathbf{H}^{-1}(z) = \mathbf{I} - z\int_0^\infty \mathbf{H}^T(x)\mathbf{\Psi}_*(x)\frac{dx}{x + z} \tag{41}$$

which for $z \in [0, \infty)$ yields the non-linear integral equation

$$\mathbf{H}^{-1}(\mu) = \mathbf{I} - \mu \int_0^\infty \mathbf{H}^T(x)\mathbf{\Psi}_*(x) \frac{dx}{x + \mu} \tag{42}$$

that is useful for a numerical calculation of $\mathbf{H}(\mu)$.

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References

[1] C. E. Siewert and E. E. Burniston, J. Math. Phys. *18*, 376 (1977).
[2] C. E. Siewert and J. T. Kriese, Z. angew. Math. Phys. *29*, 199 (1978).
[3] C. E. Siewert and J. R. Thomas, Jr. (submitted for publication).
[4] K. M. Case, Ann. Phys. *9*, 1 (1960).
[5] C. E. Siewert, E. E. Burniston and J. R. Thomas, Jr. Phys. Fluids *16*, 1532 (1973).
[6] R. J. Mason, Phys. Fluids *13*, 1467 (1970).
[7] N. I. Muskhelishvili, *Singular Integral Equations*, Noordhoff, Groningen, The Netherlands (1953).

Abstract

A system of singular integral equations and a set of integral constraints are shown to be uniquely solvable to yield the H matrix useful for half-space applications in time-dependent studies of the theory of rarefied gas dynamics. In addition some useful relationships concerning the H matrix are established.

Zusammenfassung

Es wird gezeigt, dass ein System von singulären Integralgleichungen unter gegebenen Bedingungen eindeutig gelöst werden kann, und eine H-Matrix liefert die für Halbraum-Anwendungen in zeitabhängigen Untersuchungen in der Theorie der verdünnten Gase brauchbar ist. Daneben werden noch einige nützliche Relationen, die die H-Matrix betreffen, abgeleitet.

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