TRANSPORT THEORY AND STATISTICAL PHYSICS, 8(4), 219-240 (1979)

SOUND-WAVE PROPAGATION IN A RAREFIED GAS

J. R. Thomas, Jr.

Mechanical Engineering Department
V.P.I. and State University
Blacksburg, Virginia 24061

and

C. E. Siewert

Department of Nuclear Engineering
North Carolina State University
Raleigh, North Carolina 27650

ABSTRACT

The elementary solutions of a linearized and modeled Boltzmann equation are used to study sound propagation at high frequencies. The problem is modeled as a half space bounded by an oscillating plate which reflects molecules diffusely. Accurate numerical results are obtained.

I. INTRODUCTION

Under normal conditions sound propagation in gases is known to be adequately described by the Navier-Stokes equations. At frequencies comparable to the collision frequency of the molecules, or in the region very near a solid boundary, however, this is not the case and the study of sound propagation requires kinetic theory and the Boltzmann equation.

The first serious attempt at an analytical solution of this problem was due to Wang Chang[1], who studied the dispersion law arising from the higher-order hydrodynamic equations. The results were only a slight extension of the Navier-Stokes theory. In 1952 Wang Chang and Uhlenbeck[2] published results based on a linearization of the Boltzmann equation about an equilibrium distribution and expansion of the distribution function in moments. The roots of the resulting dispersion relations were expanded in powers of the ratio of wave frequency to collision frequency; thus, the results are not really valid for high sound frequencies. This method was extended by Perkeris et al.[3,4] to trucations involving up to 483 moments; however, they followed the example of Wang Chang and Uhlenbeck in expanding in a power series limited to lower frequencies.

Experiments have been performed by Greenspan[5,6] and Meyer and Sessler[7] whose results are in substantial mutual agreement. Their data give results for the phase speed and attenuation as a function of the ratio $r = f_c/f$ where f_c is the collision frequency of the molecules and f the sound frequency. For values of r greater than about 4, the agreement between the theory described here and the experimental values is very good, but it becomes increasingly worse as r decreases as would be expected from the power series nature of the theory. A surprising result is that for $r < 0.5$, the Navier-Stokes theory seems to fit the data better than the Burnett and super-Burnett theories, although all three are considerably in error[7] for $r < 0.2$.

A fresh attack on the theoretical problem was mounted by Sirovich and Thurber[8,9] based on the kinetic model equations of Gross and Jackson[10] and Sirovich[11]. These models predict a cutoff frequency beyond which plane wave solutions do not exist[12]; thus, their study was based on analytic continuation of the dispersion relation beyond this critical frequency. Their results agreed to within about 25% of the data of Meyer and Sessler[7].

All of the theoretical work described here proceeds from the dispersion function for the equation under consideration, presuming a plane wave propagating through an infinite medium. However, the experiments always involve a source and a receiver usually spaced very closely, often less than a mean-free-path apart[6,7]. Thus, a better model for the experimental problem would be as a gas between two parallel surfaces, one of which is vibrating, i.e., a boundary value problem. A more tractable problem results if one of the surfaces, e.g., the receiver, is ignored; this is the approach taken by Weitzner[13], Buckner and Ferziger[14], and Richardson and Sirovich[15]. Of these, only Buckner and Ferziger present numerical results in reasonable agreement with the experimental data. These authors used the 3 and 5-moment models of Gross and Jackson[10] and replaced the half-space problem with that of an infinite medium by choosing a source distribution with an adjustable parameter; this parameter was to be chosen so as to minimize the solution in the half space $x < 0$ (where the true solution should vanish). In addition, they did not impose the condition of particle conservation at the wall, and due to an oversight, calculated results with a physically incorrect source distribution. Nevertheless, their results are in good agreement with the data for very low and very high frequencies; in the transition regime ($r \approx 1$) the attenuation coefficient differs from the data by about 30%.

Ostrowsky and Kleitman[16] found that the decay of the sound disturbance at large distances from an oscillating boundary is like $\exp(-x^{2/3})$. Sirovich and Thurber[17] explain this as being due to the constant collision frequency inherent in the BGK model. This behavior may be due to the discrete roots of the dispersion relation rather than the continuum contribution which has been shown to dominate for all frequencies beyond a certain cutoff value, at least according to the linear BGK model[18,19,20].

It would thus appear that a solution of this problem, correctly posed, and with correct boundary conditions, would fill an important gap. The opportunity for achieving such a solution results from recent work of Siewert and Burniston[21] and Siewert[25], who derived the elementary solutions of the linear, time-dependent BGK equation[22]. In the present work we solve the correctly posed boundary value problem in a half space, assuming a diffusely reflecting vibrating wall. Results are found to be in substantial agreement with those of Loyalka and Cheng[20], who solved the same problem using a sophisticated numerical technique. Both our results and those of Loyalka and Cheng[20] agree well with experimental data[6,7].

Section II of the paper gives the formulation of the problem, which is solved in Sec. III. Numerical results and conclusions are given in Secs. IV and V, respectively.

II. BASIC FORMULATION

In a recent paper[21], the linearized and modeled Boltzmann equation[22,23]

$$\left(\frac{\partial}{\partial t} + c_x \frac{\partial}{\partial x} + 1\right) h(x,\underset{\sim}{c},t) = (\pi)^{-3/2} \int h(x,\underset{\sim}{c}',t)[1 + 2\, \underset{\sim}{c}\cdot\underset{\sim}{c}' + \frac{2}{3}(c'^2 - \frac{3}{2})(c^2 - \frac{3}{2})] \exp(-c'^2) d^3c' \quad (1)$$

was investigated and various elementary solutions were reported. We now wish to use the established elementary solutions to solve the problem of sound-wave propagation in a semi-infinite medium. Here, $h(x,\underset{\sim}{c},t)$ represents the perturbation of the distribution function from the Maxwellian distribution, $\underset{\sim}{c}$, with components c_x, c_y, and c_z and magnitude c, is the velocity, t is the time and x is the space variable. Loyalka and Cheng[20] use this same equation as the basis for their study of this problem.

For the case of an oscillating plate located at x = 0, we seek a solution of Eq. (1) subject to

$$\lim_{x\to\infty} h(x,\underset{\sim}{c},t) < \infty \quad (2a)$$

and

$$h(0,\underset{\sim}{c},t) = (c_x + \frac{\sqrt{\pi}}{2}) \exp(i\omega t) + \frac{2}{\pi}\int_{-\infty}^{\infty} dc_y' \int_{-\infty}^{\infty} dc_z' \int_0^{\infty} dc_x'\, h(0,-c_x',c_y',c_z',t) \exp(-c'^2)\, c_x',$$

$$c_x > 0, \quad (2b)$$

where u_o, the amplitude of the oscillation, has been taken equal to 1/2 and ω is the normalized[20] frequency of oscillation.

Since we are concerned with temperature-density effects, we can take "moments" of Eq. (1) to obtain equations dependent only on x and c_x. Thus, we let

$$\psi_1(x,c_x,t) = (\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-(c_y^2 + c_z^2)]\, h(x,\underset{\sim}{c},t)\, dc_y\, dc_z, \quad (3a)$$

and

$$\psi_2(x,c_x,t) = (\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-(c_y^2 + c_z^2)] h(x,\underset{\sim}{c},t)(c_y^2 + c_z^2 - 1)\, dc_y\, dc_z, \quad (3b)$$

so that the density perturbation

$$\Delta N(x,t) = \pi^{-3/2} \int h(x,\underset{\sim}{c},t) \exp(-c^2)\, d^3c \quad (4)$$

and the temperature perturbation

$$\Delta T(x,t) = \tfrac{2}{3}\, \pi^{-3/2} \int h(x,\underset{\sim}{c},t)(c^2 - 3/2) \exp(-c^2)\, d^3c \quad (5)$$

can be expressed as

$$\Delta N(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi_1(x,\mu,t) \exp(-\mu^2)\, d\mu \quad (6)$$

and

$$\Delta T(x,t) = \frac{2}{3\pi} \int_{-\infty}^{\infty} [(\mu^2 - \tfrac{1}{2})\, \psi_1(x,\mu,t) + \psi_2(x,\mu,t)] \exp(-\mu^2)\, d\mu, \quad (7)$$

where we have used μ for c_x. If we now multiply Eq. (1) by $\exp(-c_y^2 - c_z^2)$ and integrate from $-\infty$ to ∞ over c_y and c_z, and then multiply Eq. (1) by $(c_y^2 + c_z^2 - 1) \exp(-c_y^2 - c_z^2)$ and integrate similarly, we find that the resulting two equations can be expressed as

$$(\frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x} + 1) \underset{\sim}{\Psi}(x,\mu,t)$$

$$= (\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} [\underset{\sim}{Q}(\mu) \underset{\sim}{Q}^T(\mu') + 2\mu\mu' \underset{\sim}{P}] \underset{\sim}{\Psi}(x,\mu',t) \exp(-\mu'^2) d\mu', \tag{8}$$

where

$$\underset{\sim}{Q}(\mu) = \begin{vmatrix} (\frac{2}{3})^{\frac{1}{2}} (\mu^2 - \frac{1}{2}) & 1 \\ (\frac{2}{3})^{\frac{1}{2}} & 0 \end{vmatrix}, \quad \underset{\sim}{P} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \tag{9}$$

and $\underset{\sim}{\Psi}(x,\mu,t)$ is a vector with components $\psi_1(x,\mu,t)$ and $\psi_2(x,\mu,t)$.

The boundary conditions on $\underset{\sim}{\Psi}(x,\mu,t)$ can readily be established by taking the appropriate moments of Eqs. (2) and using Eqs. (3). We thus seek a solution of Eq. (8) subject to

$$\lim_{x\to\infty} \underset{\sim}{\Psi}(x,\mu,t) < \underset{\sim}{\infty} \tag{10a}$$

and

$$\underset{\sim}{\Psi}(0,\mu,t) = \sqrt{\pi}(\mu + \frac{\sqrt{\pi}}{2}) \begin{vmatrix} 1 \\ 0 \end{vmatrix} \exp(i\omega t)$$

$$+ 2 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \int_0^{\infty} \mu' \underset{\sim}{\Psi}(0,-\mu',t) \exp(-\mu'^2)\, d\mu', \ \mu > 0. \tag{10b}$$

In order to construct the desired solution, we write

$$\underset{\sim}{\Psi}(x,\mu,t) = \sqrt{\pi}\, \underset{\sim}{\Psi}_1(x,\mu,t) + \sqrt{\pi}\, E\, \underset{\sim}{\Psi}_0(x,\mu,t), \tag{11}$$

where $\underset{\sim}{\Psi}_0(x,\mu,t)$ and $\underset{\sim}{\Psi}_1(x,\mu,t)$ both are bounded, for $x > 0$, solutions of Eq. (8) constrained to satisfy the boundary conditions

$$\underset{\sim}{\Psi}_{\beta}(0,\mu,t) = \mu^{\beta} \exp(i\omega t) \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \quad \mu > 0, \quad \beta = 0 \text{ and } 1. \tag{12}$$

We find, on substituting Eq. (11) into Eq. (10b), the required expression for the constant E:

$$E = \frac{4\, U_{1,1} + \sqrt{\pi}}{2 - 4\, U_{0,1}}, \tag{13}$$

where $U_{0,1}$ and $U_{1,1}$ are, respectively, the upper elements of

$$\underset{\sim}{U}_{\beta} = \exp(-i\omega t) \int_0^{\infty} \mu' \exp(-\mu'^2)\, \underset{\sim}{\Psi}_{\beta}(0,-\mu',t)\, d\mu', \quad \beta = 0 \text{ and } 1. \tag{14}$$

Of course, once $\underset{\sim}{\Psi}(x,\mu,t)$ is established, such quantities as the xx component of the perturbed pressure tensor

$$\Delta P_{xx}(x,t) = \frac{1}{\pi^{3/2}} \int h(x,\underset{\sim}{c},t)\, c_x^2 \exp(-c^2)\, d^3c \tag{15}$$

and the perturbed gas pressure[22]

$$\Delta P(x,t) = \frac{1}{3\pi^{3/2}} \int h(x,\underset{\sim}{c},t)\, c^2 \exp(-c^2)\, d^3c \tag{16}$$

are readily available. We find

$$\Delta P_{xx}(x,t) = \frac{1}{\pi} \begin{vmatrix} 1 \\ 0 \end{vmatrix}^{T} \int_{-\infty}^{\infty} \underset{\sim}{\Psi}(x,\mu,t) \exp(-\mu^2)\, \mu^2\, d\mu \tag{17}$$

and

$$\Delta P(x,t) = \frac{1}{3\pi} \int_{-\infty}^{\infty} \begin{vmatrix} 1 + \mu^2 \\ 1 \end{vmatrix}^{T} \underset{\sim}{\Psi}(x,\mu,t) \exp(-\mu^2)\, d\mu . \tag{18}$$

In addition, Eqs. (6) and (7) give, respectively, the perturbed density and temperature distributions in terms of $\underset{\sim}{\psi}(x,\mu,t)$.

III. SOLUTION

Since the elementary solutions of Eq. (8) are given in Ref. 21, we can express the two vectors required in Eq. (11) as

$$\underset{\sim}{\Psi}_\beta(x,\mu,t) = \exp(i\omega t)\left\{\sum_{\alpha=1}^{\varkappa(i\omega)} A_\beta(\nu_\alpha)\underset{\sim}{\Phi}(\nu_\alpha,\mu;i\omega)\exp[-(i\omega+1)x/\nu_\alpha]\right.$$

$$\left. + \int_0^\infty \underset{\sim}{\Phi}(\nu,\mu;i\omega)\,\underset{\sim}{A}_\beta(\nu)\exp[-(i\omega+1)x/\nu]\,d\nu\right\},\quad \beta = 0 \text{ and } 1, \tag{19}$$

where

$$\underset{\sim}{\Phi}(\nu,\mu;i\omega) = \theta\,\nu\,\mathrm{Pv}\left(\frac{1}{\nu-\mu}\right)\underset{\sim}{Q}(\mu)(\underset{\sim}{I} + \gamma\nu\mu\underset{\sim}{D})$$

$$+ \delta(\nu-\mu)\exp(\nu^2)\,\underset{\sim}{Q}^{-T}(\nu)\,\underset{\sim}{\lambda}(\nu;i\omega), \tag{20}$$

and

$$\underset{\sim}{\Phi}(\nu_\alpha,\mu;i\omega) = \theta\,\nu_\alpha\left(\frac{1}{\nu_\alpha-\mu}\right)\underset{\sim}{Q}(\mu)(\underset{\sim}{I} + \gamma\,\nu_\alpha\,\mu\,\underset{\sim}{D})\,\underset{\sim}{M}(\nu_\alpha)\,, \tag{21}$$

with

$$\theta = \frac{1}{\sqrt{\pi}(i\omega+1)} \quad\text{and}\quad \gamma = \frac{2i\omega}{i\omega+1}\,. \tag{22}$$

Here the discrete eigenvalues ν_α are the "positive" zeros of $\Lambda(z;i\omega) = \det \underset{\sim}{\Lambda}(z;i\omega)$, where

$$\underset{\sim}{\Lambda}(z;i\omega) = \underset{\sim}{I} + z\int_{-\infty}^{\infty}\underset{\sim}{\Psi}(\mu)\,\frac{d\mu}{\mu-z}\,, \tag{23}$$

with the characteristic matrix given by

$$\underset{\sim}{\Psi}(\mu) = \theta \exp(-\mu^2)\, \underset{\sim}{Q}^T(\mu)\underset{\sim}{Q}(\mu)(\underset{\sim}{I} + \gamma \mu^2 \underset{\sim}{D}) \ . \tag{24}$$

In addition, $\underset{\sim}{M}(\nu_\alpha)$ is a null vector of $\underset{\sim}{\Lambda}(\nu_\alpha;i\omega)$,

$$\underset{\sim}{\lambda}(\nu;i\omega) = \underset{\sim}{I} + \nu\, P \int_{-\infty}^{\infty} \underset{\sim}{\Psi}(\mu) \frac{d\mu}{\mu - \nu} \tag{25}$$

and

$$\underset{\sim}{D} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \ . \tag{26}$$

We use $\varkappa(i\omega)$ to denote the number of ± pairs of discrete eigenvalues, and we use the convention that ν_α (rather than $-\nu_\alpha$) is such that $\underset{\sim}{\Psi}_0(x,\mu,t)$ and $\underset{\sim}{\Psi}_1(x,\mu,t)$, as given by Eq. (19), remain bounded as x approaches infinity. In a previous paper[19] we reported explicit expressions for the discrete eigenvalues and, as did Buckner and Ferziger[14], concluded that $\varkappa(i\omega)$ could be 0, 1, or 2; i.e., $\varkappa(i\omega) = 2$ for $0 < \omega < 0.646453\ldots$, $\varkappa(i\omega) = 1$ for $0.646453\ldots < \omega < 2.14517\ldots$, and $\varkappa(i\omega) = 0$ for $\omega > 2.14517\ldots$.

The expansion coefficients appearing in Eq. (19) can be determined by constraining the solutions to meet the boundary conditions given by Eq. (12). Thus we must solve

$$\mu^\beta \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \sum_{\alpha=1}^{\varkappa(i\omega)} A_\beta(\nu_\alpha)\, \underset{\sim}{\Phi}(\nu_\alpha,\mu;i\omega)$$

$$+ \int_0^\infty \underset{\sim}{\Phi}(\nu,\mu;i\omega)\, \underset{\sim}{A}_\beta(\nu)\, d\nu\,, \quad \mu > 0 \ . \tag{27}$$

In Ref. 21 a half-range completeness proof was given for the case $\varkappa(i\omega) = 0$, which insures that Eq. (27) has a solution for $\varkappa(i\omega) = 0$.

In addition, Siewert and Kriese[24] have deduced the half-range adjoint functions so that Eq. (27) can formally be solved, for general index $\varkappa$ (iω), in terms of the $\underset{\sim}{H}$ matrix[21]. We find that we can write the continuum coefficients as

$$\underset{\sim}{A}_0(\nu) = \left(\frac{2z_1}{\theta}\right) \left(\frac{1}{z_1 + \nu}\right) \underset{\sim}{L}^{-1}(\nu) \, \underset{\sim}{\pi}^{-1}(\nu) \, \underset{\sim}{H}^{-T}(\nu)$$

$$\cdot \, [\underset{\sim}{H}(z_1) + \underset{\sim}{D}\, \underset{\sim}{H}(-z_1)]^{-1} \begin{vmatrix} 0 \\ 1 \end{vmatrix} \tag{28}$$

and

$$\underset{\sim}{A}_1(\nu) = \left(\frac{z_1}{\theta}\right)\left(\frac{1}{z_1 + \nu}\right) \underset{\sim}{L}^{-1}(\nu) \, \underset{\sim}{\pi}^{-1}(\nu) \, \underset{\sim}{H}^{-T}(\nu)$$

$$\cdot \, [(\nu + z_1) \underset{\sim}{I} - 2 z_1 \underset{\sim}{K}] \, (\underset{\sim}{I} - \underset{\sim}{H}_0^T) \begin{vmatrix} 0 \\ 1 \end{vmatrix}, \tag{29}$$

where

$$\underset{\sim}{L}(\nu) = \underset{\sim}{\lambda}(\nu;i\omega) \, \underset{\sim}{\Psi}^{-1}(\nu) \underset{\sim}{\lambda}(\nu;i\omega) + \pi^2 \nu^2 \underset{\sim}{\Psi}(\nu) , \tag{30}$$

$$\underset{\sim}{H}_0^T = \int_0^\infty \underset{\sim}{H}^T(\mu) \, \hat{\underset{\sim}{\Psi}}(\mu) \, d\mu \quad , \qquad \hat{\underset{\sim}{\Psi}}(\mu) = \underset{\sim}{\pi}(\mu) \, \underset{\sim}{\Psi}(\mu) \, \underset{\sim}{\pi}^{-1}(\mu) , \tag{31}$$

$$\underset{\sim}{\Pi}(z) = \begin{vmatrix} 1 & 0 \\ 0 & \dfrac{z_1 - z}{z_1} \end{vmatrix} \quad \text{and} \quad \gamma^{\frac{1}{2}} z_1 = i . \tag{32}$$

In addition, the $\underset{\sim}{K}$ matrix is given by

$$\underset{\sim}{K} = [\underset{\sim}{H}(z_1) + \underset{\sim}{D}\, \underset{\sim}{H}(-z_1)]^{-1} \underset{\sim}{D}\, \underset{\sim}{H}(-z_1) , \tag{33}$$

and the $\underset{\sim}{H}$ matrix is a generalization of Chandrasekhar's H function[21].

We can also use the orthogonality relations discussed by

Siewert and Kriese[24] to find the discrete coefficients $A_0(\nu_\alpha)$ and $A_1(\nu_\alpha)$. We find, for $\alpha = 1, 2, \cdots, \varkappa(i\omega)$,

$$A_0(\nu_\alpha) = \left(\frac{2\ z_1}{\theta}\right)\left(\frac{1}{z_1 + \nu_\alpha}\right) \frac{\nu_\alpha}{N(\nu_\alpha)} \underset{\sim}{M}^T(\nu_\alpha)\, \underset{\sim}{\pi}(-\nu_\alpha)\, \underset{\sim}{H}^{-T}(\nu_\alpha)$$

$$\cdot[\underset{\sim}{H}(z_1) + \underset{\sim}{D}\,\underset{\sim}{H}(-z_1)]^{-1}\begin{vmatrix} 0 \\ 1 \end{vmatrix} \tag{34}$$

and

$$A_1(\nu_\alpha) = \left(\frac{z_1}{\theta}\right)\left(\frac{1}{z_1 + \nu_\alpha}\right) \frac{\nu_\alpha}{N(\nu_\alpha)} \underset{\sim}{M}^T(\nu_\alpha)\, \underset{\sim}{\pi}(-\nu_\alpha)\, \underset{\sim}{H}^{-T}(\nu_\alpha)$$

$$[(\nu_\alpha + z_1)\,\underset{\sim}{I} - 2\ z_1\,\underset{\sim}{K}]\ (\underset{\sim}{I} - \underset{\sim}{H}_0^T)\begin{vmatrix} 0 \\ 1 \end{vmatrix}, \tag{35}$$

where the normalization integral $N(\nu_\alpha)$ is given by

$$N(\nu_\alpha) = \nu_\alpha^2 \underset{\sim}{M}^T(\nu_\alpha)\, \underset{\sim}{\pi}(\nu_\alpha)\, \underset{\sim}{\pi}(-\nu_\alpha)\, \underset{\sim}{\Lambda}'(\nu_\alpha)\, \underset{\sim}{M}(\nu_\alpha)\ . \tag{36}$$

Since Eqs. (28), (29) (34), and (35) explicitly express the desired expansion coefficients in terms of the $\underset{\sim}{H}$ matrix, the function $\underset{\sim}{\Psi}(x,\mu,t)$ is readily available and can be used, for example, in Eqs. (6), (7), (17), and (18) to give, respectively, the perturbed density, temperature, xx component of pressure, and the pressure. On substituting Eq. (11) into Eqs. (6), (7), (17), and (18) and carrying out the indicated integration, we find we can write

$$\Delta N(x,t) = \frac{1}{\sqrt{\pi}} \exp(i\omega t) \begin{vmatrix} 0 \\ 1 \end{vmatrix}^T \left[\sum_{\alpha=1}^{\varkappa(i\omega)} A(\nu_\alpha)\, \underset{\sim}{M}(\nu_\alpha) \exp[-(i\omega+1)x/\nu_\alpha] \right.$$

$$\left. + \int_0^\infty \underset{\sim}{A}(\nu) \exp[-(i\omega+1)x/\nu]\, d\nu \right], \tag{37}$$

$$\Delta T(x,t) = \sqrt{\frac{2}{3\pi}}\, \exp(i\omega t) \begin{vmatrix} 1 \\ 0 \end{vmatrix}^T \left[\sum_{\alpha=1}^{\varkappa(i\omega)} A(\nu_\alpha) \underset{\sim}{M}(\nu_\alpha) \exp[-(i\omega+1)x/\nu_\alpha] \right.$$

$$\left. + \int_0^\infty \underset{\sim}{A}(\nu) \exp[-(i\omega+1)x/\nu]\, d\nu \right] \quad (38)$$

$$\Delta P_{xx}(x,t) = \frac{1}{\sqrt{\pi}} \exp(i\omega t) \left(\frac{i\omega}{i\omega + 1}\right)^2 \begin{vmatrix} 0 \\ 1 \end{vmatrix}^T$$

$$\left[\sum_{\alpha=1}^{\varkappa(i\omega)} \nu_\alpha^2 A(\nu_\alpha) \underset{\sim}{M}(\nu_\alpha) \exp[-(i\omega+1)x/\nu_\alpha] \right.$$

$$\left. + \int_0^\infty \nu^2 \underset{\sim}{A}(\nu) \exp[-(i\omega+1)x/\nu]\, d\nu \right], \quad (39)$$

and

$$\Delta P(x,t) = \frac{1}{\sqrt{6\pi}} \exp(i\omega t) \begin{vmatrix} 1 \\ \sqrt{\frac{3}{2}} \end{vmatrix}^T$$

$$\left[\sum_{\alpha=1}^{\varkappa(i\omega)} A(\nu_\alpha) \underset{\sim}{M}(\nu_\alpha) \exp[-(i\omega+1)x/\nu_\alpha] \right.$$

$$\left. + \int_0^\infty \underset{\sim}{A}(\nu) \exp[-(i\omega+1)x/\nu]\, d\nu \right] \quad (40)$$

Here

$$\underset{\sim}{A}(\nu) = \underset{\sim}{A}_1(\nu) + E\,\underset{\sim}{A}_0(\nu) \quad \text{and} \quad A(\nu_\alpha) = A_1(\nu_\alpha) + E\,A_0(\nu_\alpha)\,. \qquad (41)$$

Although the constant E can be computed from Eqs. (13) and (14), we prefer to use the $\underset{\sim}{R}$ matrix of Siewert and Kriese[24] to obtain

$$E = \frac{z_1 \begin{vmatrix} 0 \\ 1 \end{vmatrix}^T \underset{\sim}{W}_1 \begin{vmatrix} 0 \\ 1 \end{vmatrix}}{\begin{vmatrix} 0 \\ 1 \end{vmatrix}^T \underset{\sim}{W}_0 \begin{vmatrix} 0 \\ 1 \end{vmatrix}}\,, \qquad (42)$$

where

$$\underset{\sim}{W}_0 = [\underset{\sim}{H}^{-T}(z_1) - \underset{\sim}{H}^{-T}(-z_1)]\,[\underset{\sim}{H}(z_1) + \underset{\sim}{D}\,\underset{\sim}{H}(-z_1)]^{-1} \qquad (43)$$

and

$$\underset{\sim}{W}_1 = [(\underset{\sim}{I} - \underset{\sim}{H}_0) - \underset{\sim}{H}^{-T}(z_1) + \{\underset{\sim}{H}^{-T}(z_1) - \underset{\sim}{H}^{-T}(-z_1)\}\,\underset{\sim}{K}](\underset{\sim}{I} - \underset{\sim}{H}_0^T)\,. \qquad (44)$$

IV. NUMERICAL RESULTS

It is clear from Eqs. (28)-(44) that the desired physical quantities ΔN, ΔT, ΔP_{xx}, and ΔP can all be computed once the $\underset{\sim}{H}$ matrix[21] is known. From the work of Siewert and Burniston[21], we note that $\underset{\sim}{H}(z)$ is the solution of the nonlinear matrix integral equation

$$\underset{\sim}{H}^{-1}(z) = \underset{\sim}{I} - z \int_0^\infty \underset{\sim}{H}^T(\mu)\,\hat{\underset{\sim}{\Psi}}(\mu)\,\frac{d\mu}{\mu+z} \qquad (45)$$

Table I. The Matrix $\underset{\sim}{H}^*(\eta)$ for $\omega = 5.2$.

η	$H^*_{11}(\eta)$		$H^*_{12}(\eta)$	
0.0	0.10000 (1)	0.0	0.0	0.0
0.05	0.10019 (1)	-0.12804 (-1)	-0.76675 (-3)	0.40696 (-2)
0.10	0.10028 (1)	-0.20651 (-1)	-0.13939 (-2)	0.56982 (-2)
0.15	0.10035 (1)	-0.26999 (-1)	-0.21028 (-2)	0.65944 (-2)
0.20	0.10039 (1)	-0.32510 (-1)	-0.29100 (-2)	0.70827 (-2)
0.25	0.10043 (1)	-0.37476 (-1)	-0.38171 (-2)	0.73052 (-2)
0.30	0.10046 (1)	-0.42060 (-1)	-0.48235 (-2)	0.73393 (-2)
0.35	0.10049 (1)	-0.46365 (-1)	-0.59283 (-2)	0.72325 (-2)
0.40	0.10050 (1)	-0.50464 (-1)	-0.71315 (-2)	0.70168 (-2)
0.45	0.10052 (1)	-0.54406 (-1)	-0.84342 (-2)	0.67145 (-2)
0.50	0.10053 (1)	-0.58233 (-1)	-0.98383 (-2)	0.63419 (-2)
0.55	0.10054 (1)	-0.61977 (-1)	-0.11347 (-1)	0.59116 (-2)
0.60	0.10054 (1)	-0.65664 (-1)	-0.12964 (-1)	0.54334 (-2)
0.65	0.10054 (1)	-0.69320 (-1)	-0.14694 (-1)	0.49154 (-2)
0.70	0.10053 (1)	-0.72964 (-1)	-0.16545 (-1)	0.43642 (-2)
0.75	0.10052 (1)	-0.76617 (-1)	-0.18524 (-1)	0.37857 (-2)
0.80	0.10051 (1)	-0.80299 (-1)	-0.20640 (-1)	0.31851 (-2)
0.85	0.10049 (1)	-0.84030 (-1)	-0.22905 (-1)	0.25674 (-2)
0.90	0.10047 (1)	-0.87832 (-1)	-0.25332 (-1)	0.19374 (-2)
0.95	0.10044 (1)	-0.91726 (-1)	-0.27939 (-1)	0.13001 (-2)

which can be solved straightforwardly by an iterative procedure. The results of such a computation are displayed in Table I for $\omega = 5.2$. In Table I we give values for $\underset{\sim}{H}^*(\eta)$, $0 \le \eta < 1$, where

$$\underset{\sim}{H}^*(\eta) = (1+\eta)\, \underset{\sim}{H}\left(\frac{\eta}{1-\eta}\right), \quad 0 \le \eta < 1. \tag{46}$$

Using the computed values of $\underset{\sim}{H}(z)$, we calculated $\Delta P_{xx}(x,0)$ and $\Delta P(x,0)$, over a range of ω-values. Sample results for ΔP_{xx}

Table I. Continued.

η	$H^*_{21}(\eta)$		$H^*_{22}(\eta)$	
0.0	0.0	0.0	0.10000 (1)	0.0
0.05	-0.17960 (-3)	0.41929 (-2)	0.10032 (1)	-0.20399 (-1)
0.10	0.38120 (-3)	0.60217 (-2)	0.10049 (1)	-0.34572 (-1)
0.15	0.12264 (-2)	0.71280 (-2)	0.10061 (1)	-0.46737 (-1)
0.20	0.22523 (-2)	0.78126 (-2)	0.10070 (1)	-0.57728 (-1)
0.25	0.34112 (-2)	0.82042 (-2)	0.10076 (1)	-0.67923 (-1)
0.30	0.46772 (-2)	0.83721 (-2)	0.10081 (1)	-0.77541 (-1)
0.35	0.60347 (-2)	0.83573 (-2)	0.10083 (1)	-0.86721 (-1)
0.40	0.74741 (-2)	0.81862 (-2)	0.10084 (1)	-0.95562 (-1)
0.45	0.89897 (-2)	0.78762 (-2)	0.10084 (1)	-0.10414
0.50	0.10578 (-1)	0.74385 (-2)	0.10082 (1)	-0.11250
0.55	0.12238 (-1)	0.68805 (-2)	0.10079 (1)	-0.12070
0.60	0.13970 (-1)	0.62063 (-2)	0.10075 (1)	-0.12877
0.65	0.15775 (-1)	0.54176 (-2)	0.10069 (1)	-0.13675
0.70	0.17655 (-1)	0.45137 (-2)	0.10063 (1)	-0.14467
0.75	0.19615 (-1)	0.34923 (-2)	0.10055 (1)	-0.15255
0.80	0.21659 (-1)	0.23490 (-2)	0.10046 (1)	-0.16042
0.85	0.23793 (-1)	0.10774 (-2)	0.10035 (1)	-0.16830
0.90	0.26025 (-1)	-0.33110 (-3)	0.10024 (1)	-0.17623
0.95	0.28384 (-1)	-0.18877 (-2)	0.10010 (1)	-0.18421

are given in Table II for ω = 0.5, 1.0, and 5.2. These values were chosen to check the results in each of the three regions discussed in Sec. III; i.e. cases for which there are 2, 1, and 0 pairs of discrete eigenvalues, respectively. For convenience, the results are given at equally-spaced x points. However, the same results were computed at the x values used by Loyalka and Cheng[20] for purposes of comparison. The agreement for both real and imaginary parts of ΔP_{xx} is very good for small x-values, with differences of less than 1% except near a zero of either the real

Table II. The Pressure Perturbation for ω=0.50(left) and ω=1.0 (right)

x	$Re(\Delta P_{xx})$	$Im(\Delta P_{xx})$	$Re(\Delta P_{xx})$	$Im(\Delta P_{xx})$
0.0	0.4538	0.3783 (-01)	0.4793	0.3566 (-01)
0.2	0.4503	-0.1175 (-01)	0.4659	-0.6190 (-01)
0.4	0.4404	-0.5962 (-01)	0.4300	-0.1494
0.6	0.4249	-0.1049	0.3776	-0.2224
0.8	0.4044	-0.1469	0.3142	-0.2790
1.0	0.3798	-0.1854	0.2444	-0.3188
1.2	0.3517	-0.2198	0.1722	-0.3423
1.4	0.3208	-0.2502	0.1011	-0.3507
1.6	0.2875	-0.2763	0.3370 (-01)	-0.3456
1.8	0.2526	-0.2982	-0.2790 (-01)	-0.3288
2.0	0.2164	-0.3158	-0.8209 (-01)	-0.3023
2.2	0.1796	-0.3291	-0.1278	-0.2684
2.4	0.1425	-0.3383	-0.1644	-0.2289
2.6	0.1055	-0.3436	-0.1917	-0.1860
2.8	0.6910 (-01)	-0.3450	-0.2098	-0.1415
3.0	0.3360 (-01)	-0.3428	-0.2190	-0.9706 (-01)
3.2	-0.6732 (-03)	-0.3371	-0.2202	-0.5417 (-01)
3.4	-0.3342 (-01)	-0.3282	-0.2141	-0.1408 (-01)
3.6	-0.6440 (-01)	-0.3164	-0.2017	0.2218 (-01)
3.8	-0.9336 (-01)	-0.3019	-0.1841	0.5385 (-01)
4.0	-0.1201	-0.2849	-0.1625	0.8038 (-01)
4.2	-0.1445	-0.2659	-0.1380	0.1015
4.4	-0.1664	-0.2450	-0.1117	0.1171
4.6	-0.1856	-0.2225	-0.8470 (-01)	0.1272
4.8	-0.2022	-0.1988	-0.5791 (-01)	0.1321
5.0	-0.2160	-0.1741	-0.3217 (-01)	0.1323
5.2	-0.2271	-0.1488	-0.8213 (-02)	0.1283
5.4	-0.2354	-0.1230	0.1339 (-01)	0.1205
5.6	-0.2411	-0.9716 (-01)	0.3220 (-01)	0.1098
5.8	-0.2441	-0.7145 (-01)	0.4791 (-01)	0.9672 (-01)
6.0	-0.2445	-0.4613 (-01)	0.6036 (-01)	0.8198 (-01)
6.2	-0.2425	-0.2143 (-01)	0.6951 (-01)	0.6623 (-01)
6.4	-0.2382	0.2418 (-02)	0.7542 (-01)	0.5010 (-01)
6.6	-0.2317	0.2522 (-01)	0.7825 (-01)	0.3413 (-01)
6.8	-0.2232	0.4680 (-01)	0.7825 (-01)	0.1884 (-01)
7.0	-0.2128	0.6699 (-01)	0.7574 (-01)	0.4625 (-02)
7.2	-0.2008	0.8565 (-01)	0.7107 (-01)	-0.8160 (-02)
7.4	-0.1873	0.1027	0.6463 (-01)	-0.1927 (-01)
7.6	-0.1725	0.1179	0.5682 (-01)	-0.2852 (-01)
7.8	-0.1567	0.1314	0.4804 (-01)	-0.3583 (-01)
8.0	-0.1399	0.1430	0.3869 (-01)	-0.4117 (-01)
8.2	-0.1225	0.1526	0.2912 (-01)	-0.4459 (-01)
8.4	-0.1046	0.1603	0.1967 (-01)	-0.4619 (-01)
8.6	-0.8650 (-01)	0.1662	0.1064 (-01)	-0.4612 (-01)
8.8	-0.6828 (-01)	0.1701	0.2263 (-02)	-0.4456 (-01)
9.0	-0.5016 (-01)	0.1722	-0.5257 (-02)	-0.4174 (-01)
9.2	-0.3232 (-01)	0.1724	-0.1177 (-01)	-0.3787 (-01)
9.4	-0.1492 (-01)	0.1710	-0.1719 (-01)	-0.3321 (-01)
9.6	0.1880 (-02)	0.1679	-0.2144 (-01)	-0.2799 (-01)
9.8	0.1794 (-01)	0.1633	-0.2453 (-01)	-0.2244 (-01)

Table II (cont'd). The Pressure Perturbation for ω=5.2

$Re(\Delta P_{xx})$	$Im(\Delta P_{xx})$
0.5023	0.1003 (-01)
0.2552	-0.3530
-0.7490 (-01)	-0.3376
-0.2234	-0.1470
0.2020	0.2836 (-01)
-0.1047	0.1133
-0.1031 (-01)	0.1152
0.4459 (-01)	0.7367 (-01)
0.5824	0.2553 (-01)
0.4579 (-01)	-0.8429 (-02)
0.2447 (-01)	-0.2328 (-01)
0.5952 (-02)	-0.2352 (-01)
-0.5070 (-02)	-0.1647 (-01)
-0.8914 (-02)	-0.8171 (-02)
-0.8151 (-02)	-0.1901 (-02)
-0.5506 (-02)	0.1490 (-02)
-0.2835 (-02)	0.2561 (-02)
-0.9642 (-03)	0.2316 (-02)
0.1444 (-04)	0.1614 (-03)
0.3584 (-03)	0.9556 (-03)
0.3807 (-03)	0.5183 (-03)
0.3062 (-03)	0.2879 (-03)
0.2426 (-03)	0.1811 (-03)
0.2135 (-03)	0.1221 (-03)
0.2030 (-03)	0.6967 (-04)
0.1899 (-03)	0.1419 (-04)
0.1629 (-03)	-0.3757 (-04)
0.1226 (-03)	-0.7573 (-04)
0.7660 (-04)	-0.9500 (-04)
0.3347 (-04)	-0.9603 (-04)
-0.5261 (-06)	-0.8368 (-04)
-0.2297 (-04)	-0.6423 (-04)
-0.3448 (-04)	-0.4314 (-04)
-0.3739 (-04)	-0.2418 (-04)
-0.3468 (-04)	-0.9198 (-05)
-0.2889 (-04)	0.1402 (-05)
-0.2196 (-04)	0.7989 (-05)
-0.1517 (-04)	0.1138 (-04)
-0.9173 (-05)	0.1245 (-04)
-0.4280 (-05)	0.1189 (-04)
-0.6131 (-06)	0.1036 (-04)
0.1926 (-05)	0.8342 (-05)
0.3488 (-05)	0.6159 (-05)
0.4204 (-05)	0.4060 (-05)
0.4278 (-05)	0.2252 (-05)
0.3940 (-05)	0.8024 (-06)
0.3333 (-05)	-0.2919 (-06)
0.2575 (-05)	-0.1014 (-05)
0.1813 (-05)	-0.1392 (-05)
0.1139 (-05)	-0.1525 (-05)

or imaginary part. However, the agreement deteriorates for increasing distance from the plate. The amplitude of ΔP_{xx} agrees to within 5% for distances up to 7 mean-free-paths from the plate.

It is, of course, possible to deduce the attenuation and dispersion of the sound disturbance from the pressure perturbation as explained by Greenspan[6]. There is considerable arbitrariness in computing the attenuation and the sound dispersion. For example, the attenuation is computed from the slope of a logarithmic plot of the amplitude of the pressure disturbance vs. x. However, it is clear from Eqs. (39) and (40) that such a plot will not be a straight line, thus making the choice of an x-interval to use in computing the slope arbitrary. However, to follow the tradition, we show in Figs. 1 and 2 a comparision of our results for the attenuation and sound dispersion with the Navier-Stokes, Burnett, and super-Burnett theories, and with the data of Meyer and Sessler[7]. It is clear that the agreement of our calculations with the experimental results of Meyer and Sessler[7] is, to within the mentioned arbitrariness, excellent. The results of Buckner and Ferziger[14], not shown on the plot, are in agreement with ours except over the range $0.1 < 1/\omega < 2$, where their results fall considerable below ours (see Loyalka and Cheng[20]).

V. CONCLUSIONS

It has been shown that the method of elementary solutions used in the present work allows the computation of the solution of the BGK model equation essentially without the distortions of numerical approximation. It is thus possible to determine if this model of the Boltzmann equation can correctly represent the physics of sound propagation. The results of Loyalka and Cheng[20], which are in complete agreement with those reported here, indicate that the BGK model predicts values of attenuation and dispersion which agree very well with experimental data[5-7]. The attenuation fits

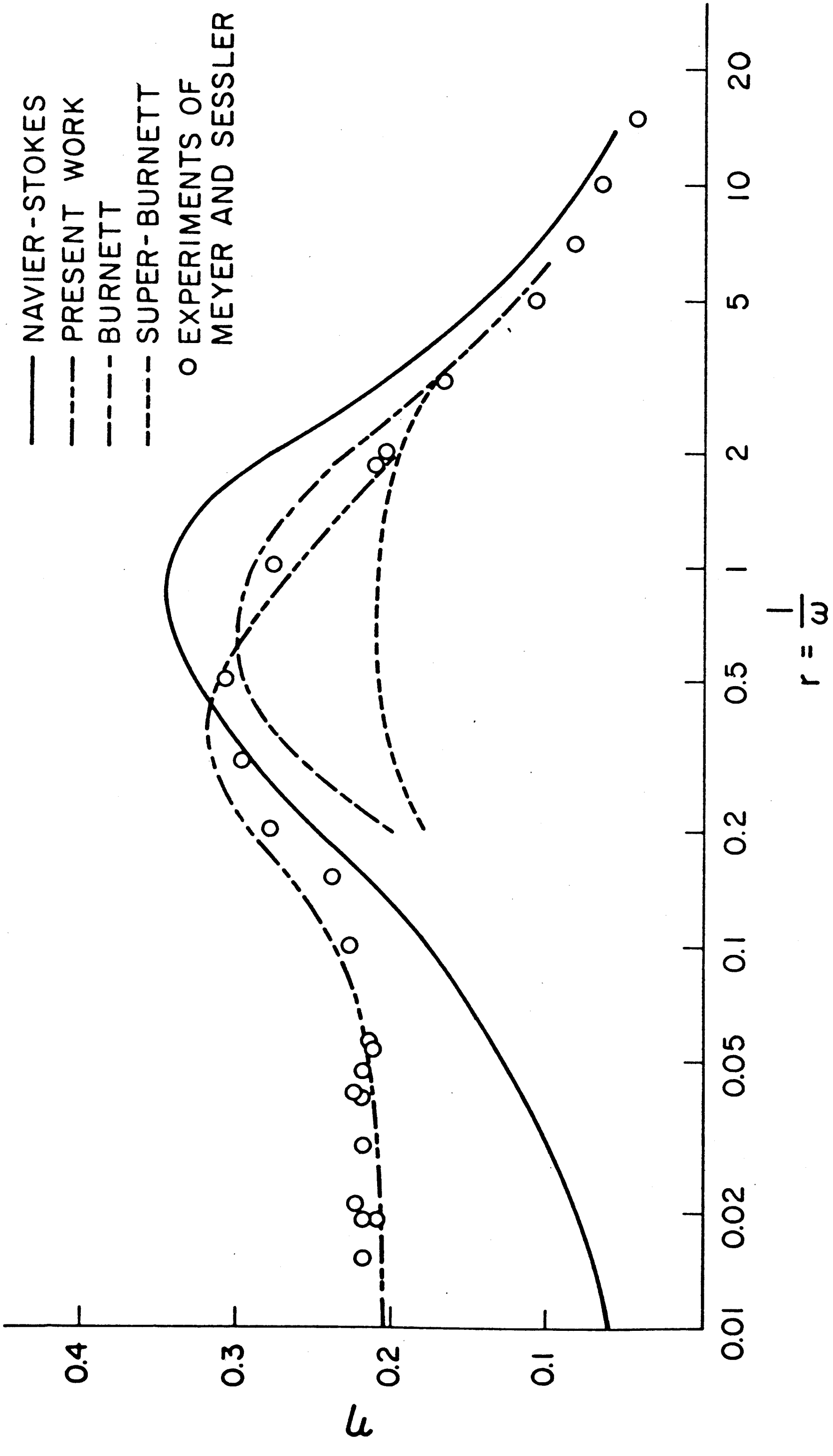

Fig. 1 -- Attenuation: Comparison with experiment and other theories.

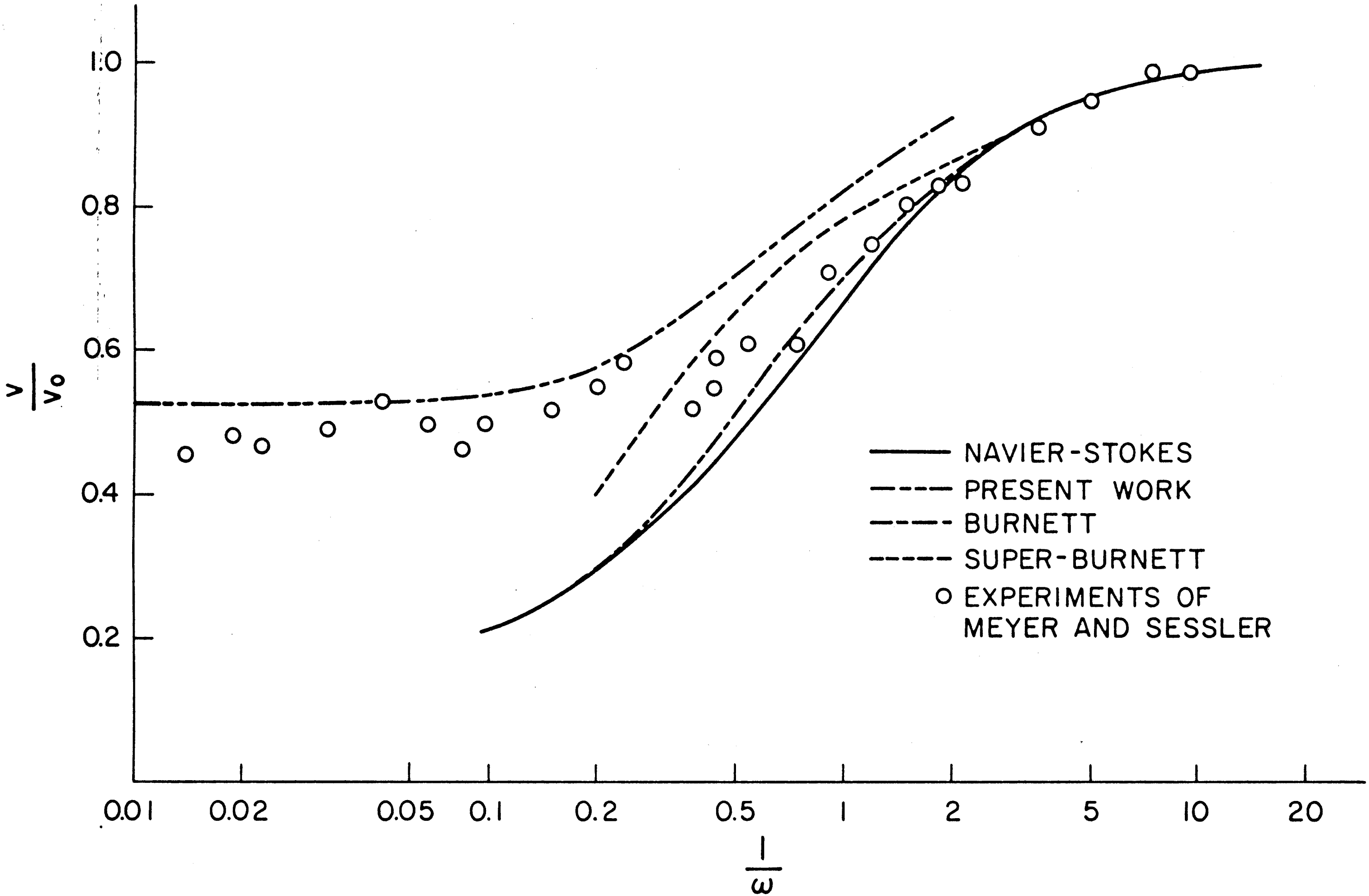

Fig. 2 -- Sound Dispersion: Comparison with experiment and other theories.

the data extremely well, while the dispersion is very slightly over-predicted by the present method. It appears that the present accurate solution of the linear BGK equation, along with that of Loyalka and Cheng[20], fits the data over the full range of ω better than any of the previously published results. We conclude that the linearized BGK model does, indeed, describe the physical processes of sound attenuation and dispersion quite well.

Ostrowsky and Kleitman[16] suggest that the BGK model should predict the sound disturbance to fall off like $\exp(-x^{2/3})$ at large distances. This behavior is not apparent in our results.

ACKNOWLEDGMENT

The authors are grateful to C. Cercignani, T. S. Chang, M. M. R. Williams, and S. K. Loyalka for helpful discussions concerning the formulation of the problem considered here.

This work was supported in part by National Science Foundation Grant ENG-7709405.

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