A concise and accurate solution for Poiseuille flow in a plane channel

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The recently developed F_N method of solving problems in particle transport theory is used to establish a concise and accurate solution for the flow of a rarefied gas between two parallel plates. The Bhatnagar, Gross, and Krook model is used, and numerical results are given for a wide range of the Knudsen number.

I. INTRODUCTION

In two basic papers in the field of rarefied gas dynamics, Cercignani and Daneri,[1] and Cercignani[2] reported on two different methods of studying the flow of a rarefied gas between two parallel plates. In both papers[1,2] the BGK[3] model was used to describe the physical problem. Cercignani and Daneri[1] used the integral form of the particle transport equation and finite difference techniques to develop numerical results applicable to a wide range of the Knudsen number, and Cercignani[2] used the method of elementary solutions[4] to reduce the problem to one of solving a Fredholm equation for the required expansion coefficient. Additional numerical results have been obtained more recently by Boffi, De Socio, Gaffuri, and Pescatore,[5] and Loyalka, Petrellis, and Storvick.[6] Here we wish to describe the F_N method[7] of solving the same problem. The method utilizes aspects of the exact elementary solutions to establish an approximate solution that is particularly concise and very economical to use from the point of view of computer-time requirements.

As discussed by Cercignani,[2] the linearized BGK model for flow in the z direction between plates a distance d apart can be written as

$$\kappa c_z + c_x(\partial/\partial x)\, h(x,\mathbf{c}) = Lh(x,\mathbf{c}), \qquad (1)$$

where $\mathbf{c}$ is the molecular velocity, $h(x,\mathbf{c})$ is the perturbation of the particle distribution function from the Maxwellian and κ is proportional to the pressure gradient that causes the flow. For the BGK model, Cercignani[2] uses the appropriate form of the collision operator L, and considers

$$Z(x,c_x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(c_y^2+c_z^2)}\, c_z\, h(x,\mathbf{c})\, dc_y\, dc_z\,, \qquad (2)$$

to be the basic unknown, and thus reduces the problem to one of solving

$$\tfrac{1}{2}\kappa\theta + \theta c_x(\partial/\partial x)\, Z(x,c_x) + Z(x,c_x) = \pi^{-1/2}\int_{-\infty}^{\infty} e^{-c_x^2}\, Z(x,c_x)\, dc_x\,, \qquad (3)$$

subject to the boundary conditions

$$Z[-(d/2)\,\mathrm{sgn}c_x, c_x] = 0\,. \qquad (4)$$

In Eq. (3) the mean-free-time is denoted by θ. In the next section we use the F_N method to deal with Eqs. (3) and (4) and thus to establish a concise result for the flow rate Q.

II. ANALYSIS

In regard to Eqs. (3) and (4), we prefer to let $\mu = c_x$, $\tau = x/\theta$, $\delta = d/\theta$ and thus to consider

$$\tfrac{1}{2}\kappa\theta + \mu(\partial/\partial\tau)\, Z(\tau,\mu) + Z(\tau,\mu) = \pi^{-1/2}\int_{-\infty}^{\infty} e^{-\mu^2}\, Z(\tau,\mu)\, d\mu\,, \qquad (5)$$

and

$$Z[-(\delta/2)\,\mathrm{sgn}\mu,\mu] = 0, \quad \mu\in(-\infty,\infty)\,. \qquad (6)$$

If we substitute

$$Z(\tau,\mu) = \tfrac{1}{2}\kappa\theta\,[\tau^2 - 2\tau\mu + 2\mu^2 - (\delta^2/4) - 2Y(\tau,\mu)]\,, \qquad (7)$$

into Eqs. (5) and (6) then we see at once that $Y(\tau,\mu)$ is the solution of

$$\mu(\partial/\partial\tau)\, Y(\tau,\mu) + Y(\tau,\mu) = \pi^{-1/2}\int_{-\infty}^{\infty} e^{-\mu^2}\, Y(\tau,\mu)\, d\mu\,, \qquad (8)$$

subject to

$$Y(-a,\mu) = Y(a,-\mu) = \mu^2 + a\mu, \quad \mu > 0\,, \qquad (9)$$

where $2a = \delta$. In order to simplify the calculation of the flow rate Q, we first wish to note several useful relationships concerning some moments of $Y(\tau,\mu)$. If we let

$$Y_\alpha(\tau) = \pi^{-1/2}\int_{-\infty}^{\infty} e^{-\mu^2}\, Y(\tau,\mu)\,\mu^\alpha\, d\mu\,, \qquad (10)$$

then we can multiply Eq. (8) by $\exp(-\mu^2)$ and integrate over all μ to deduce that $Y_1(\tau)$ is a constant, say $Y_1(a)$. Multiplying Eq. (8) by $\mu \exp(-\mu^2)$ and integrating over all μ, we find

$$(d/d\tau)\, Y_2(\tau) + Y_1(a) = 0\,, \qquad (11)$$

which, after we multiply by τ and integrate over τ from $-a$ to a, yields

$$\int_{-a}^{a} Y_2(\tau)\, d\tau = 2a Y_2(a)\,. \qquad (12)$$

If we multiply Eq. (8) by $\mu^2 \exp(-\mu^2)$ and integrate over μ, we find

$$(d/d\tau)\, Y_3(\tau) + Y_2(\tau) = \tfrac{1}{2} Y_0(\tau)\,, \qquad (13)$$

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which we can integrate over τ to find, after using Eq. (12),

$$\int_{-a}^{a} Y_0(\tau)\,d\tau = 4Y_3(a) + 4aY_2(a)\,. \qquad (14)$$

Now since the flow rate is

$$Q(a) = -\frac{1}{\kappa d a}\int_{-a}^{a} q(\tau)\,d\tau\,, \qquad (15)$$

where the macroscopic velocity is

$$q(\tau) = \pi^{-1/2}\int_{-\infty}^{\infty} e^{-\mu^2} Z(\tau,\mu)\,d\mu\,, \qquad (16)$$

we can use Eqs. (7) and (15) to express the flow rate simply in terms of surface quantities, i.e.,

$$Q(a) = \frac{a}{3} - \frac{1}{2a} + \frac{2}{a^2}\,[Y_3(a) + aY_2(a)]\,. \qquad (17)$$

If we use

$$Y_2(a) = \pi^{-1/2}\int_0^{\infty} \mu^2 e^{-\mu^2}\,Y(a,\mu)\,d\mu + \tfrac{3}{8} + \pi^{-1/2}\,\frac{a}{2}\,, \qquad (18a)$$

and

$$Y_3(a) = \pi^{-1/2}\int_0^{\infty} \mu^3 e^{-\mu^2}\,Y(a,\mu)\,d\mu - \tfrac{3}{8}\,a - \pi^{-1/2}\,, \qquad (18b)$$

in Eq. (17) we can write

$$Q(a) = \frac{a}{3} - \frac{1}{2a} + \pi^{-1/2}\left(1 - \frac{2}{a^2}\right) + \pi^{-1/2}\,\frac{2}{a^2} \times \int_0^{\infty} \mu^2 e^{-\mu^2}\,Y(a,\mu)(\mu + a)\,d\mu\,, \qquad (19)$$

so that $Q(a)$ finally is expressed in terms only of $Y(a,\mu), \mu > 0$.

We now wish to consider the boundary-value problem defined by Eqs. (8) and (9). The desired symmetrical solution, $Y(\tau,\mu) = Y(-\tau,-\mu)$, can be expressed in terms of the elementary solutions[8] as

$$Y(\tau,\mu) = A\pi^{-1/2} + \int_0^{\infty} A(\nu)[\phi(\nu,\mu)\,e^{-\tau/\nu} + \phi(-\nu,\mu)\,e^{\tau/\nu}]\,d\nu\,, \qquad (20),$$

where

$$\phi(\nu,\mu) = \pi^{-1/2}\,\nu P v\left(\frac{1}{\nu - \mu}\right) + \pi^{-1/2}\,p(\nu)\,\delta(\nu - \mu)\,. \qquad (21)$$

Here

$$p(\nu) = \pi^{1/2}\left(e^{\nu^2} - 2\nu\int_0^{\nu} e^{x^2}\,dx\right), \qquad (22)$$

and the expansion coefficients A and $A(\nu)$ are to be determined by the boundary condition, Eq. (9). To proceed with the method of elementary solutions we would substitute Eq. (20) into Eq. (9) and regularize the resulting singular integral equation to obtain ultimately a Fredholm-type integral equation for $A(\nu)$. Since we have expressed the desired flow rate $Q(a)$ in terms of $Y(a,\mu)$, [see Eq. (19)] we do not need $Y(\tau,\mu)$ for all τ, and thus we do not pursue the method of elementary solutions further. Instead, we pay special attention to establishing $Y(a,\mu)$. Since the functions $\phi(\nu,\mu)$ are orthogonal, in the sense that

$$\int_{-\infty}^{\infty} e^{-\mu^2}\,\phi(\nu,\mu)\,\phi(\nu',\mu)\,\mu\,d\mu = 0, \quad \nu \neq \nu'\,, \qquad (23a)$$

and

$$\int_{-\infty}^{\infty} e^{-\mu^2}\,\phi(\nu,\mu)\,\mu\,d\mu = 0\,, \qquad (23b)$$

we can multiply Eq.(20), evaluated at $\tau = \pm a$, by $\mu\exp(-\mu^2)\,\phi(-\nu,\mu)$ and integrate over all μ to find

$$\int_{-\infty}^{\infty} e^{-\mu^2}\,\phi(-\nu,\mu)\,Y(\mp a,\mu)\,\mu\,d\mu = A(\nu)\,N(-\nu)\,e^{\mp a/\nu}\,, \qquad (24)$$

where $N(-\nu)$ is a normalization factor that can be eliminated between the two forms of Eq. (24) to yield

$$\int_{-\infty}^{\infty} e^{-\mu^2}\,\phi(-\nu,\mu)\,Y(-a,\mu)\,\mu\,d\mu = e^{-2a/\nu}\int_{-\infty}^{\infty} e^{-\mu^2}\,\phi(-\nu,\mu)\,Y(a,\mu)\,\mu\,d\mu\,, \qquad (25)$$

or, after we use Eq. (9),

$$\int_0^{\infty} e^{-\mu^2}\,\phi(\nu,\mu)\,Y(a,\mu)\,\mu\,d\mu + e^{-2a/\nu} \times \int_0^{\infty} e^{-\mu^2}\,\phi(-\nu,\mu)\,Y(a,\mu)\,\mu\,d\mu = K(\nu)\,. \qquad (26)$$

Here the known function $K(\nu)$ is given by

$$K(\nu) = \int_0^{\infty} e^{-\mu^2}\phi(-\nu,\mu)(\mu^2 + a\mu)\,\mu\,d\mu + e^{-2a/\nu} \times \int_0^{\infty} e^{-\mu^2}\,\phi(\nu,\mu)(\mu^2 + a\mu)\,\mu\,d\mu\,. \qquad (27)$$

In a similar manner, we can multiply Eq. (20), evaluated at $\tau = a$, by $\mu\exp(-\mu^2)$, and integrate over all μ to find

$$\int_0^{\infty} e^{-\mu^2}\,Y(a,\mu)\,\mu\,d\mu = \int_0^{\infty} e^{-\mu^2}\,(\mu^2 + a\mu)\,\mu\,d\mu\,. \qquad (28)$$

Equations (26) and (28) constitute a singular integral equation and a constraint to be solved to establish $Y(a,\mu)$. It is clear that the methods of Muskhelishvili[9] could be used to convert Eqs. (26) and (28) to a Fredholm-like integral equation for $Y(a,\mu)$. However, we prefer here to introduce the F_N method[7] and thus to substitute the approximation

$$Y(a,\mu) = \mu(\mu + a)\,\theta(a)\,e^{-2a/\mu} + \sum_{\alpha=0}^{N} a_\alpha\,[1 - (-1)^\alpha\,\theta(a)\,e^{-2a/\mu}]\,\mu^\alpha\,, \quad \mu > 0\,, \qquad (29)$$

where the constants a_α are to be determined, into Eqs. (26) and (28) to obtain

$$\sum_{\alpha=0}^{N} a_\alpha\left[B_\alpha(\nu) - \theta(a)(-1)^\alpha D_\alpha(\nu) + e^{-2a/\nu}[A_\alpha(\nu) - \theta(a)(-1)^\alpha C_\alpha(\nu)]\right] = R(\nu), \qquad (30)$$

and

$$\sum_{\alpha=0}^{N} a_\alpha \, [K_\alpha - \theta(a)(-1)^\alpha T_{\alpha+1}(2a)] = K_2 + aK_1 - \theta(a)[T_3(2a) + aT_2(2a)], \tag{31}$$

where a known function is

$$R(\nu) = A_2(\nu) + aA_1(\nu) - \theta(a)[D_2(\nu) + aD_1(\nu)] + e^{-2a/\nu}\{B_2(\nu) + aB_1(\nu) - \theta(a)[C_2(\nu) + aC_1(\nu)]\}. \tag{32}$$

Here we have used the definitions

$$\nu A_\alpha(\nu) = \pi^{1/2} \int_0^\infty e^{-\mu^2} \phi(-\nu,\mu) \mu^{\alpha+1} \, d\mu, \tag{33a}$$

$$\nu B_\alpha(\nu) = \pi^{1/2} \int_0^\infty e^{-\mu^2} \phi(\nu,\mu) \mu^{\alpha+1} \, d\mu, \tag{33b}$$

$$\nu C_\alpha(\nu) = \pi^{1/2} \int_0^\infty e^{-\mu^2} \phi(-\nu,\mu) \mu^{\alpha+1} e^{-2a/\mu} \, d\mu, \tag{33c}$$

$$\nu D_\alpha(\nu) = \pi^{1/2} \int_0^\infty e^{-\mu^2} \phi(\nu,\mu) \mu^{\alpha+1} e^{-2a/\mu} \, d\mu, \tag{33d}$$

$$K_\alpha = \int_0^\infty e^{-\mu^2} \mu^{\alpha+1} \, d\mu, \tag{34}$$

and

$$T_\alpha(x) = \int_0^\infty e^{-\mu^2} e^{-x/\mu} \mu^\alpha \, d\mu. \tag{35}$$

In order to establish a solution that is accurate for all values of a, we include in Eq. (29) a term multiplied by the step function

$$\theta(a) = 1, \quad 0 \leqslant a < a_*, \tag{36a}$$

$$\theta(a) = 0, \quad a \geqslant a_*, \tag{36b}$$

where a_* is to be selected, as discussed in the next section. It is apparent that

$$K_{2n} = \frac{n!}{2}, \quad n = 0,1,2,\cdots, \tag{37a}$$

and

$$K_{2n+1} = \pi^{1/2} \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+2}}. \tag{37b}$$

We note that

$$B_0(\nu) = A_0(\nu) = \int_0^\infty e^{-\mu^2} \mu \frac{d\mu}{\mu+\nu}, \tag{38}$$

and that the remaining $B_\alpha(\nu)$ and $A_\alpha(\nu)$ can be readily generated from

$$B_\alpha(\nu) = \nu B_{\alpha-1}(\nu) - K_{\alpha-1}, \tag{39}$$

and

$$A_\alpha(\nu) = -\nu A_{\alpha-1}(\nu) + K_{\alpha-1}. \tag{40}$$

In addition

$$C_0(\nu) = \int_0^\infty e^{-\mu^2} e^{-2a/\mu} \mu \frac{d\mu}{\mu+\nu}, \tag{41}$$

$$D_0(\nu) = e^{-2a/\nu} B_0(\nu) - \int_0^\infty \mu e^{-\mu^2} \left[\frac{e^{-2a/\mu} - e^{-2a/\nu}}{\mu - \nu}\right] d\mu, \tag{42}$$

$$C_\alpha(\nu) = -\nu C_{\alpha-1}(\nu) + T_\alpha(2a), \tag{43}$$

and

$$D_\alpha(\nu) = \nu D_{\alpha-1}(\nu) - T_\alpha(2a). \tag{44}$$

If we now choose N values of $\nu \in (0,\infty)$, say ν_β, then clearly we can solve the system of algebraic equations

$$\sum_{\alpha=0}^{N} a_\alpha \Big[B_\alpha(\nu_\beta) - \theta(a)(-1)^\alpha D_\alpha(\nu_\beta) + e^{-2a/\nu_\beta} \times [A_\alpha(\nu_\beta) - \theta(a)(-1)^\alpha C_\alpha(\nu_\beta)]\Big] = R(\nu_\beta), \quad \beta = 1,2,3,\cdots,N, \tag{45a}$$

and

$$\sum_{\alpha=0}^{N} a_\alpha [K_\alpha - \theta(a)(-1)^\alpha T_{\alpha+1}(2a)] = K_2 + aK_1 - \theta(a)[T_3(2a) + aT_2(2a)] \tag{45b}$$

to find the required constants $\{a_\alpha\}$. One of the more attractive features of the F_N method is that the known coefficients in Eqs. (45) are very simply expressed. Note, for example, that the half-width a is not required in $A_\alpha(\nu)$ and $B_\alpha(\nu)$ and that the functions $A_\alpha(\nu)$ and $B_\alpha(\nu)$ are simple combinations of polynomials and the function $B_0(\nu)$. For $a \geqslant a_*$ it is thus evident that very little computer time will be required to compute the coefficients $\{a_\alpha\}$. For $a < a_*$ the coefficients in Eqs. (45) involve also the functions $C_\alpha(\nu)$, $D_\alpha(\nu)$ and $T_\alpha(2a)$; however as can be seen in the next section only a low value of N is required for $a < a_*$ to establish accurate results.

We note that the idea of using the Placzek Lemma[10] and approximating unknown surface distributions by polynomials has been used in the fields of kinetic theory[11,12] and neutron transport theory.[13] The F_N method, with $\theta(a) = 0$, clearly is related to this earlier work though it differs substantially in the way the required constants are determined.

III. NUMERICAL RESULTS

Of course to solve the system of equations given by Eq. (45) we first must select N values of $\nu_\beta \in (0,\infty)$. To have a simple and effective scheme we take the ν_β, $\beta = 1,2,3,\cdots,N$, to be the N positive zeros of the Hermite polynomial $H_{2N}(\xi)$. If we substitute Eq. (29) into Eq. (19) we find that our solution, by the F_N approximation, is

$$Q(a) = \frac{a}{3} + \pi^{-1/2}[1 + 2\theta(a)\, T_3(2a)] + \frac{2}{a}\pi^{-1/2} \times \Big[\sum_{\alpha=0}^{N} a_\alpha \, [K_{\alpha+1} - \theta(a)(-1)^\alpha T_{\alpha+2}(2a)] - \tfrac{1}{4}\pi^{1/2} + 2\theta(a) T_4(2a)\Big] + \frac{2}{a^2}\pi^{-1/2} \times \Big[\sum_{\alpha=0}^{N} a_\alpha \, [K_{\alpha+2} - \theta(a)(-1)^\alpha T_{\alpha+3}(2a)] - 1 + \theta(a)\, T_5(2a)\Big]. \tag{46}$$

TABLE I. The flow rate $Q(a)$.

$2a$	$\theta(a)$	$Q(a)$ F_0	F_2	F_4	F_6	F_8	F_{10}	"Present work"	Ref. 5	Ref. 6
0.001	1	4.2736						4.2736	4.2736	...
0.01	1	3.0495	3.0496					3.0496	3.0497	...
0.1	1	2.0314	2.0327					2.0327	2.0327	2.0327
0.5	1	1.5952	1.6018	1.6019				1.6019	1.6019	1.6018
1.0	1	1.5264	1.5385	1.5387				1.5387	1.5387	1.5386
2.0	1	1.5761	1.5944	1.5948				1.5948	1.5948	1.5948
3.0	1	1.6893	1.7099	1.7104	1.7105			1.7105	1.7105	1.7105
5.0	0	1.9504	1.9881	1.9905	1.9906	1.9906	1.9907	1.9907	1.9907	1.9907
7.0	0	2.2708	2.2932	2.2947	2.2948	2.2948	2.2949	2.2949	2.2948	2.2949
8.0	0	2.4304	2.4498	2.4510	2.4511	2.4511	2.4512	2.4512	2.4510	...
9.0	0	2.5906	2.6081	2.6090	2.6091	2.6092		2.6092	2.6090	2.6092
10.0	0	2.7514	2.7677	2.7685	2.7685	2.7686		2.7686	2.7684	2.7686
20.0	0	4.3850	4.3969	4.3973	4.3974	4.3974		4.3974	4.3971	..
30.0	0	6.0381	6.0489	6.0492	6.0492	6.0493		6.0493	6.0479	...
40.0	0	7.6976	7.7077	7.7080	7.7081			7.7081	...	...
100.0	0	17.684	17.693					17.693	...	...

In Table I we show, in addition to the results of Boffi *et al.*,[5] and Loyalka *et al.*,[6] the values obtained by using the solutions of Eq. (45) in Eq. (46). In addition to the results for various orders of the F_N approximation we list as "present work" the stable results we believe to be correct to within ± 1 in the fifth significant figure.

We have found that the approximation given by Eq. (29) with $\theta(a) = 1$ works well for all values of $2a$ listed in the table. However for $2a > 5.0$ we were able to obtain $Q(a)$ accurate to five significant figures with $\theta(a) = 0$ and thus with a greatly reduced requirement for computation time. If the desired accuracy in $Q(a)$ is reduced to four significant figures then $\theta(a) = 0$ can be used for all $2a > 1.0$. Finally we note that the F_N solution developed here is especially simple with $\theta(a) = 0$ since only $B_0(\nu)$ and the recursive formulas, Eqs. (39) and (40), are required to define the matrix elements in Eqs. (45).

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