On computing eigenvalues in radiative transfer

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The Wiener–Hopf factorization of the dispersion function is used to deduce explicit expressions for the discrete eigenvalues in the theory of radiative transfer.

I. INTRODUCTION

One of the first tasks encountered in exact[1,2] or, in some cases,[3,4] approximate analysis of the basic equation of radiative transfer[5]

$$\mu \frac{\partial}{\partial \tau} I(\tau,\mu) + I(\tau,\mu) = \frac{\omega}{2} \sum_{l=0}^{L} (2l+1) f_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I(\tau,\mu')\, d\mu', \quad (1)$$

is that of computing the discrete eigenvalues. Thus, if we substitute

$$I_\nu(\tau,\mu) = \phi(\nu,\mu)\, e^{-\tau/\nu}, \quad \nu \notin [-1,1], \quad (2)$$

into Eq. (1), then the required eigenvalues $\pm \nu_{\alpha-1}$, $\alpha = 1,2,3,\ldots,\kappa$, can readily be seen to be the zeros of the dispersion function

$$\Lambda(z) = 1 + z \int_{-1}^{1} \psi(x) \frac{dx}{x-z}, \quad (3)$$

where the characteristic function[5] is

$$\psi(\mu) = \frac{\omega}{2} \sum_{l=0}^{L} (2l+1)\, f_l\, g_l(\mu)\, P_l(\mu). \quad (4)$$

Here ω is the albedo for single scattering, and the f_l (with $f_0 = 1$) are the coefficients in a Legendre expansion of the phase function. In addition, the polynomials $g_l(\mu)$, of order l, are those introduced by Chandrasekhar,[5] i.e.,

$$g_0(\nu) = 1, \quad (5a)$$

$$g_1(\nu) = h_0 \nu, \quad (5b)$$

$$g_2(\nu) = \tfrac{1}{2}(h_0 h_1 \nu^2 - 1), \quad (5c)$$

and, in general,

$$(l+1)\, g_{l+1}(\nu) = \nu h_l\, g_l(\nu) - l g_{l-1}(\nu), \quad (6)$$

with

$$h_l = (2l+1)(1 - \omega f_l). \quad (7)$$

It is apparent that $\Lambda(z)$ is analytic in the complex plane cut from -1 to 1 along the real axis and that we can use the argument principle[6] to compute the number κ of $\pm$ pairs of zeros of $\Lambda(z)$ in the cut plane. We note, however, that although the use of the argument principle is conceptually straightforward, the calculation can pose problems for specific cases.

Before proceeding to establish some explicit expressions for the zeros of $\Lambda(z)$, we can integrate Eq. (3) to find

$$\Lambda(z) = 1 + z\psi(z) \log\left(\frac{z-1}{z+1}\right) + \omega z \sum_{l=1}^{L} (2l+1)\, f_l\, g_l(z)\, \Gamma_l(z), \quad (8)$$

where the polynomials $\Gamma_l(z)$ can be generated from

$$(2l+1)\, z\Gamma_l(z) = -\delta_{l,0} + (l+1)\, \Gamma_{l+1}(z) + l\Gamma_{l-1}(z), \quad (9)$$

with

$$\Gamma_0(z) = 0, \quad (10a)$$

$$\Gamma_1(z) = 1, \quad (10b)$$

and

$$\Gamma_2(z) = \tfrac{3}{2} z. \quad (10c)$$

II. ANALYSIS

In order to establish some useful expressions concerning $\Lambda(z)$ we first let z tend to infinity in Eq. (3) to find

$$\Lambda(\infty) = 1 - \omega \sum_{l=0}^{L} f_l W_l, \quad (11)$$

where

$$W_l = \left(\frac{2l+1}{2}\right) \int_{-1}^{1} g_l(\mu)\, P_l(\mu)\, d\mu. \quad (12)$$

We can use the recursive relation

$$(2l+1)\, \mu P_l(\mu) = (l+1)\, P_{l+1}(\mu) + l P_{l-1}(\mu), \quad (13)$$

and Eq. (6) to deduce from Eq. (12) that

$$(2l+1)\, W_{l+1} = h_l W_l, \quad (14a)$$

with

$$W_0 = 1, \quad (14b)$$

and therefore we can write

$$\Lambda(\infty) = \prod_{l=0}^{L} (1 - \omega f_l). \quad (15)$$

We can also deduce from Eq. (3) that

$$\Lambda(z) \to \Lambda(\infty) + \frac{a_2}{z^2} + \frac{a_4}{z^4} + \cdots, \quad \text{as } z \to \infty, \quad (16)$$

where

$$a_2 = -\omega \sum_{l=0}^{L} f_l \left(\frac{2l+1}{2}\right) \int_{-1}^{1} \mu^2 g_l(\mu) P_l(\mu)\, d\mu = -\omega \sum_{l=0}^{L} f_l B_l, \quad (17a)$$

and

$$a_4 = -\omega \sum_{l=0}^{L} f_l \left(\frac{2l+1}{2}\right) \int_{-1}^{1} \mu^4 g_l(\mu)\, P_l(\mu)\, d\mu = -\omega \sum_{l=0}^{L} f_l C_l. \quad (17b)$$

If we now use Eqs. (6) and (13), we find from Eq. (17a) a convenient way to compute the coefficients B_l, viz.,

$$(2l+1)\ B_{l+1} = h_l B_l + \frac{(l+2)^2}{(2l+5)(2l+3)} h_l W_l - \frac{l^2}{2l-1} W_{l-1}, \quad l \geqslant 0, \tag{18}$$

with

$$B_0 = \tfrac{1}{3}, \tag{19a}$$

$$B_1 = \tfrac{3}{5} h_0, \tag{19b}$$

and

$$B_2 = \tfrac{1}{3}\left(\tfrac{9}{7} h_0 h_1 - 1\right). \tag{19c}$$

In a similar manner we find we can express the coefficients C_l as

$$(2l+1)\ C_{l+1} = h_l C_l + \frac{(l+2)^2}{(2l+5)(2l+3)} h_l T_l - \frac{l^2}{2l-1} T_{l-1}, \quad l \geqslant 0, \tag{20a}$$

where

$$T_l = B_l + \frac{1}{2l+5}\left[\frac{(l+3)^2}{2l+7} + \frac{(l+2)^2}{2l+3}\right] W_l, \tag{20b}$$

with

$$C_0 = \tfrac{1}{5}, \tag{21a}$$

$$C_1 = \tfrac{3}{7} h_0, \tag{21b}$$

and

$$C_2 = \tfrac{2}{7}\left(\tfrac{5}{6} h_0 h_1 - 1\right). \tag{21c}$$

The work of Muskhelishvili[7] can now be used to establish a Wiener–Hopf factorization of $\Lambda(z)$, i.e.,

$$\Lambda(z) = \Lambda(\infty) X(z) X(-z) \prod_{\alpha=1}^{\kappa} (\nu_{\alpha-1}^2 - z^2), \tag{22}$$

where

$$X(z) = \frac{1}{(1-z)^\kappa} \exp\left[\frac{1}{\pi}\int_0^1 \Theta(t) \frac{dt}{t-z}\right]. \tag{23}$$

Here

$$\Theta(t) = \arg[\lambda(t) + i\pi t \psi(t)], \tag{24a}$$

or

$$\Theta(t) = \tan^{-1}\left[\frac{\pi t \psi(t)}{\lambda(t)}\right], \tag{24b}$$

with

$$\lambda(t) = 1 + t\psi(t) \ln\left(\frac{1-t}{1+t}\right) + \omega t \sum_{l=1}^{L} (2l+1) f_l\, g_l(t)\, \Gamma_l(t), \tag{25}$$

is a continuous function of t, with $\Theta(0) = 0$ and $\Theta(1) = \kappa\pi$.

Considering first the case $\kappa = 1$, we can solve Eq. (22) to find

$$\nu_0^2 = z^2 + \Lambda(z)[\Lambda(\infty) X(z) X(-z)]^{-1}, \quad \kappa = 1, \tag{26}$$

which clearly is an explicit expression for ν_0^2 for any value of z. We can find two particularly concise expressions for ν_0^2 by setting $z = 0$ or by letting $z \to \infty$ in Eq. (26):

$$\nu_0^2 = \frac{1}{\Lambda(\infty)} \exp\left[-\frac{2}{\pi}\int_0^1 \Theta(t) \frac{dt}{t}\right], \quad \kappa = 1, \tag{27}$$

and

$$\nu_0^2 = 1 - \frac{2}{\pi}\int_0^1 t\Theta(t)\, dt + \frac{\omega}{\Lambda(\infty)} \sum_{l=0}^{L} f_l B_l, \quad \kappa = 1. \tag{28}$$

For the case $\kappa = 2$, two different values of z can be used in

$$(\nu_0^2 - z^2)(\nu_1^2 - z^2) = \Lambda(z)[\Lambda(\infty) X(z) X(-z)]^{-1}, \quad \kappa = 2, \tag{29}$$

to yield two equations that can be solved simultaneously to yield ν_0^2 and ν_1^2. We use $z = 0$ and let $z \to \infty$ in Eq. (29) to find

$$\nu_0^2 = A + (A^2 - B)^{1/2}, \quad \kappa = 2, \tag{30a}$$

and

$$\nu_1^2 = A - (A^2 - B)^{1/2}, \quad \kappa = 2, \tag{30b}$$

where

$$A = 1 - \frac{1}{\pi}\int_0^1 t\Theta(t)\, dt + \frac{1}{2}\frac{\omega}{\Lambda(\infty)} \sum_{l=0}^{L} f_l B_l, \tag{31a}$$

and

$$B = \frac{1}{\Lambda(\infty)} \exp\left[-\frac{2}{\pi}\int_0^1 \Theta(t) \frac{dt}{t}\right]. \tag{31b}$$

Finally, for the case $\kappa = 3$, we deduce from Eq. (22) the three equations

$$\nu_0^2 \nu_1^2 \nu_2^2 = \frac{1}{\Lambda(\infty)} \exp\left[-\frac{2}{\pi}\int_0^1 \Theta(t)\frac{dt}{t}\right], \quad \kappa = 3, \tag{32a}$$

$$\nu_0^2 + \nu_1^2 + \nu_2^2 = 3 - \Theta_1 + \frac{\omega}{\Lambda(\infty)} \sum_{l=0}^{L} f_l B_l, \quad \kappa = 3, \tag{32b}$$

and

$$\nu_0^2\nu_1^2 + \nu_0^2\nu_2^2 + \nu_1^2\nu_2^2 = 3(1-\Theta_1) + \Theta_3 + \frac{1}{2}\Theta_1^{\,2} + (3-\Theta_1)\frac{\omega}{\Lambda(\infty)} \times \sum_{l=0}^{L} f_l B_l - \frac{\omega}{\Lambda(\infty)} \sum_{l=0}^{L} f_l C_l, \quad \kappa = 3, \tag{32c}$$

TABLE I. Numerical results.

ω	ν_0 Explicit	ν_0 Refined	ν_1 Explicit	ν_1 Refined	ν_2 Explicit	ν_2 Refined
0.1	1.030046	1.030042				
0.5	1.536814	1.536814	1.054992	1.054987		
0.95	7.480699	7.480699	1.666787	1.666787	1.019564	1.019586

where

$$\Theta_\alpha = \frac{2}{\pi}\int_0^1 t^\alpha\, \Theta(t)\, dt. \tag{33}$$

Equations (32) clearly can be solved simultaneously to yield ν_0^2, ν_1^2, and ν_2^2.

It is apparent that the foregoing procedure can be used to reduce the task of finding the zeros of $\Lambda(z)$ to one of solving a polynomial equation of order κ in the square of the desired eigenvalues. This procedure has the additional merit that it provides results for each of the desired solutions. Such results can, if necessary , be refined by using them as first approximations in an iterative solution. In order to demonstrate the accuracy of the explicit solutions given by Eqs. (27), (30), and (32) for the cases $\kappa = 1, 2$, and 3, we quote in Table I the results of Maiorino[8] who used the 80 point quadrature scheme of Gauss to compute the integrals required to evaluate the explicit solutions and a Newton–Raphson scheme to compute the refined results. The results in Table I are based on the $L = 20$ scattering law defined by[9]

$$(2l+1)\, f_l^L = \left(\frac{L+1}{2L}\right)\left[l f_{l-1}^{L-1} + (2l+1)\, f_l^{L-1} + (l+1)\, f_{l+1}^{L-1}\right], \tag{34}$$

with $f_0^L = 1$ and $f_l^L = 0$, if $l > L$.

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