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THE USE OF THE F_N METHOD FOR RADIATIVE TRANSFER PROBLEMS WITH REFLECTIVE BOUNDARY CONDITIONS

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Abstract—The F_N method is used to compute the net radiative heat flux relevant to radiative transfer in an anisotropically scattering, plane-parallel medium with specularly and diffusely reflecting boundaries.

1. INTRODUCTION

In a recent series of papers,[1-5] the F_N method relevant to radiative transfer and neutron transport theory was introduced and used to solve concisely and accurately numerous basic problems. Here we wish to demonstrate the manner in which reflective boundary conditions can be incorporated into the F_N calculations.

We consider the equation of transfer[6,7]

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \omega \sum_{l=0}^{L} \left(\frac{2l+1}{2}\right) f_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I(\tau, \mu')\, d\mu' + (1-\omega)\frac{\sigma T^4(\tau)}{\pi}, \quad (1)$$

which includes anisotropic scattering of order L. Here τ is the optical variable, μ is the direction cosine of the propagating radiation (as measured from the positive τ axis), ω is the single-scattering albedo, $T(\tau)$ is the temperature distribution in the medium and the constants f_l, $l = 0, 1, 2, \ldots, L$, with $f_0 = 1$, are the coefficients in a Legendre expansion of the phase function. For a parallel plate of thickness Δ, we seek a solution of Eq. (1) subject to boundary conditions of the form

$$I(L, \mu) = \epsilon_1\left(\frac{\sigma T_1^4}{\pi}\right) + \rho_1^s I(L, -\mu) + 2\rho_1^d \int_0^1 I(L, -\mu')\mu'\, d\mu', \quad \mu > 0, \quad (2a)$$

and

$$I(R, -\mu) = \epsilon_2\left(\frac{\sigma T_2^4}{\pi}\right) + \rho_2^s I(R, \mu) + 2\rho_2^d \int_0^1 I(R, \mu')\mu'\, d\mu', \quad \mu > 0, \quad (2b)$$

where $\tau = L$ and $\tau = R$ refer respectively to the left and right boundaries of the plate. Here, T_1 and T_2 refer to the left and right surface temperatures, σ is the Stefan–Boltzmann constant, ρ_α^s and ρ_α^d are respectively the specular and diffuse reflectivities and ϵ_1 and ϵ_2 are the emissivities.

2. ANALYSIS

We begin by expressing the radiation intensity in terms of the known elementary solutions[8] and a particular solution $I_p(\tau, \mu)$

$$I(\tau, \mu) = \sum_{\beta=0}^{\kappa-1} [A(\nu_\beta)\phi(\nu_\beta, \mu)\, e^{-\tau/\nu_\beta} + A(-\nu_\beta)\phi(-\nu_\beta, \mu)\, e^{\tau/\nu_\beta}] + \int_{-1}^{1} A(\nu)\phi(\nu, \mu)\, e^{-\tau/\nu}\, d\nu + I_p(\tau, \mu). \quad (3)$$

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Here

$$\phi(\nu,\mu)=\frac{1}{2}\omega\nu g(\nu,\mu)Pv\left(\frac{1}{\nu-\mu}\right)+\lambda(\nu)\delta(\nu-\mu), \tag{4a}$$

$$g(\nu,\mu)=\sum_{l=0}^{L}(2l+1)f_l g_l(\nu)P_l(\mu), \tag{4b}$$

$$\lambda(\nu)=1+\nu P\int_{-1}^{1}\psi(x)\frac{\mathrm{d}x}{x-\nu}, \tag{4c}$$

$$\psi(x)=\frac{1}{2}\omega g(x,x), \tag{4d}$$

and the polynomials $g_l(\nu)$ can be generated from

$$\nu h_l g_l(\nu)=(l+1)g_{l+1}(\nu)+lg_{l-1}(\nu) \tag{5a}$$

with

$$g_0(\nu)=1 \qquad \text{and} \qquad g_1(\nu)=\nu(1-\omega). \tag{5b and c}$$

In addition,

$$h_l=(2l+1)(1-\omega f_l), \tag{5d}$$

$$\phi(\nu_\beta,\mu)=\frac{1}{2}\omega\nu_\beta g(\nu_\beta,\mu)\left(\frac{1}{\nu_\beta-\mu}\right), \tag{6}$$

and the ν_β, $\beta=0,1,2,\ldots,\kappa-1$, denote the positive zeros of

$$\Lambda(z)=1+z\int_{-1}^{1}\psi(x)\frac{\mathrm{d}x}{x-z} \tag{7}$$

in the complex plane cut from -1 to 1 along the real axis. The expansion coefficients $A(\pm\nu_\beta)$ and $A(\nu)$, $\nu\in(-1,1)$, appearing in Eq. (3) are to be determined by the boundary conditions. We note that the solution given by Eq. (3) satisfies Eq. (1) exactly.

As discussed in Ref. 2, the full-range orthogonality relations concerning the functions $\phi(\xi,\mu)$ can be used to develop a system of singular integral equations and constraints for the distribution of radiation at the surfaces of the considered plate (the same singular integral equations and constraints were reported by Bowden, McCrosson, and Rhodes[9]). Thus, we let $I^*(\tau,\mu)=I(\tau,\mu)-I_p(\tau,\mu)$ and consider

$$\int_0^1\mu\phi(\xi,\mu)I^*(L,-\mu)\,\mathrm{d}\mu+\mathrm{e}^{-\Delta/\xi}\int_0^1\mu\phi(-\xi,\mu)I^*(R,\mu)\,\mathrm{d}\mu$$
$$=\int_0^1\mu\phi(-\xi,\mu)I^*(L,\mu)\,\mathrm{d}\mu+\mathrm{e}^{-\Delta/\xi}\int_0^1\mu\phi(\xi,\mu)I^*(R,-\mu)\,\mathrm{d}\mu,\quad \xi\in P, \tag{8a}$$

and

$$\int_0^1\mu\phi(\xi,\mu)I^*(R,\mu)\,\mathrm{d}\mu+\mathrm{e}^{-\Delta/\xi}\int_0^1\mu\phi(-\xi,\mu)I^*(L,-\mu)\,\mathrm{d}\mu$$
$$=\int_0^1\mu\phi(-\xi,\mu)I^*(R,-\mu)\,\mathrm{d}\mu+\mathrm{e}^{-\Delta/\xi}\int_0^1\mu\phi(\xi,\mu)I^*(L,\mu)\,\mathrm{d}\mu,\quad \xi\in P, \tag{8b}$$

where $P\Rightarrow\{\nu_\beta\}\cup(0,1)$. Equations (8) are exact; however, we wish now to introduce the F_N

method. We let

$$I(L, -\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \quad \mu > 0, \tag{9a}$$

and

$$I(R, \mu) = \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha, \quad \mu > 0. \tag{9b}$$

If we substitute Eqs. (9) into Eqs. (8) and use the boundary conditions given by Eqs. (2), we find we can evaluate analytically the required integrals to obtain

$$\sum_{\alpha=0}^{N} a_\alpha \left[B_\alpha(\xi) - \rho_1{}^s A_\alpha(\xi) - 2\rho_1{}^d \left(\frac{1}{\alpha+2}\right) A_0(\xi) \right] + e^{-\Delta/\xi} \sum_{\alpha=0}^{N} b_\alpha \left[A_\alpha(\xi) - \rho_2{}^s B_\alpha(\xi) - 2\rho_2{}^d \left(\frac{1}{\alpha+2}\right) B_0(\xi) \right] = K_1(\xi), \quad \xi \in P, \tag{10a}$$

and

$$\sum_{\alpha=0}^{N} b_\alpha \left[B_\alpha(\xi) - \rho_2{}^s A_\alpha(\xi) - 2\rho_2{}^d \left(\frac{1}{\alpha+2}\right) A_0(\xi) \right] + e^{-\Delta/\xi} \sum_{\alpha=0}^{N} a_\alpha \left[A_\alpha(\xi) - \rho_1{}^s B_\alpha(\xi) - 2\rho_1{}^d \left(\frac{1}{\alpha+2}\right) B_0(\xi) \right] = K_2(\xi), \quad \xi \in P, \tag{10b}$$

where

$$K_1(\xi) = \epsilon_1 \left(\frac{\sigma T_1{}^4}{\pi}\right) A_0(\xi) - \frac{2}{\omega\xi} \int_{-1}^{1} \mu \phi(-\xi, \mu) I_p(L, \mu)\, d\mu + e^{-\Delta/\xi} \left[\epsilon_2 \left(\frac{\sigma T_2{}^4}{\pi}\right) B_0(\xi) + \frac{2}{\omega\xi} \int_{-1}^{1} \mu \phi(-\xi, \mu) I_p(R, \mu)\, d\mu \right] \tag{11a}$$

and

$$K_2(\xi) = \epsilon_2 \left(\frac{\sigma T_2{}^4}{\pi}\right) A_0(\xi) + \frac{2}{\omega\xi} \int_{-1}^{1} \mu \phi(\xi, \mu) I_p(R, \mu)\, d\mu + e^{-\Delta/\xi} \left[\epsilon_1 \left(\frac{\sigma T_1{}^4}{\pi}\right) B_0(\xi) - \frac{2}{\omega\xi} \int_{-1}^{1} \mu \phi(\xi, \mu) I_p(L, \mu)\, d\mu \right]. \tag{11b}$$

We note that the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ can be readily computed from[3]

$$A_{\alpha+1}(\xi) = -\xi A_\alpha(\xi) + \sum_{l=0}^{L} (2l+1)(-1)^l f_l g_l(\xi) \Delta_{\alpha,l} \tag{12a}$$

with

$$A_0(\xi) = 1 - \frac{2}{\omega} \xi \psi(\xi) \log(1 + 1/\xi) + \sum_{l=1}^{L} (2l+1) f_l g_l(\xi) \Pi_l(\xi) \tag{12b}$$

and

$$B_{\alpha+1}(\xi) = \xi B_\alpha(\xi) - \sum_{l=0}^{L} (2l+1) f_l g_l(\xi) \Delta_{\alpha,l}; \tag{13a}$$

here,

$$B_0(\xi) = \frac{2}{\omega} - 2 + A_0(\xi). \tag{13b}$$

In addition,

$$\Delta_{\alpha,l} = \int_0^1 \mu^{\alpha+1} P_l(\mu)\,\mathrm{d}\mu \tag{14}$$

is readily available from

$$\Delta_{\alpha,l+2} = \left(\frac{1+\alpha-l}{4+\alpha+l}\right)\Delta_{\alpha,l} \tag{15a}$$

with

$$\Delta_{\alpha,0} = 1/(\alpha+2) \tag{15b}$$

and

$$\Delta_{\alpha,1} = 1/(\alpha+3). \tag{15c}$$

The polynomials $\Pi_l(\xi)$ required in Eq. (12b) can be generated from

$$(2l+1)\xi\Pi_l(\xi) = (-1)^l(2l+1)\Delta_{0,l} + (l+1)\Pi_{l+1}(\xi) + l\Pi_{l-1}(\xi), \tag{16}$$

with

$$\Pi_0(\xi) = 1, \tag{17a}$$

$$\Pi_1(\xi) = \xi - \tfrac{1}{2}, \tag{17b}$$

and

$$\Pi_2(\xi) = \tfrac{3}{2}\xi(\xi - \tfrac{1}{2}). \tag{17c}$$

Although Eqs. (10) cannot be satisfied for all $\xi \in P$, we can select $N+1$ different values of $\xi \in P$, say $\{\xi_j\}$, and solve the following system of $2(N+1)$ linear algebraic equations for a_α and b_α, $\alpha = 0, 1, 2, \ldots, N$:

$$\sum_{\alpha=0}^{N} a_\alpha\left[B_\alpha(\xi_j) - \rho_1{}^s A_\alpha(\xi_j) - 2\rho_1{}^d\left(\frac{1}{\alpha+2}\right)A_0(\xi_j)\right] + \mathrm{e}^{-\Delta/\xi_j}\sum_{\alpha=0}^{N} b_\alpha\left[A_\alpha(\xi_j) - \rho_2{}^s B_\alpha(\xi_j) - 2\rho_2{}^d\left(\frac{1}{\alpha+2}\right)B_0(\xi_j)\right] = K_1(\xi_j) \tag{18a}$$

and

$$\sum_{\alpha=0}^{N} b_\alpha\left[B_\alpha(\xi_j) - \rho_2{}^s A_\alpha(\xi_j) - 2\rho_2{}^d\left(\frac{1}{\alpha+2}\right)A_0(\xi_j)\right] + \mathrm{e}^{-\Delta/\xi_j}\sum_{\alpha=0}^{N} a_\alpha\left[A_\alpha(\xi_j) - \rho_1{}^s B_\alpha(\xi_j) - 2\rho_1{}^d\left(\frac{1}{\alpha+2}\right)B_0(\xi_j)\right] = K_2(\xi_j). \tag{18b}$$

We note that the F_N method yields first of all the exit distributions of radiation $I(L, -\mu)$ and $I(R, \mu)$, $\mu > 0$. However, once these quantities are established, the complete solution is given by Eq. (3) and

$$A(\pm\xi)\,e^{\mp L/\xi} = \pm\frac{1}{N(\xi)}\int_{-1}^{1} \mu\phi(\pm\xi,\mu)[I(L,\mu) - I_p(L,\mu)]\,d\mu \tag{19a}$$

or

$$A(\pm\xi)\,e^{\mp R/\xi} = \pm\frac{1}{N(\xi)}\int_{-1}^{1} \mu\phi(\pm\xi,\mu)[I(R,\mu) - I_p(R,\mu)]\,d\mu, \tag{19b}$$

where

$$N(\nu_\beta) = \tfrac{1}{2}\omega\nu_\beta^2 g(\nu_\beta,\nu_\beta)\Lambda'(\nu_\beta) \tag{20}$$

and

$$N(\nu) = \nu[\lambda^2(\nu) + \tfrac{1}{4}\pi^2\omega^2\nu^2 g^2(\nu,\nu)]. \tag{21}$$

3. NUMERICAL RESULTS

In order to demonstrate the accuracy of the F_N method we consider now the specific case of constant heat generation, i.e. $T(\tau) = T$, a constant. Thus, the appropriate particular solution is $I_p(\tau,\mu) = \sigma T^4/\pi$, and Eqs. (11) become

$$K_1(\xi) = \epsilon_1\left(\frac{\sigma T_1^4}{\pi}\right)A_0(\xi) + \frac{2}{\omega}(1-\omega)\left(\frac{\sigma T^4}{\pi}\right) + e^{-\Delta/\xi}\left[\epsilon_2\left(\frac{\sigma T_2^4}{\pi}\right)B_0(\xi) - \frac{2}{\omega}(1-\omega)\left(\frac{\sigma T^4}{\pi}\right)\right] \tag{22a}$$

and

$$K_2(\xi) = \epsilon_2\left(\frac{\sigma T_2^4}{\pi}\right)A_0(\xi) + \frac{2}{\omega}(1-\omega)\left(\frac{\sigma T^4}{\pi}\right) + e^{-\Delta/\xi}\left[\epsilon_1\left(\frac{\sigma T_1^4}{\pi}\right)B_0(\xi) - \frac{2}{\omega}(1-\omega)\left(\frac{\sigma T^4}{\pi}\right)\right]. \tag{22b}$$

We also now consider that the two surfaces have the same reflecting properties, i.e. $\rho_1^s = \rho_2^s$ and $\rho_1^d = \rho_2^d$ and $\epsilon + \rho^s + \rho^d \le 1$; thus, it is apparent that, if we choose $L = -\Delta/2$, we can express the desired solution as

$$I(\tau,\mu) = \left(\frac{\sigma T^4}{\pi}\right)\Phi(\tau,\mu) + \epsilon_1\left(\frac{\sigma T_1^4}{\pi}\right)\theta(\tau,\mu) + \epsilon_2\left(\frac{\sigma T_2^4}{\pi}\right)\theta(-\tau,-\mu). \tag{23}$$

Here $\Phi(\tau,\mu)$ satisfies Eq. (1) with $\sigma T^4/\pi = 1$, $\Phi(\tau,\mu) = \Phi(-\tau,-\mu)$, and

$$\Phi(-\Delta/2,\mu) = \rho^s\Phi(-\Delta/2,-\mu) + 2\rho^d\int_0^1 \Phi(-\Delta/2,-\mu')\mu'\,d\mu', \quad \mu > 0. \tag{24}$$

Also $\theta(\tau,\mu)$ satisfies Eq. (1) with $\sigma T^4/\pi = 0$,

$$\theta(-\Delta/2,\mu) = 1 + \rho^s\theta(-\Delta/2,-\mu) + 2\rho^d\int_0^1 \theta(-\Delta/2,-\mu')\mu'\,d\mu', \quad \mu > 0, \tag{25a}$$

and

$$\theta(\Delta/2,-\mu) = \rho^s\theta(\Delta/2,\mu) + 2\rho^d\int_0^1 \theta(\Delta/2,\mu')\mu'\,d\mu', \quad \mu > 0. \tag{25b}$$

The first basic problem clearly is symmetric and, therefore, to establish $\Phi(\tau,\mu)$ we need only consider Eq. (18a) and solve, in the F_N approximation, $(N+1)$ linear algebraic equations. For the second problem, we have the choice of solving $2(N+1)$ simultaneous equations or, after we express $\theta(\tau, \mu)$ in terms of symmetric and antisymmetric components, solving two independent systems of $(N+1)$ equations.

For our numerical calculation, we consider a scattering law, shown in Table 1, deduced from the Mie scattering theory[7,10] and relevant of $x = 3$ and $n = 1.2$, where n is the index of refraction and x is the size parameter. We consider three cases for the single-scattering albedo, $\omega = 0.2$, $\omega = 0.8$, and $\omega = 0.95$, and in Table 2 we list the discrete eigenvalues basic to these parameters. To calculate these discrete eigenvalues, an iterative solution of

$$\Lambda(z) = 1 + z\psi(z)\log\left(\frac{z-1}{z+1}\right) + \omega z \sum_{l=1}^{L} (2l+1) f_l g_l(z) \Gamma_l(z) = 0 \tag{26}$$

was achieved after using the exact expressions[11] for ν_0 and ν_1 (when appropriate) as initial values. Here the polynomials $\Gamma_l(z)$ are defined by

$$(2l+1)z\Gamma_l(z) = -\delta_{l,0} + (l+1)\Gamma_{l+1}(z) + l\Gamma_{l-1}(z) \tag{27}$$

with

$$\Gamma_0(z) = 0. \tag{28}$$

As discussed, we solved two problems, one for $\Phi(\tau, \mu)$ and the other for $\theta(\tau, \mu)$. The F_N equations for the first problem are given by

$$\sum_{\alpha=0}^{N} a_\alpha \left\{ B_\alpha(\xi_j)[1 - \rho^s \,\mathrm{e}^{-\Delta/\xi_j}] + A_\alpha(\xi_j)[\mathrm{e}^{-\Delta/\xi_j} - \rho^s] - \frac{2\rho^d}{\alpha+2}[A_0(\xi_j) + B_0(\xi_j)\,\mathrm{e}^{-\Delta/\xi_j}] \right\} = (1 - \mathrm{e}^{-\Delta/\xi_j})\frac{2}{\omega}(1-\omega) \tag{29}$$

and, for the second problem, by

$$\sum_{\alpha=0}^{N} a_\alpha^* \left[B_\alpha(\xi_j) - \rho^s A_\alpha(\xi_j) - \frac{2\rho^d}{\alpha+2} A_0(\xi_j) \right] + \mathrm{e}^{-\Delta/\xi_j} \sum_{\alpha=0}^{N} b_\alpha^* \left[A_\alpha(\xi_j) - \rho^s B_\alpha(\xi_j) - \frac{2\rho^d}{\alpha+2} B_0(\xi_j) \right] = A_0(\xi_j) \tag{30a}$$

and

$$\sum_{\alpha=0}^{N} b_\alpha^* \left[B_\alpha(\xi_j) - \rho^s A_\alpha(\xi_j) - \frac{2\rho^d}{\alpha+2} A_0(\xi_j) \right] + \mathrm{e}^{-\Delta/\xi_j} \sum_{\alpha=0}^{N} a_\alpha^* \left[A_\alpha(\xi_j) - \rho^s B_\alpha(\xi_j) - \frac{2\rho^d}{\alpha+2} B_0(\xi_j) \right] = \mathrm{e}^{-\Delta/\xi_j} B_0(\xi_j), \tag{30b}$$

Table 1. Scattering law.

l	$(2l+1)f_l$
0	1.0
1	2.35789
2	2.76628
3	2.20142
4	1.24514
5	0.51215
6	0.16096
7	0.03778
8	0.00667
9	0.00081
10	0.00000

Table 2. The discrete eigenvalues.

ω	ν_0	ν_1
0.2	1.06303332	—
0.8	2.43716171	1.05196601
0.95	5.34762059	1.14901459

where we have used the approximations

$$\Phi(-\Delta/2,-\mu)=\Phi(\Delta/2,\mu)=\sum_{\alpha=0}^{N} a_\alpha\mu^\alpha,\quad \mu>0, \tag{31}$$

$$\theta(-\Delta/2,-\mu)=\sum_{\alpha=0}^{N} a_\alpha^*\mu^\alpha,\quad \mu>0, \tag{32a}$$

and

$$\theta(\Delta/2,\mu)=\sum_{\alpha=0}^{N} b_\alpha^*\mu^\alpha,\quad \mu>0. \tag{32b}$$

To solve the above linear system of equations, we have selected $\xi_\beta=\nu_\beta$, $\beta=0,1,\ldots,\kappa-1$, and the remaining ξ_β as given by $\xi_{j+\kappa-1}=(2j-1)/[2(N-\kappa+1)]$, $j=1,2,\ldots,(N-\kappa+1)$, where N is the order of approximation.

Here we wish to report the net radiative heat flux at the boundaries, i.e.

$$q(\pm\Delta/2)=\int_{-1}^{1} I(\pm\Delta/2,\mu)\mu\,\mathrm{d}\mu, \tag{33}$$

or, in terms of the forward and backward partial fluxes, we can write

$$q(\pm\Delta/2)=q^+(\pm\Delta/2)-q^-(\pm\Delta/2). \tag{34}$$

Using Eq. (23), we note that

$$q^\pm(\pm\Delta/2)=\left(\frac{\sigma T^4}{\pi}\right)\varphi^\pm(\pm\Delta/2)+\epsilon_1\left(\frac{\sigma T_1^4}{\pi}\right)\vartheta^\pm(\pm\Delta/2)+\epsilon_2\left(\frac{\sigma T_2^4}{\pi}\right)\vartheta^\mp(\mp\Delta/2), \tag{35}$$

where

$$\varphi^\pm(\pm\Delta/2)=\int_0^1 \Phi(\pm\Delta/2,\pm\mu)\mu\,\mathrm{d}\mu \tag{36a}$$

and

$$\vartheta^\pm(\pm\Delta/2)=\int_0^1 \theta(\pm\Delta/2,\pm\mu)\mu\,\mathrm{d}\mu. \tag{36b}$$

In terms of the F_N approximation, we can write

$$\varphi^-(-\Delta/2)=\varphi^+(\Delta/2)=\sum_{\alpha=0}^{N}\left(\frac{a_\alpha}{\alpha+2}\right), \tag{37a}$$

$$\varphi^+(-\Delta/2)=\varphi^-(\Delta/2)=(\rho^s+\rho^d)\sum_{\alpha=0}^{N}\left(\frac{a_\alpha}{\alpha+2}\right), \tag{37b}$$

$$\vartheta^-(-\Delta/2)=\sum_{\alpha=0}^{N}\left(\frac{a_\alpha^*}{\alpha+2}\right), \tag{37c}$$

$$\vartheta^+(\Delta/2)=\sum_{\alpha=0}^{N}\left(\frac{b_\alpha^*}{\alpha+2}\right), \tag{37d}$$

$$\vartheta^-(\Delta/2)=(\rho^s+\rho^d)\sum_{\alpha=0}^{N}\left(\frac{b_\alpha^*}{\alpha+2}\right), \tag{37e}$$

and

$$\vartheta^+(-\Delta/2)=\frac{1}{2}+(\rho^s+\rho^d)\sum_{\alpha=0}^{N}\left(\frac{a_\alpha^*}{\alpha+2}\right). \tag{37f}$$

Table 3. Partial heat fluxes for $\omega = 0.2$ and $\Delta = 1$.

Wall reflectivity		Partial heat					
ρ^s	ρ^d	fluxes	F_0	F_1	F_3	F_5	"Exact"
		$\varphi^-(L)$	0.3776	0.4186	0.4163	0.4164	0.4164
0.0	0.5	$\vartheta^-(L)$	0.0413	0.02323	0.02461	0.02458	0.02458
		$\vartheta^+(R)$	0.2035	0.1315	0.1427	0.1426	0.1426
		$\varphi^-(L)$	0.3776	0.4158	0.4139	0.4140	0.4140
0.5	0.0	$\vartheta^-(L)$	0.0413	0.02763	0.02861	0.02863	0.02863
		$\vartheta^+(R)$	0.2035	0.1408	0.1436	0.1434	0.1434
		$\varphi^-(L)$	0.3776	0.4173	0.4152	0.4153	0.4153
0.25	0.25	$\vartheta^-(L)$	0.0413	0.02537	0.02657	0.02656	0.02656
		$\vartheta^+(R)$	0.2035	0.1400	0.1430	0.1429	0.1429

The basic quantities $\varphi^-(-\Delta/2)$, $\vartheta^-(-\Delta/2)$, and $\vartheta^+(\Delta/2)$ clearly can be used with Eqs. (37) to deduce the net radiative heat flux, as given by Eq. (35), and thus these quantities are reported in Tables 3–5 for the considered cases. In order to illustrate the effectiveness of the F_N method for this problem, we list in Tables 3–5 the results predicted by various F_N approximations. We also include the "exact" results deduced from the F_N method as N varied between 10 and 20. Because the lowest-order approximation is particularly simple and also accurate for many cases, we include in Tables 3–5 the F_0 results. Clearly the considered cases of $\omega = 0.8$ and $\omega = 0.95$ can be solved by the F_0 approximation to within 8% of error, which is adequate for many engineering applications.

We note that in contrast to the Eddington approximation,[7] the F_0 approximation, though particularly concise, includes the effect of the complete scattering law. For the cases considered the F_3 approximation is generally accurate to at least three significant figures, and for all

Table 4. Partial heat fluxes for $\omega = 0.8$ and $\Delta = 1$.

Wall reflectivity		Partial heat					
ρ^s	ρ^d	fluxes	F_0	F_1	F_2	F_3	"Exact"
		$\varphi^-(L)$	0.2203	0.2341	0.2348	0.2342	0.2342
0.0	0.5	$\vartheta^-(L)$	0.1640	0.1609	0.1633	0.1639	0.1639
		$\vartheta^+(R)$	0.3953	0.3710	0.3671	0.3678	0.3678
		$\varphi^-(L)$	0.2203	0.2330	0.2328	0.2323	0.2323
0.5	0.0	$\vartheta^-(L)$	0.1640	0.1635	0.1690	0.1696	0.1696
		$\vartheta^+(R)$	0.3953	0.3705	0.3654	0.3659	0.3659
		$\varphi^-(L)$	0.2203	0.2336	0.2339	0.2333	0.2333
0.25	0.25	$\vartheta^-(L)$	0.1640	0.1622	0.1661	0.1667	0.1666
		$\vartheta^+(R)$	0.3953	0.3706	0.3661	0.3667	0.3668

Table 5. Partial heat fluxes for $\omega = 0.95$ and $\Delta = 1$.

Wall reflectivity		Partial heat					
ρ^s	ρ^d	fluxes	F_0	F_1	F_2	F_3	"Exact"
		$\varphi^-(L)$	0.0828	0.08496	0.08502	0.08494	0.08494
0.0	0.5	$\vartheta^-(L)$	0.2890	0.2956	0.2982	0.2982	0.2982
		$\vartheta^+(R)$	0.5453	0.5344	0.5318	0.5319	0.5319
		$\varphi^-(L)$	0.0828	0.08474	0.08468	0.08462	0.08462
0.5	0.0	$\vartheta^-(L)$	0.2890	0.02979	0.3029	0.3031	0.3030
		$\vartheta^+(R)$	0.5453	0.5326	0.5277	0.5277	0.5277
		$\varphi^-(L)$	0.0828	0.08487	0.08487	0.08480	0.08480
0.25	0.25	$\vartheta^-(L)$	0.2890	0.2969	0.3006	0.3007	0.3006
		$\vartheta^+(R)$	0.5453	0.5334	0.5297	0.5297	0.5298

considered cases of $\omega = 0.8$ and $\omega = 0.95$ the F_3 approximation is accurate to four significant figures. We consider this excellent especially since the computation time, on the IBM 370/165 machine, for the F_3 approximation of $\varphi^-(-\Delta/2)$, $\vartheta^-(-\Delta/2)$ and $\vartheta^+(\Delta/2)$ for a set of given values of ρ^s, ρ^d, Δ, and ω is less than 10 seconds, which includes the calculation of the required discrete eigenvalues.

In conclusion, we note that once the constants a_α, a_α^* and b_α^* have been established the complete intensity $I(\tau, \mu)$ can be found in the manner discussed by Siewert and Maiorino.[12]

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