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An Analytical Solution to a Matrix Riemann-Hilbert Problem

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I. Introduction

Basic to exact analysis [1] of

$$\mu \frac{\partial}{\partial x} \mathbf{\Psi}(x, \mu) + \mathbf{\Psi}(x, \mu) = \frac{1}{\sqrt{\pi}} \mathbf{Q}(\mu) \int_{-\infty}^{\infty} \mathbf{Q}^T(\mu') \mathbf{\Psi}(x, \mu') \, e^{-\mu'^2} \, d\mu', \tag{1}$$

the equation formulated by Cercignani [2] to describe temperature-density variations in plane-parallel media, is the solution to the Riemann-Hilbert problem [3]

$$\mathbf{\Phi}^+(\mu) = \mathbf{G}(\mu)\mathbf{\Phi}^-(\mu), \quad \mu \in (0, \infty). \tag{2}$$

Here for the linearized BGK model

$$\mathbf{Q}(\mu) = \begin{bmatrix} \sqrt{\frac{2}{3}}\,(\mu^2 - \frac{1}{2}) & 1 \\ \sqrt{\frac{2}{3}} & 0 \end{bmatrix} \tag{3}$$

and the elements of the two-vector $\mathbf{\Psi}(x, \mu)$ are related to the temperature and density variations.

Given that the $\mathbf{G}$ matrix in Eqn. (2) can be expressed as

$$\mathbf{G}(\mu) = \mathbf{\Lambda}^+(\mu)[\mathbf{\Lambda}^-(\mu)]^{-1}, \tag{4}$$

where

$$\mathbf{\Lambda}(z) = \mathbf{I} + z \int_{-\infty}^{\infty} \mathbf{\Psi}(\mu) \frac{d\mu}{\mu - z}, \tag{5}$$

with

$$\mathbf{\Psi}(\mu) = \frac{1}{\sqrt{\pi}} \mathbf{Q}^T(\mu)\mathbf{Q}(\mu) \, e^{-\mu^2}, \tag{6}$$

we seek a 2×2 matrix $\mathbf{\Phi}(z)$ that is analytic in the complex plane cut along the positive real axis such that $\det \mathbf{\Phi}(z) \neq 0$ and such that the limiting values of $\mathbf{\Phi}(z)$, i.e. $\mathbf{\Phi}^{\pm}(\mu)$, as z approaches the cut from above and below satisfy Eqn. (2).

II. Analysis

We note first of all that we can write

$$\mathbf{Q}^{-T}(z)\mathbf{\Lambda}(z)\mathbf{Q}^{-1}(z) = \mathbf{\Pi}(z) + f(z)\mathbf{I} \tag{7}$$

where we use $-T$ to denote the inverse transpose operation,

$$\mathbf{\Pi}(z) = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{7}{4} - \frac{1}{2}z^2 \end{bmatrix}, \tag{8}$$

and

$$f(z) = \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} \frac{d\mu}{\mu - z}. \tag{9}$$

It is apparent that

$$\mathbf{W}(z) = \mathbf{Q}^{-T}(z)\mathbf{\Lambda}(z)\mathbf{Q}^{-1}(z) \tag{10}$$

can be diagonalized by a similarity transformation involving at worst

$$R(z) = \sqrt{q(z)} \tag{11}$$

where $q(z)$ is a polynomial. This idea of diagonalizing the considered matrix Riemann-Hilbert problem by non-analytic functions was reported originally by Darrozès [4]; however because this first solution is not of finite degree at infinity, the result of Darrozès is not correct. Cercignani in a recent work [5] has reported an elegant method for correcting Darrozès' result at infinity. Here we use a result, similar to that obtained by Cercignani [5], to construct a *canonical* solution of the considered matrix Riemann-Hilbert problem and subsequently the related **H** matrix [1].

We find that

$$\mathbf{S}(z) = \begin{bmatrix} \sqrt{\frac{3}{2}} & \frac{1}{2}[R(z) - z^2 - \frac{1}{2}] \\ -\sqrt{\frac{3}{2}} & \frac{1}{2}[R(z) + z^2 + \frac{1}{2}] \end{bmatrix}, \tag{12}$$

with

$$q(z) = z^4 - 3z^2 + \tfrac{25}{4}, \tag{13}$$

is such that

$$\mathbf{S}(z)\mathbf{\Lambda}(z)\mathbf{Q}(z)^{-1}\mathbf{Q}^{-T}(z)\mathbf{S}^{-1}(z) = \mathbf{\Omega}(z), \tag{14}$$

where the diagonal $\mathbf{\Omega}(z)$ has elements

$$\Omega_{1,2}(z) = \tfrac{1}{4}[\tfrac{11}{2} - z^2 \pm R(z) + 4f(z)]. \tag{15}$$

The sectionally analytic function $R(z)$ has branch points $\pm a$ and $\pm \bar{a}$, where $a = \sqrt{2} + i\sqrt{2}/2$, and here we consider the branch of $R(z)$ that is analytic in the complex plane cut along Γ, the two straight-line segments $[-\bar{a}, a]$ and $[-a, \bar{a}]$.

If we now write our desired solution as

$$\mathbf{\Phi}(z) = \mathbf{S}^{-1}(z)\mathbf{U}(z)\mathbf{S}(z), \tag{16}$$

and require

$$\mathbf{\Phi}^+(\mu) = \mathbf{G}(\mu)\mathbf{\Phi}^-(\mu), \quad \mu \in (0, \infty), \tag{17}$$

and

$$\mathbf{\Phi}^+(\tau) = \mathbf{\Phi}^-(\tau), \quad \tau \in \Gamma, \tag{18}$$

then the sectionally analytic $\mathbf{U}(z)$ must satisfy

$$\mathbf{U}^+(\mu) = \mathbf{G}_0(\mu)\mathbf{U}^-(\mu), \quad \mu \in (0, \infty), \tag{19}$$

and

$$\mathbf{U}^+(\tau)\mathbf{T} = \mathbf{T}\mathbf{U}^-(\tau), \quad \tau \in \Gamma. \tag{20}$$

Here

$$\mathbf{G}_0(\mu) = \mathbf{S}(\mu)\mathbf{G}(\mu)\mathbf{S}^{-1}(\mu) = \mathbf{\Omega}^+(\mu)[\mathbf{\Omega}^-(\mu)]^{-1} \tag{21}$$

and

$$\mathbf{T} = -\mathbf{S}^+(\tau)[\mathbf{S}^-(\tau)]^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{22}$$

and since $\mathbf{G}_0(\mu)$ is diagonal, we consider

$$\mathbf{U}(z) = \begin{bmatrix} U_1(z) & 0 \\ 0 & U_2(z) \end{bmatrix}. \tag{23}$$

If we let

$$\gamma_\alpha(\mu) = \frac{\Omega_\alpha^+(\mu)}{\Omega_\alpha^-(\mu)}, \quad \mu \in (0, \infty), \quad \alpha = 1 \text{ or } 2, \tag{24}$$

$$A(\mu) = \frac{\gamma_1(\mu)}{\gamma_2(\mu)}, \tag{25}$$

and

$$B(\mu) = \gamma_1(\mu)\gamma_2(\mu), \tag{26}$$

then we find, since $R^+(\tau) = -R^-(\tau)$, $\tau \in \Gamma$, that

$$U_\alpha^*(z) = \exp\left(\frac{1}{4\pi i}\int_0^\infty \left\{\log B(x) - (-1)^\alpha \frac{R(z)}{R(x)} [\log A(x) + 2k\pi i\Delta(x)]\right\} \frac{dx}{x - z}\right) \tag{27}$$

can be used in Eqn. (23) to yield a solution of Eqns. (19) and (20). Here k is an integer and

$$\Delta(x) = 1, \quad x \in (x_0, x_1), \\ = 0, \quad \text{otherwise}. \tag{28}$$

Since $A(x)$ and $B(x)$ have unit magnitude, we write Eqn. (27) as

$$U_\alpha^*(z) = \exp\left(\frac{1}{4\pi}\int_0^\infty \left\{\arg B(x) - (-1)^\alpha \frac{R(z)}{R(x)} [\arg A(x) + 2k\pi\Delta(x)]\right\} \frac{dx}{x - z}\right) \tag{29}$$

and use continuous values of $\arg A(x)$ and $\arg B(x)$ such that $\arg A(\infty) = \arg B(\infty) = 0$. We note that $R(z) \to z^2$ as $z \to \infty$, and thus $U_\alpha^*(z)$ will have an essential singularity at infinity unless we impose the condition

$$\int_0^\infty [\arg A(x) + 2k\pi\Delta(x)] \frac{dx}{R(x)} = 0 \tag{30}$$

or

$$\int_{x_0}^{x_1} \frac{dx}{R(x)} = -\frac{1}{2k\pi}\int_0^\infty \arg A(x) \frac{dx}{R(x)}. \tag{31}$$

We write

$$\gamma_\alpha(x) = e^{2i\vartheta_\alpha(x)} \tag{32}$$

with

$$\vartheta_\alpha(x) = \tan^{-1}\left(\frac{4\sqrt{\pi}x\, e^{-x^2}}{M(x) - (-1)^\alpha R(x)}\right) \tag{33}$$

defined such that $\vartheta_\alpha(x)$ varies continuously from 0 to π, and thus

$$\arg A(x) = 2[\vartheta_1(x) - \vartheta_2(x)] = -2\theta(x) \tag{34}$$

and

$$\arg B(x) = 2[\vartheta_1(x) + \vartheta_2(x) - 2\pi] = 2\phi(x). \tag{35}$$

Here

$$M(x) = -x^2 + \tfrac{11}{2} - 8x\, e^{-x^2}\int_0^x e^{y^2}\, dy. \tag{36}$$

It is thus apparent that the essential singularity at infinity can be removed by imposing the condition

$$\int_{x_0}^{x_1} \frac{dx}{R(x)} = \frac{1}{k\pi}\int_0^\infty \theta(x) \frac{dx}{R(x)} \tag{37}$$

which for $x_0 = 0$, clearly has a solution x_1 for any integer $k \geq 1$. We thus use $x_0 = 0$, $k = 2$ and the corresponding x_1 computed by Yuan [6]:

$$x_1 = 0.4232585948 \cdots, \quad k = 2. \tag{38}$$

Thus

$$U_\alpha^*(z) = \exp\left(\frac{1}{2\pi}\int_0^\infty \left[\phi(x) + (-1)^\alpha \frac{R(z)}{R(x)}\theta(x)\right]\frac{dx}{x - z} + (-1)^{\alpha+1}R(z)\int_0^{x_1} \frac{1}{R(x)}\frac{dx}{x - z}\right), \tag{39}$$

and to remove the singularity at $z = x_1$ we write our final results as

$$U_\alpha(z) = (z - x_1)U_\alpha^*(z). \tag{40}$$

Now since

$$\det \mathbf{\Phi}(z) = U_1(z)U_2(z), \tag{41}$$

or

$$\det \mathbf{\Phi}(z) = (z - x_1)^2 \exp\left(\frac{1}{\pi}\int_0^\infty \phi(x)\frac{dx}{x - z}\right), \tag{42}$$

it is clear that

$$\mathbf{\Phi}(z) = \mathbf{S}^{-1}(z)\begin{bmatrix} U_1(z) & 0 \\ 0 & U_2(z) \end{bmatrix}\mathbf{S}(z) \tag{43}$$

is not a canonical solution [3]. If we let $\mathbf{\Phi}_0(z)$ denote a canonical solution (with normal form at infinity) then

$$\mathbf{\Phi}(z) = \mathbf{\Phi}_0(z)\mathbf{P}_*(z) \tag{44}$$

or

$$\mathbf{\Phi}_0(z) = [z(z - x_1)]^{-2}\mathbf{\Phi}(z)\mathbf{P}(z). \tag{45}$$

Here $\mathbf{P}_*(z)$ and $\mathbf{P}(z)$ are polynomials and

$$\det \mathbf{P}(z) \propto z^2(z - x_1)^2. \tag{46}$$

Since Kriese, Chang and Siewert [1] have shown that the partial indices for the considered Riemann-Hilbert problem are both unity, we can use the normalization

$$\lim_{|z|\to\infty} z\mathbf{\Phi}_0(z) = \mathbf{\Delta} = \begin{bmatrix} \sqrt{\frac{5}{6}} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{47}$$

to deduce that

$$\mathbf{P}(z) = \mathbf{A} + \mathbf{B}z + \mathbf{C}z^2 \tag{48}$$

with

$$\mathbf{C} = \begin{bmatrix} \sqrt{\frac{5}{6}}\, U_2^*(\infty) & \frac{\sqrt{3}}{3}\, U_1^*(\infty) \\ 0 & \frac{\sqrt{2}}{2}\, U_1^*(\infty) \end{bmatrix} \tag{49}$$

and

$$U_\alpha^*(\infty) = \exp\left[(-1)^{\alpha+1}\left(\frac{1}{2\pi}\int_0^\infty \frac{x\theta(x)}{R(x)}\,dx - \int_0^{x_1} \frac{x}{R(x)}\,dx\right)\right]. \tag{50}$$

It is clear that the constants **A** and **B** must be such that

$$\mathbf{\Phi}(\xi)\mathbf{P}(\xi) = \mathbf{0}, \qquad \xi = 0 \quad \text{and} \quad \xi = x_1, \tag{51a}$$

and

$$\frac{d}{d\xi}[\mathbf{\Phi}(\xi)\mathbf{P}(\xi)] = \mathbf{0}, \qquad \xi = 0 \quad \text{and} \quad \xi = x_1. \tag{51b}$$

We can now use Eqns. (43) and (48) in Eqns. (51) for $\xi = 0$ to deduce that

$$\begin{bmatrix} \sqrt{\frac{3}{2}} & 1 \\ 0 & 0 \end{bmatrix} \mathbf{A} = \mathbf{0} \tag{52a}$$

and

$$\begin{bmatrix} \sqrt{\frac{3}{2}} & 1 \\ 0 & 0 \end{bmatrix} \mathbf{B} = \mathbf{0}. \tag{52b}$$

On considering Eqns. (51) for $\xi = x_1$, we find

$$\begin{bmatrix} 0 & 0 \\ 1 & \alpha \end{bmatrix}(\mathbf{A} + \mathbf{B}x_1) = -x_1^2 \begin{bmatrix} 0 & 0 \\ \sqrt{\frac{5}{6}}\, U_2^*(\infty) & \left(\frac{\sqrt{2}}{2}\alpha + \frac{\sqrt{3}}{3}\right) U_1^*(\infty) \end{bmatrix} \tag{53a}$$

and

$$\begin{bmatrix} 0 & 0 \\ 1 & \alpha \end{bmatrix}\mathbf{B} + \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix}(\mathbf{A} + \mathbf{B}x_1)$$
$$= -2x_1 \begin{bmatrix} 0 & 0 \\ \sqrt{\frac{5}{6}}\, U_2^*(\infty) & \left(\frac{\sqrt{2}}{2}\alpha + \frac{\sqrt{3}}{3} + \frac{\sqrt{2}}{4}\beta x_1\right) U_1^*(\infty) \end{bmatrix}, \tag{53b}$$

where

$$\alpha = -\frac{\sqrt{6}}{6}[R(x_1) + x_1^2 + \tfrac{1}{2}] \tag{54}$$

and

$$\beta = -\frac{\sqrt{6}}{6}\left[\frac{2x_1^3 - 3x_1}{R(x_1)} + 2x_1\right]. \tag{55}$$

Equations (52) and (53) can now be solved to yield

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ -\sqrt{\frac{3}{2}}a_{11} & -\sqrt{\frac{3}{2}}a_{12} \end{bmatrix} \tag{56}$$

and

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ -\sqrt{\frac{3}{2}}b_{11} & -\sqrt{\frac{3}{2}}b_{12} \end{bmatrix}, \tag{57}$$

where

$$a_{11} = \frac{\sqrt{5}}{3}\frac{x_1^2}{\sqrt{\frac{2}{3}} - \alpha}\left[\frac{\beta x_1}{\sqrt{\frac{2}{3}} - \alpha} + 1\right]U_2^*(\infty), \tag{58a}$$

$$a_{12} = \frac{\sqrt{3}}{3}\frac{x_1^2}{\sqrt{\frac{2}{3}} - \alpha}\left[\frac{\beta x_1}{\sqrt{\frac{2}{3}} - \alpha} + 1 + \frac{\beta x_1}{\sqrt{\frac{2}{3}} + \alpha}\right](\alpha + \sqrt{\tfrac{2}{3}})U_1^*(\infty), \tag{58b}$$

$$b_{11} = -\frac{\sqrt{5}}{3}\frac{x_1^2}{\sqrt{\frac{2}{3}} - \alpha}\left[\frac{\beta}{\sqrt{\frac{2}{3}} - \alpha} + \frac{2}{x_1}\right]U_2^*(\infty) \tag{58c}$$

and

$$b_{12} = -\frac{\sqrt{3}}{3}\frac{x_1^2}{\sqrt{\frac{2}{3}} - \alpha}\left[\frac{\beta}{\sqrt{\frac{2}{3}} - \alpha} + \frac{2}{x_1} + \frac{\beta}{\sqrt{\frac{2}{3}} + \alpha}\right](\sqrt{\tfrac{2}{3}} + \alpha)U_1^*(\infty). \tag{58d}$$

Since the polynomial matrix $\mathbf{P}(z)$ is now established it is apparent that

$$\mathbf{\Phi}_0(z) = [z(z - x_1)]^{-2}\mathbf{S}^{-1}(z)\mathbf{U}(z)\mathbf{S}(z)\mathbf{P}(z) \tag{59}$$

is an exact analytical solution, which is also canonical and of normal form at infinity, of the given matrix Riemann-Hilbert problem. In addition, since x_1 can be expressed in terms of inverse elliptic functions [7], we consider $\mathbf{\Phi}_0(z)$ to be a closed-form solution. As $\mathbf{\Phi}_0(z)$ is now established, and because we have imposed the normalization indicated by Eqn. (47), we have at once an exact analytical expression for the H matrix introduced by Kriese, Chang and Siewert [1], viz.

$$\mathbf{H}(z) = \mathbf{\Phi}_0^{-T}(-z)\mathbf{\Phi}_0^T(0) \tag{60}$$

or

$$\mathbf{H}(z) = \sqrt{\tfrac{12}{5}}\mathbf{S}^T(-z)\mathbf{U}^{-1}(-z)\mathbf{S}^{-T}(-z)\mathbf{\Xi}(z)\mathbf{\Phi}_0^T(0) \tag{61}$$

where

$$\mathbf{\Xi}(z) = \begin{bmatrix} \sqrt{\frac{3}{2}}\left[-a_{12} + b_{12}z + \frac{\sqrt{3}}{3}U_1^*(\infty)z^2\right] & \sqrt{\frac{3}{2}}(a_{11} - b_{11}z) \\ -a_{12} + b_{12}z - \frac{\sqrt{3}}{3}U_1^*(\infty)z^2 & a_{11} - b_{11}z + \sqrt{\frac{5}{6}}U_2^*(\infty)z^2 \end{bmatrix}. \tag{62}$$

We find that we can express $\mathbf{\Phi}_0(0)$ as

$$\mathbf{\Phi}_0(0) = \frac{1}{2x_1^2}\begin{bmatrix} a_{11}U_2''(0) - \phi_1 & a_{12}U_2''(0) - \phi_2 \\ -\sqrt{\frac{3}{2}}a_{11}U_2''(0) - \sqrt{\frac{2}{3}}\phi_1 & -\sqrt{\frac{3}{2}}a_{12}U_2''(0) - \sqrt{\frac{2}{3}}\phi_2 \end{bmatrix} \tag{63}$$

where

$$\phi_1 = x_1U_1^*(0)[\tfrac{24}{25}a_{11} + \sqrt{\tfrac{6}{5}}U_2^*(\infty)], \tag{64a}$$

$$\phi_2 = x_1U_1^*(0)\left[\tfrac{24}{25}a_{12} + \frac{4\sqrt{3}}{5}U_1^*(\infty)\right], \tag{64b}$$

$$U_1^*(0) = x_1(\tfrac{12}{5})^{1/4}\exp\left[\int_0^{x_1}\left(\frac{5}{2R(x)} - 1\right)\frac{dx}{x} - \frac{5}{4\pi}\int_0^{\infty}\frac{\theta(x)}{R(x)}\frac{dx}{x}\right] \tag{65}$$

and

$$U_2''(0) = \frac{-2x_1(\frac{12}{5})^{1/2}}{U_1^*(0)}. \tag{66}$$

In conclusion we note that Yuan [6] has numerically evaluated all of the quantities required here to compute the matrix

$$\mathbf{L}(\mu) = \frac{1}{1+\mu}\,\mathbf{H}(\mu), \quad \mu \in [0, \infty), \tag{67}$$

tabulated by Kriese, Chang and Siewert [1]. Since Yuan [6] was able to reproduce to six significant figures the mentioned tabulation [1] and to ten significant figures a tabulation by Thomas [8] we believe that a correct solution to the considered problem is established.

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Abstract

The matrix Riemann-Hilbert problem relevant to the BGK model in the kinetic theory of gases is solved analytically.

Zusammenfassung

Das Matrix-Riemann-Hilbert Problem, das in der kinetischen Theorie der Gase für das BGK-Modell auftritt, wird analytisch gelöst.

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