ON A RAYLEIGH SCATTERING PROBLEM IN STELLAR ATMOSPHERES

ABSTRACT

A rigorous calculation of the asymptotic extrapolation distance for the Milne problem in a free-electron atmosphere is presented.

Chandrasekhar (1946) formulated and developed the first solutions to one of the now-classical problems in radiative transfer. This model describes the polarized radiation field that exists in a free-electron atmosphere where the interaction of the radiation intensity is governed by the Thomson scattering process. The equation that mathematically describes this problem can be written as (Chandrasekhar 1950)

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}(\tau,\mu) + \boldsymbol{I}(\tau,\mu) = \tfrac{3}{8}\int_{-1}^{1} \boldsymbol{K}(\mu,\mu')\boldsymbol{I}(\tau,\mu')\,d\mu' \tag{1}$$

where

$$\boldsymbol{K}(\mu,\mu') = \begin{vmatrix} 2(1-\mu^2)(1-\mu'^2)+\mu^2\mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{vmatrix} . \tag{2}$$

Here $\boldsymbol{I}(\tau,\mu)$ is a vector whose two components, $I_l(\tau,\mu)$ and $I_r(\tau,\mu)$, represent the radiation intensities of the two states of polarization. In addition, τ is the optical variable and μ denotes the cosine of the propagation vector as measured from the *inward* normal to the free surface.

To solve the associated Milne problem, one must find a solution to equation (1) subject to the two boundary conditions (Chandrasekhar 1950):

$$a)\ \boldsymbol{I}(0,\mu) = 0,\ \mu > 0 \text{ (zero re-entrant radiation)} ,$$

$$b)\ \lim_{\tau\to\infty} e^{-\tau}\boldsymbol{I}(\tau,\mu) = 0 .$$

In his initial paper, Chandrasekhar (1946) found first-, second-, and third-order solutions to the Milne problem. Later (Chandrasekhar 1950), he obtained the exact solution for the emergent angular distribution, $\boldsymbol{I}(0,-\mu)$, $\mu > 0$, by passing to the infinite limit in a discrete ordinates procedure. In a recent work, Siewert and Fraley (1967) obtained the solution at any optical depth. Their approach was based on the singular eigenfunction expansion technique (Case 1960).

The purpose of this Note is to report the results of calculations of several quantities of interest in this problem. The solution to the Milne problem considered here can be written as (Siewert and Fraley 1967)

$$\boldsymbol{I}(\tau,\mu) = \tfrac{3}{8}F\Big[(\tau+\tau_0-\mu)\boldsymbol{\Phi}_{+} - q\sqrt{2}\int_0^1 \frac{e^{-\tau/\eta}}{H_l(\eta)N_1(\eta)}\boldsymbol{\Phi}_1(\eta,\mu)\,d\eta + 2\sqrt{2}\int_0^1 \frac{\overset{\gamma}{\cancel{\eta}}(\eta-c)e^{-\tau/\eta}}{H_r(\eta)N_2(\eta)}\boldsymbol{\Phi}_2(\eta,\mu)\,d\eta\Big]. \tag{3}$$

Here, $H_r(\eta)$ and $H_l(\eta)$ are two of the H-functions introduced by Chandrasekhar (1950),

$$\tau_0 = c + \frac{\sqrt{2}}{8}[H_r(1) - 4/H_r(1)] + \frac{3\sqrt{2}}{8}\int_0^1 \mu^3 H_r(\mu)\,d\mu , \tag{4a}$$

$$q = \frac{4H_l(1)H_r(1)}{2H_l^2(1)+H_r^2(1)}, \tag{4b}$$

$$N_a(\eta) = [(-1)^a + 3(1-\eta^2)(1-\eta \tanh^{-1}\eta)]^2 + \tfrac{9}{4}\pi^2\eta^2(1-\eta^2)^2, \tag{4c}$$

$$c = \frac{2H_l^2(1)-H_r^2(1)}{2H_l^2(1)+H_r^2(1)}, \tag{4d}$$

and F is the constant net flux. Also,

$$\mathbf{\Phi}_+ = \begin{vmatrix} 1 \\ 1 \end{vmatrix}, \tag{5a}$$

$$\mathbf{\Phi}_1(\eta,\mu) = \begin{vmatrix} \frac{3\eta}{2}(1-\mu^2)\frac{P}{\eta-\mu} + [-1+3(1-\eta^2)(1-\eta \tanh^{-1}\eta)]\,\delta(\eta-\mu) \\ 0 \end{vmatrix}, \tag{5b}$$

and

$$\mathbf{\Phi}_2(\eta,\mu) = \begin{vmatrix} -\frac{3\eta}{2}(\eta+\mu) \\ \frac{3\eta}{2}(1-\eta^2)\frac{P}{\eta-\mu} + [1+3(1-\eta^2)(1-\eta \tanh^{-1}\eta)]\,\delta(\eta-\mu) \end{vmatrix}. \tag{5c}$$

The symbol P is used to indicate that integrals involving these functions are to be done in the Cauchy principal-value sense; the Dirac delta function is denoted by $\delta(x)$.

TABLE 1

SEVERAL PARAMETERS IN THE MILNE PROBLEM

$H_r(1)$	1.277972611
$H_l(1)$	3.469485049
q	0.689891054
c	0.872940529

TABLE 2

THE EXTRAPOLATION DISTANCE

First order	0.57735
Second order	0.69638
Third order	0.705927
Exact	0.712109761

We note that the complete solution is expressed in terms of Chandrasekhar's H-functions. These functions have been previously tabulated (Chandrasekhar 1950); however, since the evaluation of τ_0 requires a numerical integration, we have made a more exhaustive compilation of $H_l(\eta)$ and $H_r(\eta)$. Our calculational scheme has been the usual; namely, we have solved, by successive substitution, the integral equations satisfied by the two H-functions (Chandrasekhar 1950). For the numerical integration, an 81-point improved Gaussian quadrature scheme has been used (Kronrod 1965). The calculations have been made in "double precision" on the IBM 360/75 digital computer; as a test of the accuracy, we have numerically verified the identities (Chandrasekhar 1950)

$$\int_0^1 (1-\mu^2)H_l(\mu)\,d\mu = \tfrac{4}{3} \tag{6a}$$

and

$$\int_0^1 (1-\mu^2)H_r(\mu)\,d\mu = \tfrac{4}{3}(2-\sqrt{2}). \tag{6b}$$

agreement to thirteen significant figures was obtained. We do not display these results since, to the same degree of precision, our values agree perfectly with those of Chandrasekhar (1950).[1]

Our evaluations of q, c, $H_r(1)$, and $H_l(1)$ given in Table 1; again, agreement with Chandrasekhar (1950) is precise. In addition, Table 2 lists the first-, second-, and third-order estimates of τ_0 (Chandrasekhar 1946); the exact value is that obtained from equation (4a).

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March 31, 1967
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[1] Our tabulations are available upon request.