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RADIATIVE TRANSFER IN INHOMOGENEOUS ATMOSPHERES

T. W. MULLIKIN and C. E. SIEWERT
Mathematics Department, Purdue University, West Lafayette, IA 47907, U.S.A. and
Nuclear Engineering Department, North Carolina State University, Raleigh, NC 27650, U.S.A.

Abstract—Radiative transfer in plane-parallel inhomogeneous atmospheres is discussed from the points of view of the integral and the integro-differential forms of the equation of transfer.

1. INTRODUCTION

In the extensive literature devoted to the study of radiative transfer, particular attention has been given to models with phase functions for local scattering that are independent of the optical depth. The classical methods of Fourier transforms, singular integral equations and invariance principles, for example, have all been used (Hopf, 1934; Case, 1960; Chandrasekhar, 1950; Carlstedt and Mullikin, 1966) to deduce exact results. Here we use the integral and the integro-differential forms of the equation of transfer to develop a system of linear singular integral equations for the surface intensities basic to a class of phase functions that vary with optical depth.

In order to simplify our presentation, we consider here a scalar equation of transfer and isotropic scattering.

2. LINEAR SINGULAR INTEGRAL EQUATIONS

We consider for a plane-parallel atmosphere the steady-state boundary-value problem defined by

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{1}{2}\omega(\tau) \int_{-1}^{1} I(\tau, \mu')\, d\mu', \quad (1)$$

$$I(0, \mu) = 2\delta(\mu - \mu_0), \quad \mu, \mu_0 > 0, \quad (2a)$$

and

$$I(a, -\mu) = 0, \quad \mu > 0. \quad (2b)$$

Here the scattering is isotropic, but the 'albedo' for single scattering $\omega(\tau)$ is allowed to vary with the optical variable τ.

From equations (1) and (2) we can readily deduce the familiar equation

$$J(\tau:\mu_0) = e^{-\tau/\mu_0} + \frac{1}{2}\int_0^a E_1(|\tau - t|)\omega(t)J(t:\mu_0)\, dt \quad (3)$$

for the source function

$$J(\tau:\mu_0) = \frac{1}{2}\int_{-1}^{1} I(\tau, \mu)\, d\mu. \quad (4)$$

We also find it convenient to consider

$$J^*(\tau:\mu_0) = e^{-(a-\tau)/\mu_0} + \frac{1}{2}\int_0^a E_1(|\tau - t|)\omega(t)J^*(t:\mu_0)\, dt, \quad (5)$$

where

$$J^*(\tau:\mu_0) = \frac{1}{2}\int_{-1}^{1} I^*(\tau, \mu)\, d\mu \quad (6)$$

and $I^*(\tau, \mu)$ is the solution of

$$\mu \frac{\partial}{\partial \tau} I^*(\tau, \mu) + I^*(\tau, \mu) = \tfrac{1}{2}\omega(\tau) \int_{-1}^{1} I^*(\tau, \mu')\, d\mu', \quad (7)$$

$$I^*(0, \mu) = 0, \quad \mu > 0, \quad (8a)$$

and

$$I^*(a, -\mu) = 2\delta(\mu - \mu_0), \quad \mu, \mu_0 > 0. \quad (8b)$$

A well known (Hopf, 1934) result is:

Theorem 1

A sufficient condition for the unique solvability in $L^1(0, a)$ of equations (3) and (5) is that $\max \omega(\tau) \le 1$ for $a < \infty$ and $\max \omega(\tau) < 1$ for $a = \infty$.

We note that for $a < \infty$ we can extend the definition of μ_0, as used to establish equations (3) and (5), and consider $J(\tau:\mu_0)$ and $J^*(\tau:\mu_0)$ now defined as solutions of equations (3) and (5) to be analytic functions of complex μ_0, except for $\mu_0 = 0$. Also, it is apparent that

$$J^*(\tau:\mu_0) = e^{-a/\mu_0} J(\tau: -\mu_0). \quad (9)$$

We now define left and right transmission and reflection functions, physically meaningful for $0 \le \mu, \mu_0 \le 1$ but analytically extended to all $\mu \neq 0$, $\mu_0 \neq 0$, by

$$S_l(\mu, \mu_0) = \int_0^a \omega(\tau)\, e^{-\tau/\mu} J(\tau:\mu_0)\, d\tau, \quad (10a)$$

$$S_r(\mu, \mu_0) = \int_0^a \omega(\tau)\, e^{-(a-\tau)/\mu} J^*(\tau:\mu_0)\, d\tau, \quad (10b)$$

$$T_r(\mu, \mu_0) = \int_0^a \omega(\tau)\, e^{-(a-\tau)/\mu} J(\tau:\mu_0)\, d\tau \quad (10c)$$

and

$$T_l(\mu, \mu_0) = \int_0^a \omega(\tau)\, e^{-\tau/\mu} J^*(\tau:\mu_0)\, d\tau. \quad (10d)$$

We will subsequently use:

Theorem 2

The S and T functions satisfy the principles of reciprocity

$$S_l(\mu, \mu_0) = S_l(\mu_0, \mu), \quad (11a)$$

$$S_r(\mu, \mu_0) = S_r(\mu_0, \mu), \quad (11b)$$

$$T_r(\mu, \mu_0) = T_l(\mu_0, \mu), \quad (11c)$$

$$T_r(\mu, \mu_0) = e^{-a/\mu} S_l(-\mu, \mu_0), \quad (11d)$$

and

$$S_r(\mu, \mu_0) = e^{-a/\mu} T_l(-\mu, \mu_0). \quad (11e)$$

Proof

Equations (11d) and (11e) follow from basic definitions and equation (9). We illustrate the method of proof for equations (11a)–(11c) by establishing equation (11c). Equation (3) can be rewritten as

$$\sqrt{\omega(\tau)}\, J(\tau:\mu_0) = \sqrt{\omega(\tau)}\, e^{-\tau/\mu_0} + \frac{1}{2}\int_0^a \sqrt{\omega(\tau)\omega(t)} \times E_1(|\tau - t|)\sqrt{\omega(t)}\, J(t:\mu_0)\, dt. \quad (12)$$

Thus equations (3) and (5) yield the representations

$$\sqrt{\omega(\tau)}\, J(\tau:\mu_0) = (I - \mathscr{L})^{-1} (\sqrt{\omega(\tau)}\, e^{-\tau/\mu_0}) \quad (13a)$$

and

$$\sqrt{\omega(\tau)}\, J^*(\tau:\mu_0) = (I - \mathscr{L})^{-1} (\sqrt{\omega(\tau)}\, e^{-(a-\tau)/\mu_0}), \quad (13b)$$

where $\mathscr{L}$ denotes a linear integral operator that is self-adjoint in $L^2[0, a]$. Using inner product notation, we write

$$T_r(\mu, \mu_0) = \langle \sqrt{\omega}\, e^{-(a-\tau)/\mu}, (I - \mathscr{L})^{-1} \sqrt{\omega}\, e^{-\tau/\mu_0} \rangle, \quad (14a)$$

$$T_r(\mu, \mu_0) = \langle (I - \mathscr{L})^{-1} \sqrt{\omega}\, e^{-(a-\tau)/\mu}, \sqrt{\omega}\, e^{-\tau/\mu_0} \rangle \quad (14b)$$

or

$$T_r(\mu, \mu_0) = \langle \omega J^*, e^{-\tau/\mu_0} \rangle, \quad (14c)$$

and thus

$$T_r(\mu, \mu_0) = T_l(\mu_0, \mu). \quad (14d)$$

Equations (11a) and (11b) are proven in a similar way.

We now assume a special form for $\omega(\tau)$, viz.

$$\omega(\tau) = \int_0^\infty c(\eta)\, e^{-\tau/\eta}\, d\eta \quad (15)$$

with $c(\eta)$ such that $0 \leq \omega(\tau) \leq 1$ for $0 \leq \tau \leq a$. A more general class of $\omega(\tau)$ could be used in the following analysis, e.g. a Fourier representation

$$\omega(\tau) = \frac{1}{2\pi}\int_{-\infty}^\infty \hat{\omega}(\xi)\, e^{-i\tau\xi}\, d\xi; \quad (16)$$

however, we wish here to obtain equations based on real-valued functions for $0 \leq \mu, \mu_0 \leq 1$.

We now state

Theorem 3

With $\omega(\tau)$ as shown in equation (15), the unique solutions $J(\tau:\mu)$ and $J^*(\tau:\mu)$ to equations (3) and (5), with μ_0 changed to μ, satisfy the singular integral equations

$$J(\tau:\mu) = e^{-\tau/\mu} + \frac{1}{2}\int_0^\infty c(\eta) p \times \int_0^1 \left[\frac{J(\tau:p) - J(\tau:\sigma)}{p - \sigma} + \frac{J(\tau:p) - e^{-a/p} J^*(\tau:\sigma)}{p + \sigma}\right] d\sigma\, d\eta, \quad (17)$$

and

$$J^*(\tau:\mu) = e^{-(a-\tau)/\mu} + \frac{1}{2}\int_\mu^\infty c(\eta) q \times \int_0^1 \left[\frac{e^{-a/\eta} J^*(\tau:q) - e^{-a/\mu} J(\tau:\sigma)}{q + \sigma} + e^{-a/\eta}\frac{[J^*(\tau:q) - J^*(\tau:\sigma)]}{q - \sigma}\right] d\sigma\, d\eta + \frac{1}{2}\int_0^\mu c(\eta) q_- \int_0^1 \left\{ e^{-a/\mu} \times \frac{[J(\tau:q_-) - J(\tau:\sigma)]}{q_- - \sigma} + \frac{e^{-a/\mu} J(\tau:q_-) - e^{-a/\eta} J^*(\tau:\sigma)}{q_- + \sigma}\right\} \times d\sigma\, d\eta \quad (18)$$

where, for $0 \le \mu \le 1$, the second argument of J and J^* is kept positive by using equation (9) and

$$p = \frac{\mu\eta}{\mu + \eta}, \quad q = \frac{\mu\eta}{\eta - \mu}, \quad q_- = \frac{\mu\eta}{\mu - \eta}. \tag{19}$$

[We observe that the variable τ appears only as a parameter in equations (17) and (18).]

Proof

We write the solution of equation (3), with μ_0 changed to μ, as

$$J(\tau:\mu) = (I - L)^{-1}\,\mathrm{e}^{-\tau/\mu} = \mathrm{e}^{-\tau/\mu} + (I - L)^{-1}\,L\,\mathrm{e}^{-\tau/\mu}. \tag{20}$$

Equation (17) now follows directly from the observation that

$$L\,\mathrm{e}^{-\tau/\mu} = \frac{1}{2}\int_0^\infty c(\eta)p \int_0^1 \left[\frac{\mathrm{e}^{-\tau/p} - \mathrm{e}^{-\tau/\sigma}}{p - \sigma} + \frac{\mathrm{e}^{-\tau/p} - \mathrm{e}^{-a/p}\,\mathrm{e}^{-(a-\tau)/\sigma}}{p + \sigma}\right] \mathrm{d}\sigma\,\mathrm{d}\eta \tag{21}$$

and the use of equations (3) and (5). Equation (18) follows from equations (9) and (17).

We can now use the definitions given by equations (10) to deduce:

Corollary

The functions $S_l(\mu, \mu_0)$ and $T_r(\mu, \mu_0)$ satisfy

$$S_l(\mu, \mu_0) = \int_0^a \omega(\tau)\,\mathrm{e}^{-\tau[(1)/\mu + (1)/\mu_0]}\,\mathrm{d}\tau + \frac{1}{2}\int_0^\infty c(\eta)p \int_0^1 \left[\frac{S_l(p, \mu_0) - S_l(\sigma, \mu_0)}{p - \sigma} + \frac{S_l(p, \mu_0) - \mathrm{e}^{-a/p}\,T_r(\sigma, \mu_0)}{p + \sigma}\right] \times \mathrm{d}\sigma\,\mathrm{d}\eta \tag{22}$$

and

$$\begin{aligned} T_r(\mu, \mu_0) = {} & \int_0^a \mathrm{e}^{-\tau/\mu_0}\,\mathrm{e}^{-(a-\tau)/\mu}\,\omega(\tau)\,\mathrm{d}\tau \\ & + \frac{1}{2}\int_\mu^\infty c(\eta)q \times \int_0^1 \left[\frac{\mathrm{e}^{-a/\eta}\,T_r(q, \mu_0) - \mathrm{e}^{-a/\mu}\,S_l(\sigma, \mu_0)}{q + \sigma} + \mathrm{e}^{-a/\eta}\,\frac{[T_r(q, \mu_0) - T_r(\sigma, \mu_0)]}{q - \sigma}\right] \mathrm{d}\sigma\,\mathrm{d}\eta \\ & + \frac{1}{2}\int_0^\mu c(\eta)q_- \times \int_0^1 \left[\mathrm{e}^{-a/\mu}\,\frac{[S_l(q_-, \mu_0) - S_l(\sigma, \mu_0)]}{q_- - \sigma} + \frac{\mathrm{e}^{-a/\mu}\,S_l(q_-, \mu_0) - \mathrm{e}^{-a/\eta}\,T_r(\sigma, \mu_0)}{q_- + \sigma}\right] \times \mathrm{d}\sigma\,\mathrm{d}\eta. \end{aligned} \tag{23}$$

Equations for S_r and T_l are obtained from equations (22) and (23) by replacing S_r by T_r, S_l by T_l,

$$\int_0^a \omega(\tau)\mathrm{e}^{-\tau/\mu}\,\mathrm{e}^{-\tau/\mu_0}\,\mathrm{d}\tau$$

by

$$\int_0^a \omega(\tau)\,\mathrm{e}^{-\tau/\mu}\,\mathrm{e}^{-(a-\tau)/\mu_0}\,\mathrm{d}\tau$$

and

$$\int_0^a \mathrm{e}^{-\tau/\mu_0}\,\mathrm{e}^{-(a-\tau)/\mu}\,\omega(\tau)\,\mathrm{d}\tau$$

by

$$\int_0^a \mathrm{e}^{-(a-\tau)/\mu_0}\,\mathrm{e}^{-(a-\tau)/\mu}\,\omega(\tau)\,\mathrm{d}\tau.$$

Equations (17), (18), (22) and (23) are necessary conditions on the unique solutions to equations (3) and (5). This proves existence of solutions. We have only partial results concerning the sufficiency of these same equations for the unique determination of the desired functions. Thus

Theorem 4

For $a = \infty$ the equation

$$S_l(\mu, \mu_0) = \int_0^\infty \omega(\tau)\,\mathrm{e}^{-\tau[(1)/\mu + (1)/\mu_0]}\,\mathrm{d}\tau + \frac{1}{2}\int_0^\infty c(\eta)p \times \int_0^1 \left[\frac{S_l(p, \mu_0) - S_l(\sigma, \mu_0)}{p - \sigma} + \frac{S_l(p, \mu_0)}{p + \sigma}\right] \mathrm{d}\sigma\,\mathrm{d}\eta \tag{24}$$

has a unique solution if

$$\frac{1}{2}\int_0^\infty \frac{\eta}{1 + \eta}\left[\pi + \sqrt{\frac{1 + \eta}{\eta}} + \ln(1 + 2\eta)\right] c(\eta)\,\mathrm{d}\eta < 1. \tag{25}$$

Proof

This is an obvious extension of a result due to Martin (1971) for the special case $c(\eta) = \omega_0\,\delta(\eta - s)$.

3. THE ELEMENTARY SOLUTIONS FOR HALF-SPACE PROBLEMS

We now would like to discuss the construction of solutions to basic half-space problems directly from the integro-differential form of the equation of transfer

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{1}{2}\omega_0 \mathrm{e}^{-\tau/s} \int_{-1}^{1} I(\tau, \mu')\,\mathrm{d}\mu'. \quad (26)$$

We seek a solution of equation (26) that vanishes as $\tau \longrightarrow \infty$ such that

$$I(0, \mu) = F(\mu), \quad \mu > 0, \quad (27)$$

where $F(\mu)$ is considered prescribed. To find elementary solutions of equation (26) we consider

$$I_\nu(\tau, \mu) = F(\nu, \mu)\mathrm{e}^{-\tau/p(\nu)} + G(\nu, \mu)\mathrm{e}^{-\tau/\nu}, \quad (28)$$

where ν, $p(\nu)$, $F(\nu, \mu)$ and $G(\nu, \mu)$ are to be determined. On substituting equation (28) into equation (26) and imposing the normalization conditions

$$\int_{-1}^{1} G(\nu, \mu)\,\mathrm{d}\mu = 1 \quad (29a)$$

and

$$\int_{-1}^{1} F(\nu, \mu)\,\mathrm{d}\mu = 0, \quad (29b)$$

we find

$$[p(\nu) - \mu]F(\nu, \mu)\mathrm{e}^{-\tau/p(\nu)} + \frac{1}{\nu} p(\nu)(\nu - \mu) \times G(\nu, \mu)\mathrm{e}^{-\tau/\nu} = \tfrac{1}{2}\omega_0 p(\nu)\mathrm{e}^{-\tau[(1)/\nu + (1)/s]} \quad (30)$$

Thus if we let

$$p(\nu) = \frac{\nu s}{\nu + s}, \quad (31)$$

$$(\nu - \mu)G(\nu, \mu) = 0 \quad (32)$$

and

$$[p(\nu) - \mu]F(\nu, \mu) = \tfrac{1}{2}\omega_0 p(\nu), \quad (33)$$

then equation (28) will be a solution of equation (26). Considering $\nu \in (0, 1)$, we can solve equations (32) and (33) subject to equations (29) to find

$$F(\nu, \mu) = \tfrac{1}{2}\omega_0 p(\nu) Pv \left[\frac{1}{p(\nu) - \mu}\right] - \alpha(\nu)\,\delta[p(\nu) - \mu] \quad (34)$$

and

$$G(\nu, \mu) = \delta(\nu - \mu), \quad (35)$$

with

$$\alpha(\nu) = \omega_0 p(\nu) \tanh^{-1} p(\nu). \quad (36)$$

Thus we can write a solution of equation (26) that vanishes as $\tau \longrightarrow \infty$ as

$$I(\tau, \mu) = \int_0^1 A(\nu)\Phi_\tau(\nu, \mu)\mathrm{e}^{-\tau/\nu}\,\mathrm{d}\nu, \quad (37)$$

where $A(\nu)$ is to be determined from the boundary condition, equation (27), and

$$\Phi_\tau(\nu, \mu) = \mathrm{e}^{-\tau/s} F(\nu, \mu) + G(\nu, \mu). \quad (38)$$

In order to establish a convenient orthogonality relation concerning the generalized functions $\Phi_\tau(\xi, \mu)$, we multiply equation (38), for $\nu = \xi \in (0, 1)$ and with μ changed to $-\mu$, by $p(\xi) + \mu$ to find

$$\left[1 + \frac{\mu}{p(\xi)}\right]\Phi_\tau(\xi, -\mu) = \frac{\omega_0}{2}\mathrm{e}^{-\tau/s} - \frac{\xi}{s}\,\delta(\mu + \xi). \quad (39)$$

We can now multiply equation (39) by $\Phi_\tau(\xi', \mu)$ and integrate over μ to find

$$\int_{-1}^{1}\left[1 + \frac{\mu}{p(\xi)}\right]\Phi_\tau(\xi', \mu)\,\Phi_\tau(\xi, -\mu)\,\mathrm{d}\mu = \frac{\omega_0}{2}\mathrm{e}^{-\tau/s} - \frac{\xi}{s}\Phi_\tau(\xi', -\xi). \quad (40)$$

If we interchange ξ and ξ' in equation (40) and subtract the resulting equation from equation (40) we obtain

$$\int_{-1}^{1} \mu\Phi_\tau(\xi', \mu)\,\Phi_\tau(\xi, -\mu)\,\mathrm{d}\mu = 0, \quad \xi, \xi' \in (0, 1). \quad (41)$$

Thus we observe that equations (37) and (41) require that

$$\int_{-1}^{1} \mu\Phi_0(\xi, \mu)I(0, -\mu)\,\mathrm{d}\mu = 0, \quad \xi \in (0, 1), \quad (42)$$

or

$$\int_0^1 \mu\Phi_0(\xi, \mu)I(0, -\mu)\,\mathrm{d}\mu = \int_0^1 \mu\Phi_0(\xi, -\mu)F(\mu)\,\mathrm{d}\mu, \quad \xi \in (0, 1). \quad (43)$$

We now would like to use the F_N method (Siewert and Benoist, 1979; Siewert, 1978) in order to establish some numerical results. Thus we introduce the approximation

$$I(0, -\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \quad \mu > 0, \quad (44)$$

into equation (43) for the case of isotropic incident radiation, $F(\mu) = 1$, to find

$$\sum_{\alpha=0}^{N} a_\alpha B_\alpha(\xi) = 1 - p(\xi)\ln\left[1 + \frac{1}{p(\xi)}\right], \quad \xi \in (0, 1), \tag{45}$$

where

$$B_0(\xi) = \frac{2\xi}{\omega_0 p(\xi)} - 1 - p(\xi)\ln\left[1 + \frac{1}{p(\xi)}\right] \tag{46}$$

and

$$B_\alpha(\xi) = p(\xi)B_{\alpha-1}(\xi) - \frac{1}{\alpha+1} + \frac{2}{\omega_0 s}\xi^{\alpha+1}, \quad \alpha \geq 1. \tag{47}$$

We now evaluate equation (45) at

$$\xi = \xi_\beta, \quad \beta = 0, 1, 2, \ldots, N,$$

where $\{\xi_\beta\}$ are the positive zeros of the Legendre polynomial $P_{2(N+1)}(\xi)$ to obtain the following system of linear algebraic equations that can readily be solved to establish the required constants $\{a_\alpha\}$:

$$\sum_{\alpha=0}^{N} a_\alpha B_\alpha(\xi_\beta) = 1 - p(\xi_\beta)\ln\left[1 + \frac{1}{p(\xi_\beta)}\right], \quad \beta = 0, 1, 2, \ldots, N. \tag{48}$$

In Table 1 we list some results typical of those communicated privately by R. D. M. Garcia for the albedo

$$A^* = 2\int_0^1 I(0, -\mu)\mu \, d\mu = 2\sum_{\alpha=0}^{N} \frac{a_\alpha}{\alpha+2}. \tag{49}$$

We include in Table 1 Monte Carlo and ANISN results communicated privately by W. L. Dunn and G. C. Pomraning, respectively. The 'exact' values shown in Table 1 are those deduced from F_N calculations as N varied from 10 to 15.

We have also considered half-space problems for $F(\mu) = \delta(\mu - \mu_0)$, and the mentioned scheme was able to reproduce to four significant figures the results of Martin (1971) for $I(0, -\mu)$, $\mu \in (0, 1)$.

4. ELEMENTARY SOLUTIONS FOR FINITE ATMOSPHERES

For applications based on equation (26) in regard to finite atmospheres, $\tau \in [0, a]$, we require solutions in addition to those given by equation (28) for $\nu \in [0, 1]$.

First we observe that elementary solutions are given by equation (28) for ν such that $\nu \in [-1, 1]$ and $p(\nu) \in [-1, 1]$. Both these conditions are satisfied for $\nu \in [0, 1]$ and ν in the set V defined by

$$-\frac{s}{s+1} \leq \nu \leq 0, \quad \text{for} \quad 0 \leq s \leq \infty, \tag{50}$$

plus

$$-1 \leq \nu \leq -\frac{s}{1-s}, \quad \text{if} \quad 0 \leq s \leq \tfrac{1}{2}. \tag{51}$$

Thus we now write

$$I(\tau, \mu) = \int_0^1 A(\nu)\Phi_\tau(\nu, \mu)e^{-\tau/\nu} d\nu + \int_V A(\nu)\Phi_\tau(\nu, \mu)e^{-\tau/\nu} d\nu. \tag{52}$$

After the change of variables

$$\nu = -p(\eta), \tag{53}$$

we find that as ν ranges over V, η ranges over the set H defined by

$$0 \leq \eta \leq 1, \quad \text{for} \quad 0 \leq s \leq \infty, \tag{54}$$

plus

$$-1 \leq \eta \leq -\frac{s}{1-s}, \quad \text{if} \quad 0 \leq s \leq \tfrac{1}{2}. \tag{55}$$

This change of variables in equation (28) gives elementary solutions

$$I_{-p(\eta)}(\tau, \mu) = \theta_\tau(\eta, \mu)\, e^{\tau/p(\eta)}, \tag{56}$$

where

$$\theta_\tau(\eta, \mu) = e^{-\tau/s}\omega_0\left[\frac{1}{2}\eta P\nu\left(\frac{1}{\eta+\mu}\right) - \eta \tanh^{-1}\eta\,\delta(\eta+\mu)\right] + \delta[p(\eta) + \mu]. \tag{57}$$

Table 1. The half-space albedo for $\omega_0 = 1.0$

s	F_5	F_6	F_7	'Exact'	Monte Carlo	ANISN
0.5	0.1922	0.1921	0.1922	0.1922	0.192	0.1923
1.0	0.2658	0.2659	0.2659	0.2659	0.266	0.2661
1.5	0.3122	0.3122	0.3122	0.3122	0.312	0.3125
2.0	0.3458	0.3458	0.3458	0.3458	0.346	0.3461

We write equation (52) as

$$I(\tau, \mu) = \int_0^1 A(\nu)\Phi_\tau(\nu, \mu)\,e^{-\tau/\nu}\,d\nu + \int_H B(\nu)\theta_\tau(\nu, \mu)\,e^{\tau/p(\nu)}\,d\nu. \quad (58)$$

The functions $A(\nu)$ and $B(\nu)$ are to be determined from the boundary conditions

$$I(0, \mu) = F_1(\mu), \quad \mu > 0, \quad (59)$$

and

$$I(a, -\mu) = F_2(\mu), \quad \mu > 0. \quad (60)$$

The procedure used to establish equation (41) can now be used to deduce a similar result concerning the generalized functions $\theta_\tau(\xi, \mu)$, viz.

$$\int_{-1}^1 \mu\theta_\tau(\xi', \mu)\theta_\tau(\xi, -\mu)\,d\mu = 0, \quad \xi, \xi' \in (0, 1). \quad (61)$$

In addition, we can use the explicit expressions for $\Phi_\tau(\xi, \mu)$ and $\theta_\tau(\xi, \mu)$ to show that

$$\Delta_\tau(\xi, \xi') = \int_{-1}^1 \mu\Phi_\tau(\xi, \mu)\theta_\tau(\xi', -\mu)\,d\mu, \quad \xi \in (0, 1), \quad \xi' \in H, \quad (62)$$

has the useful property

$$e^{\tau/p(\xi')}\,\Delta_\tau(\xi, \xi') = e^{\tau/\xi}\,\Delta_0(\xi, \xi'), \quad \xi \in (0, 1), \quad \xi' \in H. \quad (63)$$

Thus if we change μ to $-\mu$ in equation (58), multiply by $\mu\Phi_\tau(\xi, \mu)$, $\xi \in (0, 1)$, and integrate over μ from -1 to 1 we find

$$\int_{-1}^1 \mu\Phi_\tau(\xi, \mu)I(\tau, -\mu)\,d\mu = \int_H B(\nu)\,e^{\tau/p(\nu)}\,\Delta_\tau(\xi, \nu)\,d\nu. \quad (64)$$

Considering now equation (64) for $\tau = 0$ and $\tau = a$ and making use of equation (63), we deduce that

$$\int_{-1}^1 \mu\Phi_0(\xi, \mu)I(0, -\mu)\,d\mu = e^{-a/\xi}\int_{-1}^1 \mu\Phi_a(\xi, \mu)I(a, -\mu)\,d\mu, \quad \xi \in (0, 1). \quad (65)$$

In a similar manner we find a second equation to be

$$\int_{-1}^1 \mu\theta_a(\xi, \mu)I(a, -\mu)\,d\mu = e^{-a/p(\xi)}\int_{-1}^1 \mu\theta_0(\xi, \mu)I(0, -\mu)\,d\mu, \quad \xi \in H. \quad (66)$$

We can write equations (65) and (66) as

$$\int_0^1 \mu\Phi_0(\xi, \mu)I(0, -\mu)\,d\mu + e^{-a/\xi}\int_0^1 \mu\Phi_a(\xi, -\mu)I(a, \mu)\,d\mu = K_1(\xi), \quad \xi \in (0, 1), \quad (67)$$

and

$$\int_0^1 \mu\theta_a(\xi, -\mu)I(a, \mu)\,d\mu + e^{-a/p(\xi)}\int_0^1 \mu\theta_0(\xi, \mu)I(0, -\mu)\,d\mu = K_2(\xi), \quad \xi \in H, \quad (68)$$

where the two known functions are

$$K_1(\xi) = \int_0^1 \mu\Phi_0(\xi, -\mu)F_1(\mu)\,d\mu + e^{-a/\xi}\int_0^1 \mu\Phi_a(\xi, \mu)F_2(\mu)\,d\mu \quad (69a)$$

and

$$K_2(\xi) = \int_0^1 \mu\theta_a(\xi, \mu)F_2(\mu)\,d\mu + e^{-a/p(\xi)}\int_0^1 \mu\theta_0(\xi, -\mu)F_1(\mu)\,d\mu. \quad (69b)$$

In the notation of Section 2, the unknowns $\mu I(0, -\mu)$ and $\mu I(a, \mu)$ in equations (67) and (68) are expressed by

$$\mu I(0, -\mu) = \mu F_2(\mu)\,e^{-a/\mu} + \frac{1}{2}\int_0^1 [S_l(\mu, \mu_0)F_1(\mu_0) + T_l(\mu, \mu_0)F_2(\mu_0)]\,d\mu_0 \quad (70a)$$

and

$$\mu I(a, \mu) = \mu F_1(\mu)\,e^{-a/\mu} + \frac{1}{2}\int_0^1 [T_r(\mu, \mu_0)F_1(\mu_0) + S_r(\mu, \mu_0)F_2(\mu_0)]\,d\mu_0. \quad (70b)$$

We note that equations (67) and (68) agree with those obtained by using the Corollary of Section 2 in equation (70) for the special case

$$c(\eta) = \omega_0 \delta(\eta - s). \tag{71}$$

As discussed in Section 2, we know that equations (67) and (68) can be solved; however to resolve the matter of uniqueness remains a challenging task. In carrying out some numerical experiments we have encountered complications that are not fully understood.

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