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Technical Notes

A Multiregion Calculation in the Theory of Neutron Diffusion

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ABSTRACT

The method of elementary solutions and the F_N method are used to compute and report especially accurate numerical results relevant to a previously established four-region test problem in the theory of neutron diffusion.

INTRODUCTION

The method of elementary solutions introduced in the theory of neutron diffusion by Case[1] and the so-called Placzek lemma[2] were both used in two recent publications[3,4] to develop the F_N method of solving approximately, though concisely and accurately, basic problems in neutron transport theory. Subsequently, the F_N method was used to establish accurate numerical results for several two- and three-media problems in plane geometry,[5] for three basic problems in spheres,[6] for multiregion problems in media requiring many terms in a Legendre expansion of the scattering law,[7] and for three- and four-region critical problems in plane geometry.[8,9]

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Here we use essentially the same analytical approach to solve the four-region test problem introduced some years ago by Reed[10] and discussed subsequently by Pitkaranta[11] and Martin and Duderstadt.[12] The previous papers[10-12] reported final results only in graphical form. To establish benchmark results for this problem, we report here in tabular form the neutron flux as a function of position, which is accurate, we believe, to four significant figures.

We seek a solution to the neutron transport equation

$$\mu \frac{\partial}{\partial x} \psi(x,\mu) + \psi(x,\mu) = \frac{1}{2} c_i \int_{-1}^{1} \psi(x,\mu')d\mu' + S_i \ , \quad x \in [x_0, x_k] \ , \tag{1}$$

where

x = distance measured in mean-free-paths

c_i = mean number of secondary neutrons for $x \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, k$

S_i = constant source for $x \in [x_{i-1}, x_i]$.

For this particular four-region problem, $x_0 = 0$, $x_1 = 100$, $x_2 = 105$, $x_3 = 106$, $x_4 = 108$, $c_1 = c_2 = 0$, $c_3 = c_4 = 0.9$, $S_1 = S_3 = 1$, and $S_2 = S_4 = 0$, and we impose the boundary conditions

$$\psi(0,\mu) = \psi(0,-\mu) \ , \quad \mu > 0 \ , \tag{2a}$$

and

$$\psi(x_4,-\mu) = 0 \ , \quad \mu > 0 \ . \tag{2b}$$

In addition, we require $\psi(x,\mu)$ to be continuous (except for $\mu = 0$) at each interface $x = x_i$, $i = 1$, 2, and 3. Since the region 3.0 cm $\leqslant z \leqslant$ 5.0 cm, as defined in Ref. 10, is vacuum and has no sources, it is clear that it can be omitted in the analysis. In this work, we denote 5.0 cm $\leqslant z \leqslant$ 6.0 cm as Region 3 and 6.0 cm $\leqslant z \leqslant$ 8.0 cm as Region 4. Regions 1 and 2 are the same here as in Ref. 10.

Since Regions 1 and 2 have $c = 0$, we can readily establish that

$$\psi(x,\mu) = 1 + f(\mu)\exp(-x/\mu) \ , \quad x \in [x_0, x_1] \ , \tag{3a}$$

and

$$\psi(x,\mu) = [f(\mu) + \exp(x_1/\mu)]\exp(-x/\mu) \ , \quad x \in [x_1, x_2] \ , \tag{3b}$$

where $f(-\mu) = f(\mu)$ and

$$f(\mu) = \exp(-x_2/\mu)\psi(x_2,-\mu) - \exp(-x_1/\mu) \ , \quad \mu > 0 \ . \tag{4}$$

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We thus need now only to consider the two-region problem $x \in [x_2, x_4]$ with

$$\psi(x_2,\mu) = \exp[-(x_2 - x_1)/\mu] - \exp[-(x_1 + x_2)/\mu] + \exp(-2x_2/\mu)\psi(x_2,-\mu) \ , \quad \mu > 0 \ , \tag{5a}$$

and

$$\psi(x_4,-\mu) = 0 \ , \quad \mu > 0 \ . \tag{5b}$$

ANALYSIS

If we write for $x \in [x_{i-1}, x_i]$, $i = 3$ and 4,

$$\psi(x,\mu) = \mathcal{A}_i(\nu_0)\phi(\nu_0,\mu)\exp(-x/\nu_0) + \mathcal{A}_i(-\nu_0)\phi(-\nu_0,\mu)\exp(x/\nu_0) + \int_{-1}^{1} \mathcal{A}_i(\nu)\phi(\nu,\mu)\exp(-x/\nu)d\nu + \left(\frac{1}{1-c}\right)\delta_{i,3} \ , \tag{6}$$

then, as discussed previously,[7] we can readily deduce, for $\xi \in \nu_0 \cup (0,1)$, that

$$\int_0^1 \mu[\phi(\xi,\mu)\psi(x_2,-\mu) - \phi(-\xi,\mu)\psi(x_2,\mu)]d\mu + \exp(-\Delta_3/\xi) \int_0^1 \mu[\phi(-\xi,\mu)\psi(x_3,\mu) - \phi(\xi,\mu)\psi(x_3,-\mu)]d\mu = \xi[1 - \exp(-\Delta_3/\xi)] \ , \tag{7a}$$

$$\int_0^1 \mu[\phi(\xi,\mu)\psi(x_3,\mu) - \phi(-\xi,\mu)\psi(x_3,-\mu)]d\mu + \exp(-\Delta_3/\xi) \int_0^1 \mu[\phi(-\xi,\mu)\psi(x_2,-\mu) - \phi(\xi,\mu)\psi(x_2,\mu)]d\mu = \xi[1 - \exp(-\Delta_3/\xi)] \ , \tag{7b}$$

$$\int_0^1 \mu[\phi(\xi,\mu)\psi(x_3,-\mu) - \phi(-\xi,\mu)\psi(x_3,\mu)]d\mu + \exp(-\Delta_4/\xi) \int_0^1 \mu\phi(-\xi,\mu)\psi(x_4,\mu)d\mu = 0 \ , \tag{7c}$$

and

$$\int_0^1 \mu\phi(\xi,\mu)\psi(x_4,\mu)d\mu + \exp(-\Delta_4/\xi) \int_0^1 \mu[\phi(-\xi,\mu)\psi(x_3,-\mu) - \phi(\xi,\mu)\psi(x_3,\mu)]d\mu = 0 \ . \tag{7d}$$

Here $\Delta_i = x_i - x_{i-1}$, and we use $c = c_3 = c_4$ so that

$$\phi(\nu,\mu) = \frac{1}{2} c\nu P\nu \left(\frac{1}{\nu - \mu}\right) + [1 - c\nu \tanh^{-1}\nu]\delta(\nu - \mu) \tag{8a}$$

and

$$\phi(\nu_0,\mu) = \frac{1}{2} c\nu_0 \left(\frac{1}{\nu_0 - \mu}\right) \ , \tag{8b}$$

where

$$1 = c\nu_0 \tanh^{-1}\left(\frac{1}{\nu_0}\right) \ . \tag{9}$$

If we now introduce, for $i = 2$ and 3, the approximations

$$\psi(x_i,-\mu) = \sum_{\alpha=0}^{N} a_{i,\alpha}\mu^\alpha \ , \quad \mu > 0 \ , \tag{10a}$$

and

$$\psi(x_{i+1},\mu) = \sum_{\alpha=0}^{N} b_{i+1,\alpha}\mu^\alpha \ , \quad \mu > 0 \ , \tag{10b}$$

into Eqs. (7a) through (7d), we find a special case of the F_N equations previously presented,[7] i.e.,

$$\sum_{\alpha=0}^{N} \{a_{2,\alpha}B_\alpha(\xi) + \exp(-\Delta_3/\xi)[b_{3,\alpha}A_\alpha(\xi) - a_{3,\alpha}B_\alpha(\xi)]\} = \frac{2}{c}[1 - \exp(-\Delta_3/\xi)] + \frac{2}{c\xi}\int_0^1 \mu\phi(-\xi,\mu)\psi(x_2,\mu)d\mu \ , \tag{11a}$$

$$\sum_{\alpha=0}^{N} [b_{3,\alpha}B_\alpha(\xi) - a_{3,\alpha}A_\alpha(\xi) + \exp(-\Delta_3/\xi)a_{2,\alpha}A_\alpha(\xi)] = \frac{2}{c}[1 - \exp(-\Delta_3/\xi)] + \frac{2}{c\xi}\exp(-\Delta_3/\xi)\int_0^1 \mu\phi(\xi,\mu)\psi(x_2,\mu)d\mu \ , \tag{11b}$$

$$\sum_{\alpha=0}^{N} [a_{3,\alpha}B_\alpha(\xi) - b_{3,\alpha}A_\alpha(\xi) + \exp(-\Delta_4/\xi)b_{4,\alpha}A_\alpha(\xi)] = 0 \ , \tag{11c}$$

and

$$\sum_{\alpha=0}^{N} \{b_{4,\alpha}B_\alpha(\xi) + \exp(-\Delta_4/\xi)[a_{3,\alpha}A_\alpha(\xi) - b_{3,\alpha}B_\alpha(\xi)]\} = 0 \ . \tag{11d}$$

Here

$$A_\alpha(\xi) = -\xi A_{\alpha-1}(\xi) + \frac{1}{\alpha + 1} \ , \quad \alpha \geq 1 \ , \tag{12a}$$

and

$$B_\alpha(\xi) = \xi B_{\alpha-1}(\xi) - \frac{1}{\alpha + 1} \ , \quad \alpha \geq 1 \ , \tag{12b}$$

with

$$B_0(\xi) + 2 - \frac{2}{c} = A_0(\xi) = 1 - \xi \ln\left(1 + \frac{1}{\xi}\right) \ . \tag{12c}$$

If we now use Eq. (10a) in Eq. (5a) we find, for $\mu > 0$,

$$\psi(x_2,\mu) = \exp(-\Delta_2/\mu) - \exp[-(x_1 + x_2)/\mu] + \exp(-2x_2/\mu)\sum_{\alpha=0}^{N} a_{2,\alpha}\mu^\alpha \ . \tag{13}$$

A simpler expression can be used for $\psi(x_2,\mu)$, instead of Eq. (13), since the last two terms in the right side of Eq. (13) are negligible:

$$\psi(x_2,\mu) \simeq \exp(-\Delta_2/\mu) \ , \quad \mu > 0 \ . \tag{14}$$

Equation (14) can now be used to find explicit forms of the right sides of Eqs. (11a) and (11b). We obtain for Eq. (11a)

$$\frac{2}{c}[1 - \exp(-\Delta_3/\xi)] + \int_0^1 \mu \exp(-\Delta_2/\mu)\frac{d\mu}{\mu + \xi} \tag{15a}$$

and for Eq. (11b)

$$\frac{2}{c}[1 - \exp(-\Delta_3/\xi)] + \exp[-(\Delta_2 + \Delta_3)/\xi]B_0(\xi) - \exp(-\Delta_3/\xi)\int_0^1 \mu \frac{\exp(-\Delta_2/\mu) - \exp(-\Delta_2/\xi)}{\mu - \xi} d\mu \ . \tag{15b}$$

Equations (11a) through (11d) evaluated at $N + 1$ selected values of $\xi \in \nu_0 \cup (0,1)$, say ξ_β, generate $4(N + 1)$ linear algebraic equations for the unknowns $a_{2,\alpha}$, $a_{3,\alpha}$, $b_{3,\alpha}$, and $b_{4,\alpha}$.

To establish the angular distribution for all x, we use the full-range orthogonality properties of the eigenfunctions $\phi(\xi,\mu)$ to find the expansion coefficients required in Eq. (6), for $\xi \in \nu_0 \cup (0,1)$:

$$\mathcal{A}_3(\xi)N(\xi)\exp(-x_2/\xi) = -\frac{1}{2}c\xi\int_0^1 \mu\frac{\exp(-\Delta_2/\mu)-\exp(-\Delta_2/\xi)}{\mu-\xi}d\mu + \frac{1}{2}c\xi\exp(-\Delta_2/\xi)B_0(\xi) - \frac{1}{2}c\xi\sum_{\alpha=0}^{N}a_{2,\alpha}A_\alpha(\xi) - \xi \ , \quad (16a)$$

$$\mathcal{A}_3(-\xi)N(-\xi)\exp(x_3/\xi) = \frac{1}{2}c\xi\sum_{\alpha=0}^{N}[b_{3,\alpha}A_\alpha(\xi) - a_{3,\alpha}B_\alpha(\xi)] + \xi \ , \quad (16b)$$

$$\mathcal{A}_4(\xi)N(\xi)\exp(-x_3/\xi) = \frac{1}{2}c\xi\sum_{\alpha=0}^{N}[b_{3,\alpha}B_\alpha(\xi) - a_{3,\alpha}A_\alpha(\xi)] \ , \quad (16c)$$

and

$$\mathcal{A}_4(-\xi)N(-\xi)\exp(x_4/\xi) = \frac{1}{2}c\xi\sum_{\alpha=0}^{N}b_{4,\alpha}A_\alpha(\xi) \ , \quad (16d)$$

where $N(\pm\xi)$, $\xi = \nu_0$, or $\nu \in (0,1)$, are the full-range normalization factors.[1]

NUMERICAL RESULTS

We solved Eqs. (11a) through (11d), for various orders of the F_N approximation by selecting ξ_β according to the scheme

$$\xi_0 = \nu_0 \ , \quad \text{all } N \ , \quad (17a)$$

and

$$\xi_\beta = \frac{2\beta-1}{2N} \ , \quad \beta = 1, 2, \ldots, N \ , \quad N \neq 0 \ . \quad (17b)$$

Of main interest in this problem is the flux

$$\phi(x) = \int_{-1}^{1}\psi(x,\mu)d\mu \ . \quad (18)$$

We find

$$\phi(x) = 2 + \sum_{\alpha=0}^{N}a_{2,\alpha}E_{\alpha+2}(x_2 - x) - E_2(x_1 - x) \ , \quad x \in [x_0,x_1] \ , \quad (19a)$$

$$\phi(x) = E_2(x - x_1) + \sum_{\alpha=0}^{N}a_{2,\alpha}E_{\alpha+2}(x_2 - x) \ , \quad x \in [x_1,x_2] \ , \quad (19b)$$

and, for $x \in [x_{i-1},x_i]$,

$$\phi(x) = \mathcal{A}_i(\nu_0)\exp(-x/\nu_0) + \mathcal{A}_i(-\nu_0)\exp(x/\nu_0) + \int_{-1}^{1}\mathcal{A}_i(\nu)\exp(-x/\nu)d\nu + \left(\frac{2}{1-c}\right)\delta_{i,3} \ , \quad i = 3 \text{ and } 4 \ , \quad (19c)$$

where $E_n(x)$ is the exponential integral and $\mathcal{A}_i(\nu_0)$, $\mathcal{A}_i(-\nu_0)$, and $\mathcal{A}_i(\nu)$ are available from Eqs. (16a) through (16d) once the solution of Eqs. (11a) through (11d) is known.

We show in Fig. 1 the normalized flux $\phi(z)/\phi(0)$, as calculated by the F_4 approximation. The flux is known to be constant for 3.0 cm $\leqslant z \leqslant$ 5.0 cm, as discussed before. In Table I, we report the normalized flux for some selected values of z and various orders of the F_N approximation. Note that the F_4 approximation is already excellent.

In summary, we conclude that the F_N method yields remarkably good results for this particular application. Among the best features of the method are the accurate results that

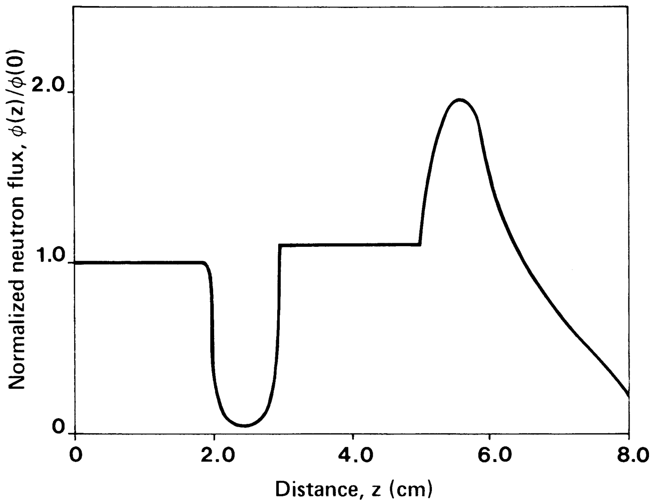

Fig. 1. The normalized neutron flux $\phi(z)/\phi(0)$.

TABLE I

The Normalized Flux $\phi(z)/\phi(0)$

z (cm)	Region	F_2	F_4	F_8	"Exact"[a]
0.0		1.000	1.000	1.000	1.000
1.0		1.000	1.000	1.000	1.000
1.9		0.9995	0.9995	0.9995	0.9995
1.95	1	0.9902	0.9902	0.9902	0.9902
1.99		0.8372	0.8372	0.8372	0.8372
2.0		0.5010	0.5010	0.5010	0.5010
2.0		0.5010	0.5010	0.5010	0.5010
2.05		0.2602	0.2602	0.2602	0.2602
2.09		0.1798	0.1798	0.1798	0.1798
2.1		0.1651	0.1651	0.1651	0.1651
2.5	2	0.03012	0.03009	0.03009	0.03009
2.9		0.3514	0.3520	0.3520	0.3520
2.95		0.5640	0.5650	0.5650	0.5650
2.99		0.9174	0.9151	0.9151	0.9151
3.0		1.118	1.106	1.106	1.106
5.0		1.106	1.106	1.106	1.106
5.1	3	1.441	1.441	1.441	1.441
5.5		1.941	1.941	1.941	1.941
6.0		1.632	1.633	1.634	1.634
6.0		1.648	1.633	1.634	1.634
7.0	4	0.7081	0.7085	0.7085	0.7085
8.0		0.2223	0.2225	0.2225	0.2225

[a]The "Exact" values were deduced from F_N calculations as N varied from 8 to 10.

can be reached with relatively low orders of the approximation and the minor computation effort required.

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