THE COMPLETE SOLUTION FOR THE SCATTERING OF POLARIZED LIGHT IN A RAYLEIGH AND ISOTROPICALLY SCATTERING ATMOSPHERE

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Abstract. The F_N method is used to solve, in a concise manner, the complete problem concerning the diffusion of polarized light in a plane-parallel Rayleigh and isotropically scattering atmosphere.

1. Introduction

In the classical treatment of the scattering of polarized light established by Chandrasekhar (1950), the three-vector $\mathbf{I}(\tau, \mu, \phi)$, with components $I_l(\tau, \mu, \phi)$, $I_r(\tau, \mu, \phi)$ and $U(\tau, \mu, \phi)$, satisfies for a Rayleigh and isotropically scattering atmosphere the equation of transfer

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu, \phi) + \mathbf{I}(\tau, \mu, \phi) =$$

$$= \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \mathbf{P}(\mu, \phi : \mu', \phi') \mathbf{I}(\tau, \mu', \phi') \, \mathrm{d}\mu' \, \mathrm{d}\phi' \tag{1}$$

and, typically, boundary conditions of the form

$$\mathbf{I}(0, \mu, \phi) = \pi\delta(\mu - \mu_0)\delta(\phi - \phi_0)\mathbf{F}, \qquad \mu, \mu_0 > 0; \phi, \phi_0 \in [0, 2\pi], \tag{2a}$$

and

$$\mathbf{I}(\tau_0, -\mu, \phi) = \frac{\lambda_0}{\pi} \mathbf{E} \int_0^{2\pi} \int_0^1 \mathbf{I}(\tau_0, \mu', \phi')\mu' \, \mathrm{d}\mu' \, \mathrm{d}\phi', \qquad \mu > 0; \phi \in [0, 2\pi], \tag{2b}$$

where $\tau \in [0, \tau_0]$ is the optical variable, μ is the direction cosine as measured from the *positive* τ-axis, $\mathbf{F}$ with components F_l, F_r and F_U is a prescribed constant, λ_0 is the

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coefficient for Lambert reflection at the 'ground' and

$$\mathbf{E} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

Note that we have changed μ to $-\mu$ in regard to Chandrasekhar's notation and that $\mathbf{I}(\tau, \mu, \phi)$ is the complete field and not just the diffuse component.

Thus we use Chandrasekhar's phase matrix in the form

$$\mathbf{P}(\mu, \phi:\mu', \phi') = \mathbf{Q}[c\mathbf{P}^{(0)}(\mu:\mu') + (1-c)\mathbf{E} + c(1-\mu^2)^{1/2}(1-\mu'^2)^{1/2} \times \\ \times \mathbf{P}^{(1)}(\mu, \phi:\mu', \phi') + c\mathbf{P}^{(2)}(\mu, \phi:\mu', \phi')], \tag{4}$$

where

$$c = \frac{1-\gamma}{1+2\gamma}. \tag{5}$$

The depolarization factor can also be expressed (Chandrasekhar, 1950) in terms of γ – i.e.,

$$\varrho_n = \frac{2\gamma}{1+\gamma}, \tag{6}$$

with $\gamma \leqslant \frac{1}{3}$. In addition,

$$\mathbf{P}^{(0)}(\mu:\mu') = \frac{3}{4}\begin{bmatrix} 2(1-\mu^2)(1-\mu'^2) + \mu^2\mu'^2 & \mu^2 & 0 \\ \mu'^2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{7}$$

$$\mathbf{P}^{(1)}(\mu, \phi:\mu', \phi') = \frac{3}{4}\begin{bmatrix} 4\mu\mu' \cos(\phi'-\phi) & 0 & -2\mu \sin(\phi'-\phi) \\ 0 & 0 & 0 \\ 2\mu' \sin(\phi'-\phi) & 0 & \cos(\phi'-\phi) \end{bmatrix}, \tag{8}$$

$$\mathbf{P}^{(2)}(\mu, \phi:\mu', \phi') = \\ = \frac{3}{4}\begin{bmatrix} \mu^2\mu'^2 \cos 2(\phi'-\phi) & -\mu^2 \cos 2(\phi'-\phi) & -\mu^2\mu' \sin 2(\phi'-\phi) \\ -\mu'^2 \cos 2(\phi'-\phi) & \cos 2(\phi'-\phi) & \mu' \sin 2(\phi'-\phi) \\ \mu\mu'^2 \sin 2(\phi'-\phi) & -\mu \sin 2(\phi'-\phi) & \mu\mu' \cos 2(\phi'-\phi) \end{bmatrix}, \tag{9}$$

and

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{10}$$

As observed by Chandrasekhar, the fourth component of the radiation field $V(\tau, \mu, \phi)$ is uncoupled from the three elements of $\mathbf{I}(\tau, \mu, \phi)$. In fact, to establish $V(\tau, \mu, \phi)$ we must solve only the scalar equation of transfer

$$\mu \frac{\partial}{\partial \tau} V(\tau, \mu, \phi) + V(\tau, \mu, \phi) = \\ = \frac{\omega}{8\pi}(5c-2) \int_0^{2\pi} \int_{-1}^{1} [\mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2} \cos(\phi - \phi')] \times \\ \times V(\tau, \mu', \phi') \, \mathrm{d}\mu' \, \mathrm{d}\phi', \tag{11}$$

subject to the boundary conditions

$$V(0, \mu, \phi) = \pi F_V \delta(\mu - \mu_0)\delta(\phi - \phi_0), \\ \mu, \mu_0 > 0; \phi, \phi_0 \in [0, 2\pi], \tag{12a}$$

and

$$V(\tau_0, -\mu, \phi) = 0, \qquad \mu > 0, \phi \in [0, 2\pi]. \tag{12b}$$

If we express $V(\tau, \mu, \phi)$ as

$$V(\tau, \mu, \phi) = \tfrac{3}{8} F_V [\mu\mu_0 V^{(0)}(\tau, \mu) + (1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2} \times \\ \times \cos(\phi - \phi_0) V^{(1)}(\tau, \mu)] + \pi F_V \delta(\mu - \mu_0) \times \\ \times e^{-\tau/\mu} \left[\delta(\phi - \phi_0) - \frac{1}{2\pi} - \frac{1}{\pi} \cos(\phi - \phi_0) \right] \tag{13}$$

then it is apparent that $V^{(0)}(\tau, \mu)$ and $V^{(1)}(\tau, \mu)$ are solutions of

$$\mu \frac{\partial}{\partial \tau} V^{(0)}(\tau, \mu) + V^{(0)}(\tau, \mu) = \frac{\omega}{4}(5c-2) \int_{-1}^{1} \mu'^2 V^{(0)}(\tau, \mu') \, \mathrm{d}\mu', \tag{14}$$

with

$$V^{(0)}(0, \mu) = \frac{4}{3\mu_0^2} \delta(\mu - \mu_0), \qquad \mu, \mu_0 > 0, \tag{15a}$$

and

$$V^{(0)}(\tau_0, -\mu) = 0, \qquad \mu > 0, \tag{15b}$$

and

$$\mu \frac{\partial}{\partial \tau} V^{(1)}(\tau, \mu) + V^{(1)}(\tau, \mu) = \frac{\omega}{8}(5c-2) \int_{-1}^{1} (1-\mu'^2) V^{(1)}(\tau, \mu') \, \mathrm{d}\mu', \tag{16}$$

with

$$V^{(1)}(0, \mu) = \tfrac{8}{3}(1-\mu_0^2)^{-1} \delta(\mu - \mu_0), \qquad \mu, \mu_0 > 0, \tag{17a}$$

and

$$V^{(1)}(\tau_0, -\mu) = 0, \qquad \mu > 0. \tag{17b}$$

The radiative transfer problems defined by Equations (14)–(17) are of the general form we consider in Section 3.

Chandrasekhar (1950) also deduced that the problem for the three-vector $\mathbf{I}(\tau, \mu, \phi)$ could be reduced to one two-vector problem and two scalar problems. We therefore write

$$\begin{aligned}\mathbf{I}(\tau, \mu, \phi) = \mathbf{\Xi I}(\tau, \mu) &+ \\ &+ \tfrac{1}{4}\mathbf{Q}[(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}\phi^{(1)}(\tau, \mu)\mathbf{P}^{(1)}(\mu, \phi : \mu_0, \phi_0) + \\ &+ \phi^{(2)}(\tau, \mu)\mathbf{P}^{(2)}(\mu, \phi : \mu_0, \phi_0)]\mathbf{F} + \\ &+ \pi\delta(\mu-\mu_0)\,\mathrm{e}^{-\tau/\mu}\Bigg[\delta(\phi-\phi_0) - \frac{1}{2\pi}\mathbf{U} - \frac{2}{3\pi}(1+2\mu_0^2)^{-1} \times \\ &\times \mathbf{Q}\mathbf{P}^{(1)}(\mu, \phi : \mu_0, \phi_0) - \frac{4}{3\pi}(1+\mu_0^2)^{-2} \times \\ &\times \mathbf{Q}\mathbf{P}^{(2)}(\mu, \phi : \mu_0, \phi_0)\Bigg]\mathbf{F},\end{aligned} \tag{18}$$

where

$$\mathbf{\Xi} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{19a, b}$$

and $\mathbf{I}(\tau, \mu)$ is a two-vector that satisfies

$$\mu\frac{\partial}{\partial\tau}\mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \frac{\omega}{2}\mathbf{Q}(\mu)\int_{-1}^{1}\mathbf{Q}^T(\mu')\mathbf{I}(\tau, \mu')\,\mathrm{d}\mu', \tag{20}$$

with

$$\mathbf{I}(0, \mu) = \frac{1}{2}\begin{bmatrix} F_l \\ F_r \end{bmatrix}\delta(\mu-\mu_0), \qquad \mu, \mu_0 > 0, \tag{21a}$$

and

$$\mathbf{I}(\tau_0, -\mu) = \lambda_0\mathbf{D}\int_0^1 \mathbf{I}(\tau_0, \mu')\mu'\,\mathrm{d}\mu', \qquad \mu > 0, \tag{21b}$$

where

$$\mathbf{Q}(\mu) = \tfrac{3}{2}(c+2)^{-1/2}\begin{bmatrix} c\mu^2 + \tfrac{2}{3}(1-c) & (2c)^{1/2}(1-\mu^2) \\ \tfrac{1}{3}(c+2) & 0 \end{bmatrix} \tag{22}$$

and

$$\mathbf{D} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{23}$$

In addition, $\phi^{(1)}(\tau, \mu)$ and $\phi^{(2)}(\tau, \mu)$ satisfy

$$\mu \frac{\partial}{\partial \tau} \phi^{(1)}(\tau, \mu) + \phi^{(1)}(\tau, \mu) = = \frac{3\omega c}{8} \int_{-1}^{1} (1 - \mu'^2)(1 + 2\mu'^2)\phi^{(1)}(\tau, \mu')\, \mathrm{d}\mu', \tag{24}$$

with

$$\phi^{(1)}(0, \mu) = \tfrac{8}{3}[(1 + 2\mu_0^2)(1 - \mu_0^2)]^{-1}\delta(\mu - \mu_0), \qquad \mu, \mu_0 > 0, \tag{25a}$$

and

$$\phi^{(1)}(\tau_0, -\mu) = 0, \qquad \mu > 0, \tag{25b}$$

and

$$\mu \frac{\partial}{\partial \tau} \phi^{(2)}(\tau, \mu) + \phi^{(2)}(\tau, \mu) = \frac{3\omega c}{16} \int_{-1}^{1} (1 + \mu'^2)^2 \phi^{(2)}(\tau, \mu')\, \mathrm{d}\mu', \tag{26}$$

with

$$\phi^{(2)}(0, \mu) = \tfrac{16}{3}(1 + \mu_0^2)^{-2}\delta(\mu - \mu_0), \qquad \mu, \mu_0 > 0, \tag{27a}$$

and

$$\phi^{(2)}(\tau_0, -\mu) = 0, \qquad \mu > 0. \tag{27b}$$

We note that $\phi^{(1)}(\tau, \mu)$ and $\phi^{(2)}(\tau, \mu)$ are also defined by radiative transfer problems of the general form considered in Section 3.

It is apparent that the formulation of the basic problems to be solved given here differs from that of Chandrasekhar only in that we allow ω to be less than unity and we work in terms of the complete intensity rather than the diffuse field.

2. The Two-Vector Problem

From the foregoing section it is clear that the most difficult task in constructing the solution we seek is that of solving the two-vector problem defined by Equations (20) and (21). The solution by the F_N method of this problem has recently been reported (cf. Siewert, 1979; or Maiorino and Siewert, 1980), and thus here we give only a summary of the basic equations.

If the exit distributions are approximated (cf. Maiorino and Siewert, 1980) by

$$\mathbf{I}(0, -\mu) = \mathbf{Q}(\mu) \sum_{\alpha=0}^{N} \mathbf{a}_\alpha \mu^\alpha, \qquad \mu > 0, \tag{28a}$$

and

$$\mathbf{I}(\tau_0, \mu) = \frac{1}{2}\begin{bmatrix} F_l \\ F_r \end{bmatrix} \delta(\mu - \mu_0)\, \mathrm{e}^{-\tau_0/\mu} + \mathbf{Q}(\mu) \sum_{\alpha=0}^{N} \mathbf{b}_\alpha \mu^\alpha, \qquad \mu > 0, \tag{28b}$$

then the constants $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$ can be found by solving the following F_N equations at selected values of $\xi \in \eta_0 \cup (0,1)$:

$$\sum_{\alpha=0}^{N} \mathbf{\Delta}_\alpha(\xi)\mathbf{a}_\alpha + \mathrm{e}^{-\tau_0/\xi} \sum_{\alpha=0}^{N} [\mathbf{\Gamma}_\alpha(\xi) - \lambda_0 \mathbf{B}_0^T(\xi)\mathbf{D}\mathbf{Y}_\alpha]\mathbf{b}_\alpha = \mathbf{K}_1(\xi) \tag{29a}$$

and

$$\sum_{\alpha=0}^{N} [\mathbf{\Delta}_\alpha(\xi) - \lambda_0 \mathbf{A}_0^T(\xi)\mathbf{D}\mathbf{Y}_\alpha]\mathbf{b}_\alpha + \mathrm{e}^{-\tau_0/\xi} \sum_{\alpha=0}^{N} \mathbf{\Gamma}_\alpha(\xi)\mathbf{a}_\alpha = \mathbf{K}_2(\xi), \tag{29b}$$

where

$$\mathbf{Y}_\alpha = \left(\frac{1}{\alpha+2}\right)\mathbf{Q}_0 + \left(\frac{1}{\alpha+4}\right)\mathbf{Q}_2, \tag{30}$$

$$\mathbf{K}_1(\xi) = \tfrac{1}{2}\mu_0 \left\{ [1 - \mathrm{e}^{-\tau_0(1/\mu_0 + 1/\xi)}] \frac{\mathbf{M}^T(\xi)\mathbf{Q}^T(\mu_0)}{\xi + \mu_0} + \right.$$
$$\left. + \lambda_0\, \mathrm{e}^{-\tau_0(1/\mu_0 + 1/\xi)} \mathbf{B}_0^T(\xi)\mathbf{D} \right\} \begin{bmatrix} F_l \\ F_r \end{bmatrix}, \tag{31a}$$

$$\mathbf{K}_2(\xi) = \tfrac{1}{2}\mu_0 \left\{ [\mathrm{e}^{-\tau_0/\xi} - \mathrm{e}^{-\tau_0/\mu_0}] \frac{\mathbf{M}^T(\xi)\mathbf{Q}^T(\mu_0)}{\xi - \mu_0} + \lambda_0\, \mathrm{e}^{-\tau_0/\mu_0} \mathbf{A}_0^T(\xi)\mathbf{D} \right\} \begin{bmatrix} F_l \\ F_r \end{bmatrix}, \tag{31b}$$

$$\mathbf{A}_0(\xi) = \left[\mathbf{Q}_0 + \tfrac{1}{3}\mathbf{Q}_2 + \xi(\xi - \tfrac{1}{2})\mathbf{Q}_2 - \xi \ln\left(1 + \frac{1}{\xi}\right)\mathbf{Q}(\xi) \right] \mathbf{M}(\xi), \tag{32a}$$

$$\mathbf{B}_0(\xi) = \left[\frac{2}{\omega}\mathbf{\Delta}(\xi) - 2\mathbf{Q}_0 - \tfrac{2}{3}\mathbf{Q}_2 \right] \mathbf{M}(\xi) + \mathbf{A}_0(\xi), \tag{32b}$$

$$\mathbf{\Delta}(\xi) = \mathbf{Q}^{-T}(\xi)[\mathbf{I} + \omega\xi^2 \mathbf{Q}_2^T(\mathbf{Q}_0 + \tfrac{1}{3}\mathbf{Q}_2)], \tag{33}$$

$$\mathbf{\Gamma}_0(\xi) = \mathbf{M}^T(\xi)[\mathbf{R}_0 - \xi\mathbf{K}(\xi)], \tag{34a}$$

$$\mathbf{\Gamma}_\alpha(\xi) = -\xi\mathbf{\Gamma}_{\alpha-1}(\xi) + \mathbf{M}^T(\xi)\mathbf{R}_\alpha, \tag{34b}$$

$$\mathbf{\Delta}_0(\xi) = \mathbf{M}^T(\xi)\left[\frac{2}{\omega}\mathbf{I} - \mathbf{R}_0 - \xi\mathbf{K}(\xi) \right], \tag{35a}$$

and

$$\mathbf{\Delta}_\alpha(\xi) = \xi\mathbf{\Delta}_{\alpha-1}(\xi) - \mathbf{M}^T(\xi)\mathbf{R}_\alpha, \tag{35b}$$

where

$$\mathbf{Q}(\xi) = \mathbf{Q}_0 + \xi^2\mathbf{Q}_2, \tag{36a}$$

$$\mathbf{R}_\alpha = \left(\frac{1}{\alpha+1}\right)\mathbf{Q}_0^T\mathbf{Q}_0 + \left(\frac{1}{\alpha+3}\right)(\mathbf{Q}_2^T\mathbf{Q}_0 + \mathbf{Q}_0^T\mathbf{Q}_2) + \left(\frac{1}{\alpha+5}\right)\mathbf{Q}_2^T\mathbf{Q}_2 \tag{36b}$$

and

$$\mathbf{K}(\xi) = \mathbf{Q}_0^T\mathbf{Q}_0 \ln\left(1 + \frac{1}{\xi}\right) + (\mathbf{Q}_2^T\mathbf{Q}_0 + \mathbf{Q}_0^T\mathbf{Q}_2)\left[\xi^2 \ln\left(1 + \frac{1}{\xi}\right) + \frac{1}{2} - \xi\right] + $$
$$+ \mathbf{Q}_2^T\mathbf{Q}_2\left[\xi^4 \ln\left(1 + \frac{1}{\xi}\right) + \frac{1}{4} - \frac{\xi}{3} + \frac{\xi^2}{2} - \xi^3\right]. \tag{37}$$

We use $\mathbf{M}(\xi) = \mathbf{I}$, $\xi \in (0, 1)$; whereas $\mathbf{M}(\eta_0)$ is a null-vector of the dispersion matrix (cf. Bond and Siewert, 1971)

$$\mathbf{\Lambda}(\eta_0)\mathbf{M}(\eta_0) = \mathbf{0}, \tag{38}$$

where

$$\mathbf{\Lambda}(z) = \mathbf{I} + \tfrac{1}{2}\omega z \int_{-1}^{1} \mathbf{Q}^T(x)\mathbf{Q}(x)\frac{\mathrm{d}x}{x - z}. \tag{39}$$

As discussed previously (Maiorino and Siewert, 1980), the set of linear algebraic equations obtained by evaluating Equations (29) at selected values of $\xi \in \eta_0 \cup (0, 1)$ can be solved economically to yield, by way of Equations (28), accurate numerical results for $\mathbf{I}(0, -\mu)$ and $\mathbf{I}(\tau_0, \mu)$, $\mu > 0$.

3. The Scalar Problems

In order to develop solutions by the F_N method for the required scalar functions $V^{(0)}(\tau, \mu)$, $V^{(1)}(\tau, \mu)$, $\phi^{(1)}(\tau, \mu)$ and $\phi^{(2)}(\tau, \mu)$ discussed in Section 1 we now consider the general class of problems defined by the equations

$$\mu\frac{\partial}{\partial\tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\omega}{2}\int_{-1}^{1} f(\mu')I(\tau, \mu')\,\mathrm{d}\mu', \tag{40}$$

$$I(0, \mu) = N\delta(\mu - \mu_0), \qquad \mu, \mu_0 > 0, \tag{41a}$$

and

$$I(\tau_0, -\mu) = 0, \qquad \mu > 0, \tag{41b}$$

where N is an arbitrary normalization factor, and we consider that $f(\mu) = f(-\mu)$, $f(\mu) \geqslant 0$, for $\mu \in [0, 1]$, and that

$$\Lambda(\infty) = 1 - \omega\int_0^1 f(\mu)\,\mathrm{d}\mu \neq 0. \tag{42}$$

By analogy with other scalar problems (McCormick and Kuščer, 1973) it is clear that a solution to Equation (40) can be expressed as

$$I(\tau, \mu) = [A(\eta_0)\phi(\eta_0, \mu)\,\mathrm{e}^{-\tau/\eta_0} + A(-\eta_0)\phi(-\eta_0, \mu)\,\mathrm{e}^{\tau/\eta_0}]\delta_{\kappa,1} + $$
$$+ \int_{-1}^{1} A(\eta)\phi(\eta, \mu)\,\mathrm{e}^{-\tau/\eta}\,\mathrm{d}\eta, \tag{43}$$

where

$$\phi(\xi, \mu) = \frac{\omega\xi}{2} f(\xi) Pv\left(\frac{1}{\xi - \mu}\right) + \lambda(\xi)\delta(\xi - \mu) \tag{44}$$

and

$$\lambda(\eta) = 1 + \frac{\omega}{2}\eta P \int_{-1}^{1} f(x)\frac{\mathrm{d}x}{x - \eta}. \tag{45}$$

We note that $f(1) > 0 \Rightarrow \kappa = 1$, and if $f(1) = 0$, then $\kappa = 1$ if $\lambda(1) < 0$ and $\kappa = 0$ if $\lambda(1) > 0$. We do not consider here the possibility that $\lambda(t)$ and $f(t)$ can simultaneously be zero for some $t_0 \in (0, 1)$. When $\kappa = 1$ we note that η_0 is a zero of

$$\Lambda(z) = 1 + \frac{\omega}{2} z \int_{-1}^{1} f(x)\frac{\mathrm{d}x}{x - z}, \tag{46}$$

and that η_0 can be expressed (Siewert, 1980) as

$$\eta_0 = [\Lambda(\infty)]^{-1/2} \exp\left[-\frac{1}{\pi}\int_0^1 \theta(t)\frac{\mathrm{d}t}{t}\right], \tag{47}$$

where

$$\theta(t) = \tan^{-1}\left(\frac{\omega t \pi f(t)}{2\lambda(t)}\right) \tag{48}$$

is continuous for $t \in [0, 1]$ and such that $\theta(1) = \pi$.

The expansion coefficients $A(\pm\xi)$, $\xi \in P$ – i.e., $\xi \in \eta_0 \cup (0, 1)$ if $\kappa = 1$ or $\xi \in (0, 1)$ if $\kappa = 0$ – in Equation (43) are to be determined by the boundary conditions given as Equations (41); however, the fact that the functions $\phi(\xi, \mu)$ are orthogonal, i.e.,

$$\int_{-1}^{1} \mu f(\mu)\phi(\xi, \mu)\phi(\xi', \mu)\,\mathrm{d}\mu = 0, \qquad \xi \neq \xi';\ \pm\xi, \pm\xi' \in P, \tag{49}$$

can be used, as discussed by Grandjean and Siewert (1979), to deduce the singular integral equations and constraints for the surface intensities $I(0, -\mu)$ and $I(\tau_0, \mu)$, $\mu > 0$,

$$\int_0^1 \mu f(\mu)\phi(\xi, \mu)I(0, -\mu)\,\mathrm{d}\mu + \mathrm{e}^{-\tau_0/\xi} \times$$
$$\times \int_0^1 \mu f(\mu)\phi(-\xi, \mu)I(\tau_0, \mu)\,\mathrm{d}\mu = L_1(\xi) \tag{50a}$$

and

$$\int_0^1 \mu f(\mu)\phi(\xi, \mu)I(\tau_0, \mu)\,\mathrm{d}\mu + \mathrm{e}^{-\tau_0/\xi} \times$$
$$\times \int_0^1 \mu f(\mu)\phi(-\xi, \mu)I(0, -\mu)\,\mathrm{d}\mu = L_2(\xi). \tag{50b}$$

In Equations (50) the known terms, after we make use of Equations (41), are

$$L_1(\xi) = \mu_0 f(\mu_0)\phi(-\xi, \mu_0)N \tag{51a}$$

and

$$L_2(\xi) = \mu_0 f(\mu_0)\, e^{-\tau_0/\xi}\, \phi(\xi, \mu_0)N. \tag{51b}$$

If we now introduce the approximations

$$I(0, -\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \qquad \mu > 0, \tag{52a}$$

and

$$I(\tau_0, \mu) = N\delta(\mu - \mu_0)\, e^{-\tau_0/\mu} + \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha, \qquad \mu > 0, \tag{52b}$$

into Equations (50) we find the basic F_N equations to be

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\xi) + e^{-\tau_0/\xi} b_\alpha A_\alpha(\xi)] = \frac{\mu_0 f(\mu_0)}{\xi + \mu_0} [1 - e^{-\tau_0(1/\xi + 1/\mu_0)}]N \tag{53a}$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha(\xi) + e^{-\tau_0/\xi} a_\alpha A_\alpha(\xi)] = \mu_0 f(\mu_0)\left[\frac{e^{-\tau_0/\xi} - e^{-\tau_0/\mu_0}}{\xi - \mu_0}\right]N, \tag{53b}$$

where

$$A_0(\xi) = \int_0^1 \mu f(\mu) \frac{d^\mu}{\mu + \xi}, \tag{54a}$$

$$A_\alpha(\xi) = -\xi A_{\alpha-1}(\xi) + \int_0^1 \mu^\alpha f(\mu)\, d\mu, \qquad \alpha \geqslant 1. \tag{54b}$$

$$B_0(\xi) = \frac{2}{\omega} \Lambda(\infty) + A_0(\xi) \tag{55a}$$

and

$$B_\alpha(\xi) = \xi B_{\alpha-1}(\xi) - \int_0^1 \mu^\alpha f(\mu)\, d\mu, \qquad \alpha \geqslant 1\,. \tag{55b}$$

It is apparent that we now can readily solve the system of linear algebraic equations obtained by evaluating Equations (53) at selected values of $\xi \in P$ to find the constants a_α and b_α and thus to establish the surface results given by Equations (52).

4. Numerical Results

In order to evaluate the effectiveness of the F_N method for solving the complete problem for Rayleigh and isotropic scattering in a planetary atmosphere we now would like to report some numerical results. Since the atmosphere is illuminated by a beam of natural light, we have $F_l = F_r = \frac{1}{2}$ and $F_U = F_V = 0$, and thus here $V(\tau, \mu, \phi) \equiv 0$. Since the solution, by the F_N method, of the two-vector problem has been discussed recently (cf. Maiorino and Siewert, 1980) we comment here only on the two scalar problems, for $\phi^{(1)}(\tau, \mu)$ and $\phi^{(2)}(\tau, \mu)$, required to complete the solution given by Equation (18). Thus, for the $\phi^{(i)}$ problem, $i = 1$ and 2, we have used

$$f_1(\mu) = \frac{3c}{4}(1-\mu^2)(1+2\mu^2), \tag{56a}$$

$$N_1 = \tfrac{8}{3}[(1-\mu_0^2)(1+2\mu_0^2)]^{-1}, \tag{56b}$$

$$f_2(\mu) = \frac{3c}{8}(1+\mu^2)^2 \tag{56c}$$

and

$$N_2 = \tfrac{16}{3}[1+\mu_0^2]^{-2} \tag{56d}$$

to establish the F_N equations

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha^{(i)}(\xi_j) + \mathrm{e}^{-\tau_0/\xi_j} b_\alpha A_\alpha^{(i)}(\xi_j)] = \frac{2c\mu_0}{\xi_j + \mu_0}[1 - \mathrm{e}^{-\tau_0(1/\xi_j + 1/\mu_0)}] \tag{57a}$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha^{(i)}(\xi_j) + \mathrm{e}^{-\tau_0/\xi_j} a_\alpha A_\alpha^{(i)}(\xi_j)] = 2c\mu_0\left[\frac{\mathrm{e}^{-\tau_0/\xi_j} - \mathrm{e}^{-\tau_0/\mu_0}}{\xi_j - \mu_0}\right]. \tag{57b}$$

The required functions $A_\alpha^{(i)}(\xi)$ can readily be generated from

$$A_\alpha^{(1)}(\xi) = -\xi A_{\alpha-1}^{(1)}(\xi) + \frac{3c}{4}\left[\frac{1}{\alpha+1} + \frac{1}{\alpha+3} - \frac{2}{\alpha+5}\right] \tag{58a}$$

with

$$A_0^{(1)}(\xi) = \frac{3c}{4}\left[(2\xi^5 - \xi^3 - \xi)\ln\left(1 + \frac{1}{\xi}\right) + \frac{14}{15} + \frac{\xi^2}{3} + \xi^3 - 2\xi^4\right], \tag{58b}$$

and

$$A_\alpha^{(2)}(\xi) = -\xi A_{\alpha-1}^{(2)}(\xi) + \frac{3c}{8}\left[\frac{1}{\alpha+1} + \frac{2}{\alpha+3} + \frac{1}{\alpha+5}\right] \tag{59a}$$

with

$$A_0^{(2)}(\xi) = \frac{3c}{8}\left[-(\xi^5 + 2\xi^3 + \xi)\ln\left(1 + \frac{1}{\xi}\right) + \frac{28}{15} - \frac{5\xi}{4} + \frac{7\xi^2}{3} - \frac{\xi^3}{2} + \xi^4\right]. \tag{59b}$$

The functions $B_\alpha^{(i)}(\xi)$ follow from Equations (55), (58) and (59), with

$$\Lambda(\infty) = 1 - \tfrac{7}{10}\omega c. \tag{60}$$

TABLE I

The Stokes parameters $I(0, -\mu, \phi)$, $Q(0, -\mu, \phi)$ and $U(0, -\mu, \phi)$

μ	$\phi = 0°$	$\phi = 90°$	$\phi = 180°$
0.06	0.3952	0.2849	0.4080
	−0.1291 (−1)	0.1110	−0.8 (−4)
	0.0	−0.1069	0.0
0.10	0.3763	0.2778	0.3963
	−0.1475 (−1)	0.1059	0.5296 (−2)
	0.0	−0.1002	0.0
0.20	0.3320	0.2584	0.3662
	−0.2146 (−1)	0.9388 (−1)	0.1278 (−1)
	0.0	−0.8559 (−1)	0.0
0.32	0.2871	0.2371	0.3327
	−0.3036 (−1)	0.8185 (−1)	0.1526 (−1)
	0.0	−0.7128 (−1)	0.0
0.52	0.2291	0.2082	0.2833
	−0.4387 (−1)	0.6687 (−1)	0.1041 (−1)
	0.0	−0.5219 (−1)	0.0
0.72	0.1867	0.1857	0.2382
	−0.5378 (−1)	0.5616 (−1)	−0.2356 (−2)
	0.0	−0.3571 (−1)	0.0
0.92	0.1592	0.1678	0.1912
	−0.5647 (−1)	0.4819 (−1)	−0.2440 (−1)
	0.0	−0.1743 (−1)	0.0
1.00	0.1616	0.1616	0.1616
	−0.4556 (−1)	0.4556 (−1)	−0.4556 (−1)
	0.0	0.0	0.0

To define the values of ξ in Equations (57) we have selected, for $\kappa = 1$, $\xi_0 = \eta_0$ and the remaining $\xi_j = (2j - 1)/(2N)$, $j = 1, 2, \ldots, N$; for $\kappa = 0$ we have used $\xi_{j-1} = (2j - 1)/[2(N + 1)]$, $j = 1, 2, \ldots, (N + 1)$. Thus, for each problem we solved a set of $2(N + 1)$ linear algebraic equations to deduce $\{a_\alpha\}$ and $\{b_\alpha\}$ and thus to establish $\phi^{(1)}$ and $\phi^{(2)}$ at the boundaries.

The quantities $\mathbf{I}(0, -\mu)$, $\mathbf{I}(\tau_0, \mu)$, $\phi^{(1)}(0, -\mu)$, $\phi^{(1)}(\tau_0, \mu)$, $\phi^{(2)}(0, -\mu)$ and $\phi^{(2)}(\tau_0, \mu)$, $\mu > 0$, can be used with Equation (18) to deduce I_l, I_r and U, or the Stokes parameters $I = I_l + I_r$, $Q = I_l - I_r$ and U, at the boundaries. In Tables I and II we list the 'converged' F_N results for the Stokes parameters corresponding to $\tau_0 = 5$, $\mu_0 = 0.4$, $\phi_0 = 0°$, $\lambda_0 = 0.2$, $c = 0.8$ and $\omega = 0.9$. We note that our 'converged' results (typically obtained from $N \sim$ 10–12) in Table I agree, to the number of significant figures reported, with a calculation communicated by Kawabata (1980).

TABLE II

The Stokes parameters $I(\tau_0, \mu, \phi)$, $Q(\tau_0, \mu, \phi)$ and $U(\tau_0, \mu, \phi)$

μ	$\phi = 0°$	$\phi = 90°$	$\phi = 180°$
0.06	0.8330 (−2)	0.8113 (−2)	0.8320 (−2)
	−0.7270 (−3)	−0.5177 (−3)	−0.7365 (−3)
	0.0	−0.788 (−4)	0.0
0.10	0.8753 (−2)	0.8521 (−2)	0.8736 (−2)
	−0.7266 (−3)	−0.5069 (−3)	−0.7434 (−3)
	0.0	−0.8437 (−4)	0.0
0.20	0.9804 (−2)	0.9534 (−2)	0.9764 (−2)
	−0.7283 (−3)	−0.4773 (−3)	−0.7680 (−3)
	0.0	−0.9927 (−4)	0.0
0.32	0.1115 (−1)	0.1083 (−1)	0.1107 (−1)
	−0.7250 (−3)	−0.4197 (−3)	−0.8028 (−3)
	0.0	−0.1215 (−3)	0.0
0.52	0.1383 (−1)	0.1340 (−1)	0.1363 (−1)
	−0.6846 (−3)	−0.1985 (−3)	−0.8858 (−3)
	0.0	−0.1934 (−3)	0.0
0.72	0.1730 (−1)	0.1668 (−1)	0.1682 (−1)
	−0.6353 (−3)	0.3088 (−3)	−0.1116 (−2)
	0.0	−0.3337 (−3)	0.0
0.92	0.2106 (−1)	0.2052 (−1)	0.2039 (−1)
	−0.9187 (−3)	0.1138 (−2)	−0.1585 (−2)
	0.0	−0.3621 (−3)	0.0
1.00	0.2212 (−1)	0.2212 (−1)	0.2212 (−1)
	−0.1540 (−2)	0.1540 (−2)	−0.1540 (−2)
	0.0	0.0	0.0

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