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ON MULTI-MEDIA CALCULATIONS IN THE THEORY OF NEUTRON DIFFUSION

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Abstract—The F_N method is used to solve the critical problem for a three-region reactor and to compute the disadvantage factor required for thermal utilization calculations. Anisotropic scattering is allowed and numerical results are given.

I. INTRODUCTION

We consider first the basic disadvantage-factor calculation discussed previously by, for example, Ferziger and Robinson (1965), Bond and Siewert (1969), Eccleston and McCormick (1970) and Laletin, Sultanov, Vlasov and Koniaev (1974). If we denote the fuel and moderator regions of a basic reactor cell by 1 and 2, respectively, then the relevant one-speed transport equations are

$$\mu \frac{\partial}{\partial x}\psi_1(x,\mu) + \psi_1(x,\mu) = \tfrac{1}{2}c_1 \sum_{l=0}^{L_1} (2l+1) f_{1,l} P_l(\mu) \times \int_{-1}^{1} P_l(\mu')\psi_1(x,\mu')\,d\mu', \quad 0 \le x \le a, \tag{1a}$$

and

$$\mu \frac{\partial}{\partial x}\psi_2(x,\mu) + \psi_2(x,\mu) = \tfrac{1}{2}c_2 \sum_{l=0}^{L_2} (2l+1) f_{2,l} P_l(\mu) \times \int_{-1}^{1} P_l(\mu')\psi_2(x,\mu')\,d\mu' + \frac{S}{\sigma_2}, \quad a \le x \le b. \tag{1b}$$

Here we allow general anisotropic scattering, S represents a constant source and we seek the disadvantage factor

$$\zeta = \frac{a}{b-a} \frac{\int_a^b \int_{-1}^{1} \psi_2(x,\mu)\,d\mu\,dx}{\int_0^a \int_{-1}^{1} \psi_1(x,\mu)\,d\mu\,dx} \tag{2}$$

where $\psi_1(x,\mu)$ and $\psi_2(x,\mu)$ satisfy equations (1) subject to

$$\psi_1(0,\mu) = \psi_1(0,-\mu), \tag{3a}$$

$$\psi_2(b,\mu) = \psi_2(b,-\mu), \tag{3b}$$

$$\psi_1(a,\mu) = \psi_2(a,\mu) \tag{3c}$$

and

$$\psi_1(a,-\mu) = \psi_2(a,-\mu) \tag{3d}$$

for $\mu \in (0, 1)$. Since $\psi_1(x,\mu)$ and

$$\psi_2^*(x,\mu) = \psi_2(x,\mu) - \frac{S}{\sigma_2(1-c_2)} \tag{4}$$

can be expressed in terms of the elementary solutions (Mika, 1961) of equations (1) we can use immediately the singular integral equations and constraints reported by Devaux, Grandjean, Ishiguro and Siewert (1979) to define the angular fluxes $\psi(0,\mu)$, $\psi(a,\mu)$ and $\psi(b,\mu)$. We thus write

$$\int_0^1 \mu\{[\phi_1(\xi,\mu) - \phi_1(-\xi,\mu)]\psi(0,\mu) + e^{-a/\xi}[\phi_1(-\xi,\mu)\psi(a,\mu) - \phi_1(\xi,\mu)\psi(a,-\mu)]\}\,d\mu = 0, \tag{5a}$$

$$\int_0^1 \mu\{[\phi_1(\xi,\mu)\psi(a,\mu) - \phi_1(-\xi,\mu)\psi(a,-\mu)] + e^{-a/\xi}[\phi_1(-\xi,\mu) - \phi_1(\xi,\mu)]\psi(0,\mu)\}\,d\mu = 0, \tag{5b}$$

$$\int_0^1 \mu\{[\phi_2(\xi,\mu)\psi(a,-\mu) - \phi_2(-\xi,\mu)\psi(a,\mu)] + e^{-(b-a)/\xi}[\phi_2(-\xi,\mu) - \phi_2(\xi,\mu)]\psi(b,\mu)\}\,d\mu = \frac{S}{\sigma_2}\xi[1 - e^{-(b-a)/\xi}] \tag{6a}$$

and

$$\int_0^1 \mu\{[\phi_2(\xi,\mu) - \phi_2(-\xi,\mu)]\psi(b,\mu) + e^{-(b-a)/\xi}[\phi_2(-\xi,\mu)\psi(a,-\mu) - \phi_2(\xi,\mu)\psi(a,\mu)]\}\,d\mu = \frac{S}{\sigma_2}\xi[1 - e^{-(b-a)/\xi}]. \tag{6b}$$

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We note that equations (5) require

$$\xi \in P_1 = \{\nu_{1,\beta}\} \cup (0, 1)$$

and equations (6) require

$$\xi \in P_2 = \{\nu_{2,\beta}\} \cup (0, 1).$$

We follow here the basic notation of Devaux *et al.* (1979) and thus do not repeat all of the required definitions. If we now introduce for $\mu > 0$ the approximations

$$\psi(0, \mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \tag{7a}$$

$$\psi(a, -\mu) = \sum_{\alpha=0}^{N} e_\alpha \mu^\alpha, \tag{7b}$$

$$\psi(a, \mu) = \sum_{\alpha=0}^{N} f_\alpha \mu^\alpha, \tag{7c}$$

and

$$\psi(b, \mu) = \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha \tag{7d}$$

into equations (5) and (6) and consider the resulting equations for selected values ξ_β then our F_N results can be obtained here by solving the following system of $4(N + 1)$ linear algebraic equations:

$$\sum_{\alpha=0}^{N} \{a_\alpha[B_\alpha^{(1)}(\xi_\beta) - A_\alpha^{(1)}(\xi_\beta)] + e^{-a/\xi_\beta}[f_\alpha A_\alpha^{(1)}(\xi_\beta) - e_\alpha B_\alpha^{(1)}(\xi_\beta)]\} = 0, \tag{8a}$$

$$\sum_{\alpha=0}^{N} \{[f_\alpha B_\alpha^{(1)}(\xi_\beta) - e_\alpha A_\alpha^{(1)}(\xi_\beta)] + e^{-a\xi_\beta} a_\alpha[A_\alpha^{(1)}(\xi_\beta) - B_\alpha^{(1)}(\xi_\beta)]\} = 0, \tag{8b}$$

$$\sum_{\alpha=0}^{N} \{[e_\alpha B_\alpha^{(2)}(\xi_\beta) - f_\alpha A_\alpha^{(2)}(\xi_\beta)] + e^{-(b-a)/\xi_\beta} b_\alpha[A_\alpha^{(2)}(\xi_\beta) - B_\alpha^{(2)}(\xi_\beta)]\} = \frac{2S}{c_2\sigma_2}[1 - e^{-(b-a)/\xi_\beta}] \tag{9a}$$

and

$$\sum_{\alpha=0}^{N} \{b_\alpha[B_\alpha^{(2)}(\xi_\beta) - A_\alpha^{(2)}(\xi_\beta)] + e^{-(b-a)/\xi_\beta}[e_\alpha A_\alpha^{(2)}(\xi_\beta) - f_\alpha B_\alpha^{(2)}(\xi_\beta)]\} = \frac{2S}{c_2\sigma_2}[1 - e^{-(b-a)/\xi_\beta}]. \tag{9b}$$

The known functions $A_\alpha^{(i)}(\xi)$ and $B_\alpha^{(i)}(\xi)$ appearing in equations (8) and (9) are those reported earlier (Siewert, 1978). From equations (1) we can readily deduce that

$$(1 - c_1)\int_0^a \int_{-1}^1 \psi_1(x, \mu)\,d\mu\,dx = -\int_{-1}^1 \psi(a, \mu)\mu\,d\mu \tag{10a}$$

and

$$(1 - c_2)\int_a^b \int_{-1}^1 \psi_2(x, \mu)\,d\mu\,dx = \frac{2S}{\sigma_2}(b - a) + \int_{-1}^1 \psi(a, \mu)\mu\,d\mu \tag{10b}$$

so that we can express the disadvantage factor as

$$\zeta = \frac{a}{b - a}\left(\frac{1 - c_1}{1 - c_2}\right) \times \left\{-1 - \frac{2S}{\sigma_2}(b - a)\left[\int_{-1}^1 \psi(a, \mu)\mu\,d\mu\right]^{-1}\right\} \tag{11}$$

or, if we use the approximations given by equations (7b and c) and let $2S(b - a) = \sigma_2$,

$$\zeta = \frac{a}{b - a}\left(\frac{1 - c_1}{1 - c_2}\right)\left\{\left[\sum_{\alpha=0}^{N}\left(\frac{1}{\alpha + 2}\right)(e_\alpha - f_\alpha)\right]^{-1} - 1\right\}. \tag{12}$$

As a continuation of a recently reported work (Neshat and Maiorino, 1980) on dissimilar media, we wish to discuss now the critical problem for a three-region reactor. We take $x = 0$ to be the plane of symmetry and consider that the fuel region $|x| \in [0, a]$ has $c_1 > 1$. We take $|x| \in [a, b]$ to be a blanket region and thus allow $c_2 \geq 1$ or $c_2 < 1$. For the reflector $|x| \in [b, c]$ we have $c_3 < 1$. We therefore seek solutions of

$$\mu \frac{\partial}{\partial x}\psi_\alpha(x, \mu) + \psi_\alpha(x, \mu) = \tfrac{1}{2}c_\alpha \sum_{l=0}^{L_\alpha} (2l + 1) f_{\alpha,l} P_l(\mu) \times \int_{-1}^1 P_l(\mu')\psi_\alpha(x, \mu')\,d\mu', \quad \alpha = 1, 2 \text{ and } 3, \tag{13}$$

subject to

$$\psi(-x, -\mu) = \psi(x, \mu), \quad \mu \in (-1, 1), \tag{14a}$$

$$\psi_1(a, \pm\mu) = \psi_2(a, \pm\mu), \quad \mu \in (0, 1), \tag{14b}$$

$$\psi_2(b, \pm\mu) = \psi_3(b, \pm\mu), \quad \mu \in (0, 1), \tag{14c}$$

and

$$\psi_3(c, -\mu) = 0, \quad \mu \in (0, 1). \tag{14d}$$

Here again we can apply the analysis of Devaux *et al.* (1979) to establish the following basic equations for this symmetric application:

$$\int_0^1 \mu\{\phi_1(-\xi,\mu)\psi(a,-\mu) - \phi_1(\xi,\mu)\psi(a,\mu) + e^{-2a/\xi}[\phi_1(\xi,\mu)\psi(a,-\mu) - \phi_1(-\xi,\mu)\psi(a,\mu)]\} \times d\mu = 0, \tag{15a}$$

$$\int_0^1 \mu\{\phi_2(-\xi,\mu)\psi(a,\mu) - \phi_2(\xi,\mu)\psi(a,-\mu) + e^{-\Delta_b/\xi}[\phi_2(\xi,\mu)\psi(b,-\mu) - \phi_2(-\xi,\mu)\psi(b,\mu)]\} \times d\mu = 0, \tag{15b}$$

$$\int_0^1 \mu\{\phi_2(-\xi,\mu)\psi(b,-\mu) - \phi_2(\xi,\mu)\psi(b,\mu) + e^{-\Delta_b/\xi}[\phi_2(\xi,\mu)\psi(a,\mu) - \phi_2(-\xi,\mu)\psi(a,-\mu)]\} \times d\mu = 0, \tag{15c}$$

$$\int_0^1 \mu\{\phi_3(-\xi,\mu)\psi(b,\mu) - \phi_3(\xi,\mu)\psi(b,-\mu) - e^{-\Delta_r/\xi}\phi_3(-\xi,\mu)\psi(c,\mu)\}\, d\mu = 0 \tag{15d}$$

and

$$\int_0^1 \mu\{-\phi_3(\xi,\mu)\psi(c,\mu) + e^{-\Delta_r/\xi}[\phi_3(\xi,\mu)\psi(b,\mu) - \phi_3(-\xi,\mu)\psi(b,-\mu)]\} \times d\mu = 0, \tag{15e}$$

where $\Delta_b = b - a$ and $\Delta_r = c - b$. We note that for equation (15a) $\xi \in P_1$, for equations (15b and c) $\xi \in P_2$ and for equations (15d and e) $\xi \in P_3$. If we now substitute the approximations, for $\mu > 0$,

$$\psi(a,\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \tag{16a}$$

$$\psi(a,-\mu) = \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha, \tag{16b}$$

$$\psi(b,\mu) = \sum_{\alpha=0}^{N} d_\alpha \mu^\alpha, \tag{16c}$$

$$\psi(b,-\mu) = \sum_{\alpha=0}^{N} e_\alpha \mu^\alpha \tag{16d}$$

and

$$\psi(c,\mu) = \sum_{\alpha=0}^{N} f_\alpha \mu^\alpha, \tag{16e}$$

into equations (15) we find the F_N equations for this problem:

$$\sum_{\alpha=0}^{N} \{a_\alpha B_\alpha^{(1)}(\xi_\beta) - b_\alpha A_\alpha^{(1)}(\xi_\beta) + e^{-2a/\xi_\beta}[a_\alpha A_\alpha^{(1)}(\xi_\beta) - b_\alpha B_\alpha^{(1)}(\xi_\beta)]\} = 0, \tag{17a}$$

$$\sum_{\alpha=0}^{N} \{a_\alpha A_\alpha^{(2)}(\xi_\beta) - b_\alpha B_\alpha^{(2)}(\xi_\beta) + e^{-\Delta_b/\xi_\beta}[e_\alpha B_\alpha^{(2)}(\xi_\beta) - d_\alpha A_\alpha^{(2)}(\xi_\beta)]\} = 0, \tag{17b}$$

$$\sum_{\alpha=0}^{N} \{e_\alpha A_\alpha^{(2)}(\xi_\beta) - d_\alpha B_\alpha^{(2)}(\xi_\beta) + e^{-\Delta_b/\xi_\beta}[a_\alpha B_\alpha^{(2)}(\xi_\beta) - b_\alpha A_\alpha^{(2)}(\xi_\beta)]\} = 0, \tag{17c}$$

$$\sum_{\alpha=0}^{N} \{d_\alpha A_\alpha^{(3)}(\xi_\beta) - e_\alpha B_\alpha^{(3)}(\xi_\beta) - e^{-\Delta_r/\xi_\beta} f_\alpha A_\alpha^{(3)}(\xi_\beta)\} = 0 \tag{17d}$$

and

$$\sum_{\alpha=0}^{N} \{-f_\alpha B_\alpha^{(3)}(\xi_\beta) + e^{-\Delta_r/\xi_\beta}[d_\alpha B_\alpha^{(3)}(\xi_\beta) - e_\alpha A_\alpha^{(3)}(\xi_\beta)]\} = 0. \tag{17e}$$

If we set $a_0 = 1$, then we can solve equations (17) iteratively to find the critical half thickness a, as discussed previously (Grandjean and Siewert, 1979; Neshat and Maiorino, 1980).

Table 1. Basic data

Case	a	b–a	$3f_{2,1}$	$5f_{2,2}$	$7f_{2,3}$	$9f_{2,4}$
1	0.1434	1.631	0	0	0	0
2	0.1434	1.631	2	0	0	0
3	0.1434	1.631	2	1.25	0	0
4	0.1434	1.631	2	1.25	0	−0.375
5	0.1434	1.631	0.97088	0	0	0
6	0.1434	1.631	0.97088	0.24428	0	0
7	0.2868	3.262	0	0	0	0
8	0.2868	3.262	2	0	0	0
9	0.2868	3.262	2	1.25	0	0
10	0.2868	3.262	2	1.25	0	−0.375
11	0.2868	3.262	0.97088	0	0	0
12	0.2868	3.262	0.97088	0.24428	0	0

Table 2. Disadvantage factor

Case	F_1	F_3	F_5	F_7	'Exact'	L-S-V-K
1	1.235	1.2305	1.2317	1.2318	1.2317	1.2314
2	1.166	1.1622	1.1633	1.1635	1.1634	1.1630
3	1.172	1.1672	1.1683	1.1684	1.1683	1.1679
4	1.172	1.1671	1.1681	1.1683	1.1682	1.1678
5	1.201	1.1974	1.1985	1.1987	1.1986	1.1982
6	1.202	1.1983	1.1994	1.1995	1.1995	1.1991
7	1.621	1.6286	1.6286	1.6284	1.6284	1.6284
8	1.353	1.3601	1.3600	1.3599	1.3599	1.3599
9	1.363	1.3685	1.3684	1.3683	1.3683	1.3682
10	1.363	1.3684	1.3683	1.3682	1.3682	1.3682
11	1.492	1.4990	1.4990	1.4988	1.4988	1.4989
12	1.493	1.5004	1.5003	1.5002	1.5002	1.5002

II. NUMERICAL RESULTS

In order to have a simple scheme for selecting the points ξ_β to be used in equations (8) we use, as suggested by Pomraning (1979),

$$\xi_\beta^{(1)} = \nu_{1,\beta}, \quad \beta = 0, 1, 2, \ldots \kappa_1 - 1,$$

$$\xi_{j+\kappa_1-1}^{(1)} = (2j - 1)/[2(N - \kappa_1 + 1)],$$

$$j = 1, 2, \ldots (N - \kappa_1 + 1),$$

where N is the order of the approximation. We choose the points $\xi_\beta^{(2)}$ to be used in equations (9) in a similar manner. We consider that

$$c_1 = 0.55370, \quad c_2 = 0.99163,$$

$$f_{1,l} = \delta_{0,l}, \quad f_{2,0} = 1, \quad f_{2,l} = 0, \quad l > 4,$$

and the additional basic data given in Table 1 are exact, and in Table 2 we list the disadvantage factor computed by various orders of the F_N approximation. We also list under L-S-V-K the results of Laletin *et al.* (1974). We believe the results we list as 'exact' are accurate to ± 1 in the last significant figure. It is apparent that the calculations of Eccleston and McCormick (1970) are in error for cases 3, 4, 6, 9, 10 and 12.

For the critical calculation we use the same scheme for choosing the points ξ_β at which to solve equations (17). For reporting purposes we consider only isotropic scattering, and thus we list in Table 3 the cases studied. In Table 4 we list our F_N results and what we conclude to be 'exact' to within ± 1 in the last significant figure. We note that our results for

Table 3. Cases studied

Case	c_1	c_2	c_3	Δ_b	Δ_r
1	1.3	0.95	0.9	1	∞
2	1.3	0.95	0.9	2	∞
3	1.3	1.1	0.9	1	∞
4	1.3	1.1	0.9	0.5	∞
5	1.3	0.95	0.9	1	3
6	1.3	0.95	0.9	2	3
7	1.3	1.1	0.9	1	3
8	1.3	1.1	0.9	0.5	3
9	1.3	0.95	0.9	1	1
10	1.3	0.95	0.9	2	1
11	1.3	1.1	0.9	1	1
12	1.3	1.1	0.9	0.5	1

Table 4. Critical half thickness

Case	F_0	F_3	F_5	F_7	'Exact'
1	0.410	0.40731	0.40733	0.40734	0.40734
2	0.380	0.38029	0.38031	0.38032	0.38032
3	0.112	0.10629	0.10626	0.10625	0.10625
4	0.278	0.27109	0.27114	0.27114	0.27114
5	0.420	0.41296	0.41299	0.41299	0.41299
6	0.385	0.38278	0.38281	0.38282	0.38282
7	0.129	0.11612	0.11609	0.11607	0.11607
8	0.298	0.28203	0.28207	0.28207	0.28207
9	0.483	0.45542	0.45544	0.45545	0.45545
10	0.412	0.40091	0.40094	0.40095	0.40095
11	0.233	0.19024	0.19018	0.19018	0.19018
12	0.423	0.36781	0.36783	0.36783	0.36783

cases 1–4 agree with the F_N calculations of Ishiguro and Fernandes (1979) and that Lawrence (1979), using an alternative calculational procedure, has obtained for cases 5–12 results in perfect agreement with our 'exact' values.

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