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The F_N Method for Radiative Transfer Problems Without Azimuthal Symmetry

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I. Introduction

In the past two years the F_N method has been developed and utilized [1–5] for many applications in neutron-transport theory and radiative transfer. Because the method has proved efficient and accurate, we now would like to report the generalizations that are required for the method to be applicable to problems in plane geometry that do not have azimuthal symmetry. For example, an anisotropically scattering plane-parallel medium illuminated by parallel rays will, in general, have an associated radiation field that is not azimuthally symmetric. We thus consider the equation of transfer [6]

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu, \varphi) + I(\tau, \mu, \varphi) = \frac{\omega}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\cos \Theta) I(\tau, \mu', \varphi')\, d\mu'\, d\varphi' \tag{1}$$

where ω is the single scattering albedo, μ is the direction cosine, as measured from the *positive* τ axis, of the propagating radiation and φ is the azimuthal angle measured with respect to a reference angle φ_r. In addition Θ is the scattering angle, and we consider phase functions that have a Legendre expansion of the form

$$p(\cos \Theta) = \sum_{l=0}^{L} (2l+1) f_l P_l(\cos \Theta), \quad f_0 = 1. \tag{2}$$

Since we intend to use the analysis developed here for multi-region problems, we consider boundary conditions of the form

$$I(L, \mu, \varphi) = F_1(\mu, \varphi), \quad \mu > 0, \varphi \in [0, 2\pi], \tag{3a}$$

and

$$I(R, -\mu, \varphi) = F_2(\mu, \varphi), \quad \mu > 0, \varphi \in [0, 2\pi], \tag{3b}$$

where $\tau = L$ and $\tau = R$ refer respectively to the left and right (or upper and lower) boundary surfaces and $F_1(\mu, \varphi)$ and $F_2(\mu, \varphi)$ are considered given. We note that here $I(\tau, \mu, \varphi)$ is the complete intensity and not just the diffuse component. Following

Chandrasekhar [6], we now express the intensity as

$$I(\tau, \mu, \varphi) = \sum_{m=0}^{L} [I_c^m(\tau, \mu) \cos m(\varphi - \varphi_r) + I_s^m(\tau, \mu) \sin m(\varphi - \varphi_r)] + R(\mu, \varphi)\, e^{-\tau/\mu}, \tag{4}$$

where

$$\int_0^{2\pi} R(\mu', \varphi') p(\cos \Theta)\, d\varphi' = 0, \quad \mu, \mu' \in [-1, 1], \varphi \in [0, 2\pi]. \tag{5}$$

We note that the form used in Eqn. (4) is a generalization of that used by Chandrasekhar [6] and that it is convenient for separating the complete problem for $I(\tau, \mu, \varphi)$ into a set of problems for the φ independent components $I_c^m(\tau, \mu)$ and $I_s^m(\tau, \mu)$.

If we use the addition theorem [7] for the Legendre polynomials we can write

$$p(\cos \Theta) = \sum_{m=0}^{L} (2 - \delta_{0,m}) \sum_{l=m}^{L} (2l + 1) f_l^m P_l^m(\mu) P_l^m(\mu') \cos m(\varphi - \varphi') \tag{6}$$

and substitute Eqn. (4) into Eqn. (1) to deduce that

$$\mu \frac{\partial}{\partial \tau} I^m(\tau, \mu) + I^m(\tau, \mu) = \frac{\omega}{2} \sum_{l=m}^{L} (2l + 1) f_l^m P_l^m(\mu) \int_{-1}^{1} P_l^m(\mu') I^m(\tau, \mu')\, d\mu'. \tag{7}$$

Here we use $I^m(\tau, \mu)$ for $I_c^m(\tau, \mu)$ and/or $I_s^m(\tau, \mu)$, and $P_l^m(\mu)$ denotes the associated Legendre function, i.e.

$$P_l^m(\mu) = (1 - \mu^2)^{m/2} \frac{d^m}{d\mu^m} P_l(\mu), \tag{8}$$

and

$$f_l^m = f_l \frac{(l - m)!}{(l + m)!}. \tag{9}$$

Considering now the boundary conditions, we observe that if we take, for $\mu > 0$,

$$R(\mu, \varphi)\, e^{-L/\mu} = F_1(\mu, \varphi) - \sum_{m=0}^{L} [I_c^m(L, \mu) \cos m(\varphi - \varphi_r) + I_s^m(L, \mu) \sin m(\varphi - \varphi_r)] \tag{10a}$$

and

$$R(-\mu, \varphi)\, e^{R/\mu} = F_2(\mu, \varphi) - \sum_{m=0}^{L} [I_c^m(R, -\mu) \cos m(\varphi - \varphi_r) + I_s^m(R, -\mu) \sin m(\varphi - \varphi_r)] \tag{10b}$$

the components $I^m(\tau, \mu)$ do, in fact, decouple since they satisfy Eqn. (7) and the boundary conditions

$$I^m(L, \mu) = \left(\frac{2 - \delta_{0,m}}{2\pi}\right) \int_0^{2\pi} F_1(\mu, \varphi) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m(\varphi - \varphi_r)\, d\varphi, \quad \mu > 0, \tag{11a}$$

and

$$I^m(R, -\mu) = \left(\frac{2 - \delta_{0,m}}{2\pi}\right) \int_0^{2\pi} F_2(\mu, \varphi) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m(\varphi - \varphi_r)\, d\varphi, \quad \mu > 0. \tag{11b}$$

In order to construct the complete solution $I(\tau, \mu, \varphi)$ we clearly must solve the problems defined by Eqns. (7) and (11) for $m = 0, 1, 2, \ldots, L$.

II. Analysis—The F_N Method

Since the analysis here in regard to the F_N method follows very closely that reported earlier for the case $m = 0$, and since we also follow closely the work of McCormick and Kuščer [8] concerning the elementary solutions of Eqn. (7) we can be brief in our development here. Also we omit the index m except where it is essential. We begin by writing a general solution [8] of Eqn. (7) as

$$I(\tau, \mu) = \sum_{\beta=0}^{\kappa-1} [A(\nu_\beta)\varphi(\nu_\beta, \mu)\, e^{-\tau/\nu\beta} + A(-\nu_\beta)\varphi(-\nu_\beta, \mu)\, e^{\tau/\nu\beta}] + \int_{-1}^{1} A(\nu)\varphi(\nu, \mu)\, e^{-\tau/\nu}\, d\nu \tag{12}$$

where we have $\kappa \pm$ pairs of zeros $(\pm \nu_\beta)$ of

$$\Lambda(z) = 1 + z \int_{-1}^{1} \psi(\mu) \frac{d\mu}{\mu - z} \tag{13}$$

with

$$\psi(\mu) = \frac{\omega}{2} \sum_{l=m}^{L} (2l + 1) f_l^m (1 - \mu^2)^{m/2} P_l^m(\mu) g_l^m(\mu). \tag{14}$$

Here the polynomials $g_l^m(\xi)$ satisfy [6]

$$(l - m + 1) g_{l+1}^m(\xi) = h_l \xi g_l^m(\xi) - (l + m)(1 - \delta_{m,l}) g_{l-1}^m(\xi), \quad l \geq m, \tag{15a}$$

with

$$g_m^m(\xi) = (2m - 1)!!. \tag{15b}$$

In addition

$$\varphi(\nu_\beta, \mu) = \frac{\omega}{2} \nu_\beta \left(\frac{1}{\nu_\beta - \mu}\right) \sum_{l=m}^{L} (2l + 1) f_l^m P_l^m(\mu) g_l^m(\nu_\beta) \tag{16a}$$

and

$$\varphi(\nu, \mu) = \frac{\omega}{2} \nu P v \left(\frac{1}{\nu - \mu}\right) \sum_{l=m}^{L} (2l + 1) f_l^m P_l^m(\mu) g_l^m(\nu) + (1 - \nu^2)^{-m/2} \lambda(\nu) \delta(\nu - \mu) \tag{16b}$$

where

$$\lambda(\nu) = 1 + \nu P \int_{-1}^{1} \psi(\mu) \frac{d\mu}{\mu - \nu}, \tag{17}$$

and

$$h_l = (2l + 1)(1 - \omega f_l). \tag{18}$$

As for the case $m = 0$, there is a full-range orthogonality theorem [8] concerning the elementary solutions, i.e.

$$(\xi - \xi') \int_{-1}^{1} \mu\varphi(\xi, \mu)\varphi(\xi', \mu)\, d\mu = 0, \quad \xi, \xi' = \pm\nu_\beta \text{ or } \in (-1, 1), \tag{19}$$

and thus we can use the technique described earlier [3] to develop the following singular integral equations and constraints concerning the desired surface intensities $I(L, -\mu)$ and $I(R, \mu)$, $\mu > 0$:

$$\int_0^1 \mu\varphi(\xi, \mu)I(L, -\mu)\, d\mu + e^{-\Delta/\xi} \int_0^1 \mu\varphi(-\xi, \mu)I(R, \mu)\, d\mu = L_1(\xi) \tag{20a}$$

and

$$\int_0^1 \mu\varphi(\xi, \mu)I(R, \mu)\, d\mu + e^{-\Delta/\xi} \int_0^1 \mu\varphi(-\xi, \mu)I(L, -\mu)\, d\mu = L_2(\xi). \tag{20b}$$

Here $\xi \in P = \{\nu_\beta\} \cup (0, 1)$, $\Delta = R - L$ and the known terms are

$$L_1(\xi) = \int_0^1 \mu\varphi(-\xi, \mu)I(L, \mu)\, d\mu + e^{-\Delta/\xi} \int_0^1 \mu\varphi(\xi, \mu)I(R, -\mu)\, d\mu \tag{21a}$$

and

$$L_2(\xi) = \int_0^1 \mu\varphi(-\xi, \mu)I(R, -\mu)\, d\mu + e^{-\Delta/\xi} \int_0^1 \mu\varphi(\xi, \mu)I(L, \mu)\, d\mu. \tag{21b}$$

If we now introduce the approximations

$$I(L, -\mu) = (1 - \mu^2)^{m/2} \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \quad \mu > 0, \tag{22a}$$

and

$$I(R, \mu) = (1 - \mu^2)^{m/2} \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha, \quad \mu > 0, \tag{22b}$$

into Eqns. (20) we obtain the F_N equations:

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\xi) + e^{-\Delta/\xi} b_\alpha A_\alpha(\xi)] = \frac{2}{\omega\xi} L_1(\xi), \quad \xi \in P, \tag{23a}$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha(\xi) + e^{-\Delta/\xi} a_\alpha A_\alpha(\xi)] = \frac{2}{\omega\xi} L_2(\xi), \quad \xi \in P, \tag{23b}$$

where

$$A_\alpha(\xi) = \frac{2}{\omega\xi} \int_0^1 \mu^{\alpha+1} \varphi(-\xi, \mu)(1-\mu^2)^{m/2}\, d\mu \tag{24a}$$

and

$$B_\alpha(\xi) = \frac{2}{\omega\xi} \int_0^1 \mu^{\alpha+1} \varphi(\xi, \mu)(1-\mu^2)^{m/2}\, d\mu. \tag{24b}$$

We can now use the explicit expressions given by Eqns. (16) and (17) to derive a set of recursive relations that is useful for computing the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$. We find

$$A_{\alpha+1}(\xi) = -\xi A_\alpha(\xi) + (-1)^m \sum_{l=m}^{L} (2l+1)(-1)^l f_l^m g_l^m(\xi) \Delta_{\alpha,l}^m \tag{25a}$$

and

$$B_{\alpha+1}(\xi) = \xi B_\alpha(\xi) - \sum_{l=m}^{L} (2l+1) f_l^m g_l^m(\xi) \Delta_{\alpha,l}^m \tag{25b}$$

with

$$\Delta_{\alpha,l}^m = \int_0^1 \mu^{\alpha+1}(1-\mu^2)^{m/2} P_l^m(\mu)\, d\mu. \tag{26}$$

For $\alpha = 0$, we find

$$B_0(\xi) = A_0(\xi) + \frac{2}{\omega}\left(\frac{h_m}{2m+1}\right) \tag{27}$$

and

$$A_0(\xi) = \sum_{l=m}^{L} (2l+1) f_l^m g_l^m(\xi) \Pi_l^m(\xi) - \frac{2}{\omega} \xi \psi(\xi) \log\left(1 + \frac{1}{\xi}\right). \tag{28}$$

Here the polynomials $\Pi_l^m(\xi)$ can be generated, for $l \geq m$, from

$$\begin{aligned}(2l+1)\xi \Pi_l^m(\xi) = {} & (-1)^{l-m}(2l+1)\Delta_{0,l}^m + (l-m+1)\Pi_{l+1}^m(\xi) \\ & + (l+m)(1-\delta_{l,m})\Pi_{l-1}^m(\xi)\end{aligned} \tag{29a}$$

with

$$\Pi_0^0(\xi) = 1 \tag{29b}$$

and

$$\Pi_{\alpha+1}^{\alpha+1}(\xi) = (2\alpha + 1)(1 - \xi^2)\Pi_\alpha^\alpha(\xi) + \frac{(2\alpha + 1)!!}{2(\alpha + 1)}\xi - \frac{2^\alpha \alpha!}{2\alpha + 3}. \tag{29c}$$

It is important to note here that we have taken the definition given by Eqn. (8) to be valid for all $\xi \in P$, i.e.

$$P_l^m(\xi) = (1 - \xi^2)^{m/2}\frac{d^m}{d\xi^m}P_l(\xi). \tag{30}$$

The explicit evaluation of the integral appearing in Eqn. (26) is given by Robin [9]; however, we prefer to use that result to deduce the following recursive relation

$$\Delta_{\alpha,l+2}^m = \left(\frac{l + 2 + m}{l + 2 - m}\right)\left(\frac{l + 1 + m}{l + 1 - m}\right)\left(\frac{1 + \alpha - l + m}{4 + \alpha + l + m}\right)\Delta_{\alpha,l}^m \tag{31a}$$

with

$$\Delta_{\alpha,m}^m = \frac{\sqrt{\pi}}{2^{\alpha+m+2}}\frac{(2m)!\,(\alpha + 1)!}{\left(\dfrac{\alpha + 1}{2}\right)!\left(\dfrac{\alpha + 2 + 2m}{2}\right)!} \tag{31b}$$

and

$$\Delta_{\alpha,m+1}^m = \frac{\sqrt{\pi}}{2^{\alpha+m+2}}\frac{(2m + 1)!\,(\alpha + 1)!}{\left(\dfrac{\alpha}{2}\right)!\left(\dfrac{\alpha + 3 + 2m}{2}\right)!}. \tag{31c}$$

Since the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ are now readily available we can select $N + 1$ values of $\xi \in P$ at which to satisfy Eqns. (23); we thus can generate $2(N + 1)$ linear algebraic equations that can be solved to establish the required constants a_α and b_α, $\alpha = 0, 1, 2, \ldots, N$.

III. Discrete Eigenvalues

From the foregoing analysis it is clear that the calculation of the discrete eigenvalues $\{\nu_\beta\}$ is essential to the F_N solution of the considered problem. In fact, because the F_N equations are so easy to generate and to solve, the computation of the discrete eigenvalues is perhaps the most difficult aspect of the method. We thus would like to report some explicit expressions, at least for the cases for which $\kappa \leq 3$, that yield exact results for the desired $\{\nu_\beta\}$. We note that McCormick and Kuščer [8] have shown that $\kappa \leq L - m + 1$ and that Shultis [10] has argued that the discrete eigenvalues are, for $\omega < 1$, simple and real. The upper bound $L - m + 1$ is very conservative, and thus we use the argument principle [11] to establish κ. Since $\Lambda(z)$ is analytic in the complex plane cut from -1 to 1 along the real axis, $\Lambda(z) = \Lambda(-z)$, $\overline{\Lambda(\bar{z})} = \Lambda(z)$ and

$\Lambda(\infty) \neq 0$, then $\pi\kappa$ is the change in the argument of

$$\lim_{\varepsilon \to 0} \Lambda(t + i\varepsilon) = \Lambda^+(t) = \lambda(t) + \pi i t \psi(t) \tag{32}$$

as t varies from zero to one; i.e.

$$\kappa = \frac{1}{\pi} \Delta_{0,1}[\arg \Lambda^+(t)]. \tag{33}$$

We define

$$\theta(t) = \arg \Lambda^+(t) \tag{34}$$

and write

$$\theta(t) = \tan^{-1} \left[\frac{\pi t \psi(t)}{\lambda(t)} \right] \tag{35}$$

where $\theta(t)$ is defined to vary continuously from 0 to $\kappa\pi$ as t varies from 0 to 1.

We note that $\Lambda(z)$, as given by Eqn. (13), can readily be evaluated numerically by using a quadrature scheme to compute the required integral (and for large values of z we prefer this method). The integral in Eqn. (13) can also be evaluated analytically to yield

$$\Lambda(z) = 1 + z\psi(z) \log\left(\frac{z-1}{z+1}\right) + \omega z \sum_{l=m}^{L} (2l+1) f_l^m g_l^m(z) \Gamma_l^m(z), \tag{36}$$

where the polynomials $\Gamma_l^m(z)$ can be generated, for $l \geq m$, from

$$(2l+1) z \Gamma_l^m(z) = -2^m m!\, \delta_{l,m} + (l+1-m)\Gamma_{l+1}^m(z) + (l+m)(1-\delta_{l,m})\Gamma_{l-1}^m(z) \tag{37}$$

with

$$\Gamma_0^0(z) = 0 \tag{38a}$$

and

$$\Gamma_{\alpha+1}^{\alpha+1}(z) = (2\alpha+1)(1-z^2)\Gamma_\alpha^\alpha(z) - 2^\alpha \alpha!\, z. \tag{38b}$$

Following now a previous work [12] based on the case $m = 0$, we write

$$\Lambda(\infty) = 1 - \omega \sum_{l=m}^{L} f_l W_l^m \tag{39}$$

where

$$W_l^m = \frac{(l-m)!}{(l+m)!} \left(\frac{2l+1}{2}\right) \int_{-1}^{1} (1-\mu^2)^{m/2} P_l^m(\mu) g_l^m(\mu)\, d\mu. \tag{40}$$

We find that

$$(2l+1) W_{l+1}^m = h_l W_l^m, \tag{41}$$

with

$$W_m^m = 1, \tag{42}$$

so that Eqn. (39) can be expressed as [10]

$$\Lambda(\infty) = \prod_{l=m}^{L} (1 - \omega f_l). \tag{43}$$

We also deduce that

$$\Lambda(z) \to \Lambda(\infty) + \frac{a_2}{z^2} + \frac{a_4}{z^4} + \cdots, \quad \text{as } z \to \infty, \tag{44}$$

where

$$a_2 = -\omega \sum_{l=m}^{L} \left(\frac{2l+1}{2}\right) f_l^m \int_{-1}^{1} \mu^2 (1-\mu^2)^{m/2} P_l^m(\mu) g_l^m(\mu)\, d\mu = -\omega \sum_{l=m}^{L} f_l B_l^m \tag{45}$$

and

$$a_4 = -\omega \sum_{l=m}^{L} \left(\frac{2l+1}{2}\right) f_l^m \int_{-1}^{1} \mu^4 (1-\mu^2)^{m/2} P_l^m(\mu) g_l^m(\mu)\, d\mu = -\omega \sum_{l=m}^{L} f_l C_l^m. \tag{46}$$

Here

$$(2l+1) B_{l+1}^m = h_l B_l^m + \frac{(l+2+m)(l+2-m)}{(2l+3)(2l+5)} h_l W_l^m - \frac{(l+m)(l-m)}{2l-1} W_{l-1}^m, \tag{47}$$

with

$$B_m^m = \frac{1}{2m+3}, \tag{48}$$

and

$$(2l+1) C_{l+1}^m = h_l C_l^m + \frac{(l+2+m)(l+2-m)}{(2l+3)(2l+5)} h_l T_l^m - \frac{(l+m)(l-m)}{2l-1} T_{l-1}^m, \tag{49}$$

with

$$T_l^m = B_l^m + \frac{1}{2l+5}\left[\frac{(l+3+m)(l+3-m)}{2l+7} + \frac{(l+2+m)(l+2-m)}{2l+3}\right] W_l^m \tag{50}$$

and

$$C_m^m = \frac{3}{(2m+3)(2m+5)}. \tag{51}$$

A Wiener-Hopf factorization of $\Lambda(z)$ is given by

$$\Lambda(z) = \Lambda(\infty) X(z) X(-z) \prod_{\alpha=1}^{\kappa} (v_{\alpha-1}^2 - z^2) \tag{52}$$

where

$$X(z) = \frac{1}{(1-z)^{\kappa}} \exp\left(\frac{1}{\pi}\int_0^1 \theta(t)\frac{dt}{t-z}\right), \tag{53}$$

and thus the results given previously for $m = 0$ [12] can be used here; for $\kappa = 1$

$$v_0^2 = \frac{1}{\Lambda(\infty)} \exp\left(-\frac{2}{\pi}\int_0^1 \theta(t)\frac{dt}{t}\right) \tag{54}$$

and

$$v_0^2 = 1 - \theta_1 + \frac{\omega}{\Lambda(\infty)} \sum_{l=m}^{L} f_l B_l^m \tag{55}$$

where

$$\theta_\alpha = \frac{2}{\pi}\int_0^1 t^\alpha \theta(t)\, dt. \tag{56}$$

For $\kappa = 2$ we have

$$v_0^2 = A + (A^2 - B)^{1/2} \tag{57a}$$

and

$$v_1^2 = A - (A^2 - B)^{1/2} \tag{57b}$$

where

$$A = 1 - \tfrac{1}{2}\theta_1 + \tfrac{1}{2}\frac{\omega}{\Lambda(\infty)} \sum_{l=m}^{L} f_l B_l^m \tag{58}$$

and

$$B = \frac{1}{\Lambda(\infty)} \exp\left(-\frac{2}{\pi}\int_0^1 \theta(t)\frac{dt}{t}\right). \tag{59}$$

When $\kappa = 3$, we write

$$v_0^2 v_1^2 v_2^2 = \frac{1}{\Lambda(\infty)} \exp\left(-\frac{2}{\pi}\int_0^1 \theta(t)\frac{dt}{t}\right), \tag{60a}$$

$$v_0^2 + v_1^2 + v_2^2 = 3 - \theta_1 + \frac{\omega}{\Lambda(\infty)} \sum_{l=m}^{L} f_l B_l^m \tag{60b}$$

and

$$\nu_0^2\nu_1^2 + \nu_0^2\nu_2^2 + \nu_1^2\nu_2^2 = 3(1-\theta_1) + \theta_3 + \tfrac{1}{2}\theta_1^2 + (3-\theta_1)\frac{\omega}{\Lambda(\infty)}\sum_{l=m}^{L} f_l B_l^m - \frac{\omega}{\Lambda(\infty)}\sum_{l=m}^{L} f_l C_l^m. \tag{60c}$$

Equations (60) clearly can be solved to yield ν_0^2, ν_1^2 and ν_2^2.

We have explicit results here clearly when $\kappa \leq 3$; however, when $\kappa > 3$ various (arbitrary) values of z can be used in Eqn. (52) to generate κ equations that can be solved iteratively to find $\{\nu_\beta\}$.

IV. Numerical Results

In order to demonstrate the F_N method for a basic problem without azimuthal symmetry we consider now a beam incident on the surface $\tau = L$. We thus write

$$F_1(\mu, \varphi) = \pi\delta(\mu - \mu_0)\delta(\varphi - \varphi_0) \tag{61a}$$

and

$$F_2(\mu, \varphi) = 0. \tag{61b}$$

If we now take $\phi_r = \phi_0$ then the terms involving $I_s^m(\tau, \mu)$ in Eqn. (4) will not be required. For our sample calculation we use the scattering law given in Table I. The

Table I
The scattering law.

l	$(2l+1)f_l$
0	1
1	2.00916
2	1.56339
3	0.67407
4	0.22215
5	0.04725
6	0.00671
7	0.00068
8	0.00005

Table II
Eigenvalues for $\omega = 0.95$.

m	$\{\nu_\beta\}$
0	4.4036458255
	1.0000526893
1	1.1322065380
2–8	—

Table III
$I^m(L, -\mu)$ for $\Delta = 1.0$, $\omega = 0.95$ and $\mu_0 = 0.5$.

μ	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$
0.05	0.3568	0.4659	0.1959	0.5548(−1)	0.1379(−1)	0.2306(−2)	0.2622(−3)	0.2166(−4)	0.1328(−5)
0.1	0.3401	0.4286	0.1735	0.4770(−1)	0.1187(−1)	0.1993(−2)	0.2272(−3)	0.1879(−4)	0.1181(−5)
0.2	0.2975	0.3522	0.1340	0.3475(−1)	0.8639(−2)	0.1456(−2)	0.1656(−3)	0.1361(−4)	0.8945(−6)
0.3	0.2517	0.2817	0.1014	0.2464(−1)	0.6083(−2)	0.1020(−2)	0.1146(−3)	0.9224(−5)	0.6294(−6)
0.4	0.2082	0.2204	0.7499(−1)	0.1684(−1)	0.4094(−2)	0.6754(−3)	0.7409(−4)	0.5750(−5)	0.4036(−6)
0.5	0.1697	0.1689	0.5394(−1)	0.1096(−1)	0.2591(−2)	0.4152(−3)	0.4379(−4)	0.3211(−5)	0.2292(−6)
0.6	0.1365	0.1263	0.3730(−1)	0.6651(−2)	0.1504(−2)	0.2295(−3)	0.2279(−4)	0.1536(−5)	0.1097(−6)
0.7	0.1083	0.9104(−1)	0.2424(−1)	0.3628(−2)	0.7617(−3)	0.1075(−3)	0.9727(−5)	0.5797(−6)	0.4025(−7)
0.8	0.8459(−1)	0.6166(−1)	0.1405(−1)	0.1637(−2)	0.3025(−3)	0.3731(−4)	0.2905(−5)	0.1430(−6)	0.9164(−8)
0.9	0.6456(−1)	0.3602(−1)	0.6134(−2)	0.4719(−3)	0.6693(−4)	0.6281(−5)	0.3646(−6)	0.1270(−7)	0.6563(−9)
1.0	0.4768(−1)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table IV
$I^m_*(R, \mu)$ for $\Delta = 1.0$, $\omega = 0.95$ and $\mu_0 = 0.5$.

μ	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$
0.05	0.1795	0.1671	0.4676(−1)	0.1108(−1)	0.2552(−2)	0.4136(−3)	0.4636(−4)	0.3786(−5)	0.2197(−6)
0.1	0.2031	0.1924	0.5453(−1)	0.1302(−1)	0.2964(−2)	0.4745(−3)	0.5268(−4)	0.4259(−5)	0.2397(−6)
0.2	0.2442	0.2412	0.7124(−1)	0.1736(−1)	0.3862(−2)	0.6004(−3)	0.6491(−4)	0.5100(−5)	0.2686(−6)
0.3	0.2703	0.2745	0.8321(−1)	0.2043(−1)	0.4395(−2)	0.6570(−3)	0.6842(−4)	0.5160(−5)	0.2521(−6)
0.4	0.2811	0.2879	0.8742(−1)	0.2118(−1)	0.4348(−2)	0.6173(−3)	0.6114(−4)	0.4364(−5)	0.1956(−6)
0.5	0.2808	0.2849	0.8465(−1)	0.1982(−1)	0.3818(−2)	0.5067(−3)	0.4697(−4)	0.3119(−5)	0.1264(−6)
0.6	0.2730	0.2686	0.7615(−1)	0.1680(−1)	0.2972(−2)	0.3607(−3)	0.3060(−4)	0.1847(−5)	0.6628(−7)
0.7	0.2598	0.2405	0.6286(−1)	0.1261(−1)	0.1982(−2)	0.2128(−3)	0.1597(−4)	0.8465(−6)	0.2604(−7)
0.8	0.2425	0.2001	0.4543(−1)	0.7803(−2)	0.1026(−2)	0.9181(−4)	0.5738(−5)	0.2513(−6)	0.6245(−8)
0.9	0.2217	0.1425	0.2434(−1)	0.3096(−2)	0.2948(−3)	0.1903(−4)	0.8568(−6)	0.2680(−7)	0.4660(−9)
1.0	0.1979	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

law is based on the Mie theory for spherical particles and corresponds to that used previously [13] for $\alpha = 2$ and $m = 1.33$, where α is the size parameter and m is the index of refraction. In Table II we list the discrete eigenvalues as computed from the exact expressions given in the foregoing section and refined by using a Newton-Raphson method for finding the zeros of $\Lambda(z)$.

Of course the approximation given by Eqn. (22b) must be modified here since $I(R, \mu)$ has a component that is a generalized function. We therefore write

$$I(L, -\mu) = (1 - \mu^2)^{m/2} \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \quad \mu > 0, \tag{62a}$$

and

$$I(R, \mu) = I_*(R, \mu) + \tfrac{1}{2}(2 - \delta_{0,m})\delta(\mu - \mu_0)\, e^{-\Delta/\mu}, \quad \mu > 0, \tag{62b}$$

where

$$I_*(R, \mu) = (1 - \mu^2)^{m/2} \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha, \quad \mu > 0. \tag{62c}$$

To find the constants a_α and b_α we now must solve the F_N equations

$$\sum_{\alpha=0}^{N} (a_\alpha B_\alpha(\xi_j) + e^{-\Delta/\xi_j} b_\alpha A_\alpha(\xi_j)) = (2 - \delta_{0,m})\mu_0 \frac{1}{\omega \xi_j} \varphi(-\xi_j, \mu_0)[1 - e^{-\Delta(1/\xi_j + 1/\mu_0)}] \tag{63a}$$

and

$$\sum_{\alpha=0}^{N} (b_\alpha B_\alpha(\xi_j) + e^{-\Delta/\xi_j} a_\alpha A_\alpha(\xi_j)) = (2 - \delta_{0,m})\mu_0 \frac{1}{\omega \xi_j} \varphi(\xi_j, \mu_0)[e^{-\Delta/\xi_j} - e^{-\Delta/\mu_0}]. \tag{63b}$$

Here $\xi_j \in P$, and to have a simple scheme of selecting the points we use $\xi_\beta = \nu_\beta$, $\beta = 0, 1, 2, \ldots, \kappa - 1$, and $\xi_\beta = (2\beta + 1 - 2\kappa)/[2(N - \kappa + 1)]$, $\beta = \kappa, \kappa + 1, \ldots, N$. It is clear that once the constants a_α and b_α are determined the complete angular intensity is readily available from Eqns. (4) and (62). We therefore list in Tables III and IV the exit distributions $I(L, -\mu)$ and $I_*(R, \mu)$, $\mu > 0$, deduced from F_N calculations as N varied from 10 to 15. We believe the 'converged' results shown in Tables III and IV are accurate to ± 1 in the last digit reported.

Finally we note that McCormick and Sanchez [14] have communicated some preliminary work that relies on the F_N method to generate solutions in order to study the effect of experimental errors on a reported solution of the inverse problem [15].

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Abstract

The formalism required to solve by the F_N method radiative transfer problems lacking azimuthal symmetry is developed, and numerical results are given.

Zusammenfassung

Es wird der Formalismus entwickelt der zur Lösung von Strahlungsproblemen mit der F_N-Methode benötigt wird, falls keine azimutale Symmetrie vorliegt. Numerische Ergebnisse werden gegeben.

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