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The Inverse Problem for a Finite Rayleigh-Scattering Atmosphere

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1. Introduction

In a recent paper [1] the half-space solution, in terms of the **H** matrix, was used to deduce the single-scattering albedo from measurements of the polarized radiation field emerging from a Rayleigh-scattering atmosphere. Here we consider a similar problem for a finite atmosphere with Lambert reflection at the ground. We find it sufficient to study the azimuthally symmetric component of the complete solution, and thus we consider the equation of transfer [2]

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \tfrac{1}{2}\omega \mathbf{Q}(\mu) \int_{-1}^{1} \mathbf{Q}^T(\mu')\, \mathbf{I}(\tau, \mu')\, d\mu', \tag{1}$$

where $\mathbf{I}(\tau, \mu)$ has components $I_l(\tau, \mu)$ and $I_r(\tau, \mu)$, τ is the optical variable, μ is the direction cosine as measured from the positive τ axis, ω is the albedo for single scattering and, for Rayleigh scattering,

$$\mathbf{Q}(\mu) = \frac{3^{1/2}}{2} \begin{bmatrix} \mu^2 & 2^{1/2}(1-\mu^2) \\ 1 & 0 \end{bmatrix}. \tag{2}$$

We allow boundary conditions of the form

$$\mathbf{I}(0, \mu) = \mathbf{F}_1(\mu), \qquad \mu > 0, \tag{3a}$$

and

$$\mathbf{I}(\tau_0, -\mu) = \mathbf{F}_2(\mu) = \lambda_0\, \mathbf{D} \int_0^1 \mathbf{I}(\tau_0, \mu')\, \mu'\, d\mu', \qquad \mu > 0, \tag{3b}$$

where λ_0 is the coefficient for Lambert reflection and

$$D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{4}$$

We seek to express ω and λ_0 in terms of $\mathbf{I}(0, -\mu)$ and $\mathbf{I}(\tau_0, \mu)$, $\mu > 0$, which we presume can be measured experimentally.

2. Analysis

If we change μ to $-\mu$ in Eqn. (1) and premultiply the resulting equation by $\mathbf{I}^T(\tau, \mu)$ and integrate over μ from -1 to 1, we find

$$T_0(\tau) = -2 \int_0^1 \mathbf{I}^T(\tau, \mu)\, \mathbf{I}(\tau, -\mu)\, d\mu + \tfrac{1}{2}\omega \mathbf{I}_0^T(\tau)\, \mathbf{I}_0(\tau) \tag{5}$$

where

$$T_0(\tau) = \int_{-1}^{1} \mathbf{I}^T(\tau, \mu)\, \mathbf{F}(\tau, -\mu)\, d\mu, \tag{6}$$

$$\mathbf{F}(\tau, \mu) = \mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) = \tfrac{1}{2}\omega \mathbf{Q}(\mu)\, \mathbf{I}_0(\tau) - \mathbf{I}(\tau, \mu) \tag{7}$$

and, in general,

$$\mathbf{I}_\alpha(\tau) = \int_{-1}^{1} \mu^\alpha\, \mathbf{Q}^T(\mu)\, \mathbf{I}(\tau, \mu)\, d\mu. \tag{8}$$

If we now differentiate Eqn. (6) and use Eqn. (7) we find

$$\frac{d}{d\tau} T_0(\tau) = \frac{d}{d\tau} \left[\frac{\omega}{4} \mathbf{I}_0^T(\tau)\, \mathbf{I}_0(\tau) - \int_0^1 \mathbf{I}^T(\tau, \mu)\, \mathbf{I}(\tau, -\mu)\, d\mu \right]. \tag{9}$$

Following a procedure recently used for scalar inverse problems [3], we now differentiate Eqn. (5) and solve the resulting equation simultaneously with Eqn. (9) to deduce that $T_0(\tau)$ is a constant. We thus integrate Eqn. (9) to obtain

$$4S_0 = \omega[\mathbf{I}_0^T(\tau_0)\, \mathbf{I}_0(\tau_0) - \mathbf{I}_0^T(0)\, \mathbf{I}_0(0)] \tag{10}$$

where

$$S_0 = \int_0^1 \mathbf{I}^T(\tau_0, \mu)\, \mathbf{F}_2(\mu)\, d\mu - \int_0^1 \mathbf{F}_1^T(\mu)\, \mathbf{I}(0, -\mu)\, d\mu. \tag{11}$$

Clearly if $\mathbf{F}_2(\mu)$ were known we could solve Eqn. (10) for ω. However, since $\mathbf{F}_2(\mu)$ depends on λ_0 we seek a second equation to relate ω and λ_0 to known surface quantities.

If now we go back and multiply Eqn. (1), with μ changed to $-\mu$, by $\mu^2 \mathbf{I}^T(\tau, \mu)$ and integrate over μ from -1 to 1 we find we can deduce that

$$2S_2 = \omega \int_0^{\tau_0} \mathbf{I}_0^T(\tau) \frac{d}{d\tau} \mathbf{I}_2(\tau)\, d\tau \tag{12}$$

where

$$S_2 = \int_0^1 \mathbf{I}^T(\tau_0, \mu)\, \mathbf{F}_2(\mu)\, \mu^2\, d\mu - \int_0^1 \mathbf{F}_1^T(\mu)\, \mathbf{I}(0, -\mu)\, \mu^2\, d\mu. \tag{13}$$

If we multiply Eqn. (1) by $\mu^\alpha \mathbf{Q}^T(\mu)$, $\alpha = 0$ and 1, and integrate over μ from -1 to 1, we obtain

$$\frac{d}{d\tau} \mathbf{I}_1(\tau) + \mathbf{\Lambda}(\infty)\, \mathbf{I}_0(\tau) = \mathbf{0} \tag{14}$$

where

$$\mathbf{\Lambda}(\infty) = \mathbf{I} - \tfrac{1}{2}\omega \int_{-1}^{1} \mathbf{Q}^T(x)\, \mathbf{Q}(x)\, dx \tag{15}$$

and

$$\frac{d}{d\tau} \mathbf{I}_2(\tau) + \mathbf{I}_1(\tau) = \mathbf{0}. \tag{16}$$

Thus Eqn. (12) can be reduced, after we use Eqns. (14) and (16), to

$$4S_2 = \omega[\mathbf{I}_1^T(\tau_0)\,\mathbf{\Lambda}^{-1}(\infty)\,\mathbf{I}_1(\tau_o) - \mathbf{I}_1^T(0)\,\mathbf{\Lambda}^{-1}(\infty)\,\mathbf{I}_1(0)], \tag{17}$$

or

$$4(1-\omega)(1-\tfrac{7}{10}\omega)\,S_2 = \omega[\mathbf{I}_1^T(\tau_0)(\mathbf{I}-\omega\mathbf{R})\,\mathbf{I}_1(\tau_0) - \mathbf{I}_1^T(0)(\mathbf{I}-\omega\mathbf{R})\,\mathbf{I}_1(0)] \tag{18}$$

where

$$\mathbf{R} = (\det\mathbf{\Delta})\,\mathbf{\Delta}^{-1} \tag{19}$$

with

$$\mathbf{\Delta} = \int_0^1 \mathbf{Q}^T(x)\,\mathbf{Q}(x)\,dx. \tag{20}$$

If we now introduce the notation

$$\boldsymbol{\pi}_\alpha = \int_0^1 \mathbf{I}(\tau_o, \mu)\,\mu^\alpha\,d\mu, \tag{21}$$

$$S_\alpha^* = \int_0^1 \mathbf{F}_1^T(\mu)\,\mathbf{I}(0,-\mu)\,\mu^\alpha\,d\mu, \tag{22}$$

$$\mathbf{\Gamma}_\alpha = \int_0^1 \mathbf{Q}^T(\mu)\,\mathbf{I}(\tau_0, \mu)\,\mu^\alpha\,d\mu, \tag{23}$$

and

$$\mathbf{E}_\alpha = \int_0^1 \mathbf{Q}^T(\mu)\,\mu^\alpha\,d\mu \tag{24}$$

then we can write Eqns. (10) and (18) so that only ω and λ_0 appear as unknowns:

$$4S_0^* = \omega[\mathbf{I}_0^T(0)\,\mathbf{I}_0(0) - (\mathbf{\Gamma}_0^T + \lambda_0\,\boldsymbol{\pi}_1^T\,\mathbf{D}\mathbf{E}_0^T)(\mathbf{\Gamma}_0 + \lambda_0\,\mathbf{E}_0\,\mathbf{D}\boldsymbol{\pi}_1)] + 4\lambda_0\,\boldsymbol{\pi}_0^T\,\mathbf{D}\boldsymbol{\pi}_1 \tag{25}$$

and

$$\begin{aligned}4(1-\omega)(1-\tfrac{7}{10}\omega)\,S_2^* = \omega[&\mathbf{I}_1^T(0)(\mathbf{I}-\omega\mathbf{R})\,\mathbf{I}_1(0) - (\mathbf{\Gamma}_1^T - \lambda_0\,\boldsymbol{\pi}_1^T\,\mathbf{D}\mathbf{E}_1^T)(\mathbf{I}-\omega\mathbf{R})\\ &\times(\mathbf{\Gamma}_1 - \lambda_0\,\mathbf{E}_1\,\mathbf{D}\boldsymbol{\pi}_1)] + 4\lambda_0\,\boldsymbol{\pi}_2^T\,\mathbf{D}\boldsymbol{\pi}_1.\end{aligned} \tag{26}$$

It is clear that we can eliminate ω between Eqns. (25) and (26) to obtain a fifth-degree polynomial equation for λ_0. Upon solving the polynomial equation for λ_0, we can readily compute ω from, say, Eqn. (25).

We note that McCormick [4] has solved the inverse problem, without Lambert reflection, for a combination of Rayleigh and isotropic scattering.

3. Numerical Results

In order to demonstrate the effectiveness of Eqns. (25) and (26) we report some numerical results. We have used the F_N method [5] to compute all the quantities required in Eqns. (25) and (26) and subsequently have solved the two equations, as described in Section 2, to obtain the

Table 1
The Computed Values of ω and λ_0.

Quantity	2SF	3SF	4SF	5SF	Exact
ω	0.88	0.891	0.8993	0.90006	0.9
λ_0	0.24	0.224	0.2018	0.19986	0.2

results shown in Table 1. For this numerical example we use $\tau_0 = 1.0$, $\omega = 0.9$, $\lambda_0 = 0.2$, $\mu_0 = 0.9$ and

$$\mathbf{F}_1(\mu) = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\delta(\mu - \mu_0). \tag{27}$$

The columns marked 2SF, 3SF, 4SF and 5SF are based on using results for the surface quantities (that would be measured in an experiment) that have been rounded to yield 2, 3, 4 and 5 correct significant figures. It is apparent from Table 1 that our exact formulas are very sensitive to errors in the surface quantities; for practical applications they thus will yield accurate values of ω and λ_0 only when exceptionally accurate experimental data becomes available.

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Abstract

Elementary considerations are used to deduce the single-scattering albedo and the coefficient in the Lambert law of ground reflection from measurements of the polarized radiation emerging from an atmosphere of finite thickness.

Zusammenfassung

Durch elementare Betrachtungen wird aus der polarisierten Strahlung, die aus einer Atmosphäre mit endlicher Dicke austritt, der einfachstreuende Albedo und der Lambert-Koeffizient der Boden-Reflexion bestimmt.

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