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RADIATIVE TRANSFER IN INHOMOGENEOUS ATMOSPHERES—NUMERICAL RESULTS

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Abstract—The F_N method is used to establish accurate numerical results for the albedo and the exit distribution of radiation relevant to an illuminated semi-infinite half space with an exponentially varying albedo for single scattering.

1. INTRODUCTION

We consider here the radiative transfer problem defined by

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{1}{2}\omega(\tau) \int_{-1}^{1} I(\tau, \mu')\, d\mu' \tag{1}$$

with

$$I(0, \mu) = F(\mu),\ \mu > 0, \tag{2a}$$

and

$$\lim_{\tau \to \infty} I(\tau, \mu) < \infty. \tag{2b}$$

Here $\tau \in [0, \infty)$ is the optical variable, $\mu \in [-1, 1]$ is the direction cosine of the propagating radiation, $I(\tau, \mu)$ is the radiation intensity, and we consider an albedo for single scattering of the form

$$\omega(\tau) = \omega_0 \, e^{-\tau/s}, \tag{3}$$

where $0 < \omega_0 \le 1$ and $s > 0$. In addition, we consider $F(\mu)$ to be the given incident distribution of radiation, and we seek the albedo

$$A^* = \left[\int_0^1 F(\mu)\mu \, d\mu\right]^{-1} \int_0^1 I(0, -\mu)\mu \, d\mu \tag{4}$$

and the exit distribution of radiation $I(0, -\mu)$, $\mu > 0$.

In a recent paper[1] Mullikin and Siewert established some elementary solutions of Eq. (1) and reported, for $\omega_0 = 1.0$ and $s = 0.5$, 1.0, 1.5, and 2.0, numerical results that were obtained by using the F_N method[2,3]. A subsequent numerical study[4] by Larsen, Pomraning, and Badham has shown that the numerical scheme used by Mullikin and Siewert does not yield satisfactory results for $\omega_0 \approx 1$ and, at the same time, large values of s, say $s \ge 10$. We report here an improved version of the F_N method that allows us to compute A^* and $I(0, -\mu)$, $\mu > 0$, to at least five significant figures for essentially all values of ω_0 and s.

2. ANALYSIS

We begin by expressing $I(\tau, \mu)$ in terms of the elementary solutions reported by Mullikin and Siewert[1]

$$I(\tau, \mu) = \int_0^1 A(\nu)\Phi_\tau(\nu, \mu)\, e^{-\tau/\nu}\, d\nu, \tag{5}$$

where

$$\Phi_\tau(\nu, \mu) = \frac{1}{2}\omega_0 p(\nu)\, e^{-\tau/s}\left[Pv\left(\frac{1}{p(\nu)-\mu}\right) - 2\tanh^{-1} p(\nu)\delta[p(\nu)-\mu]\right] + \delta(\nu-\mu) \tag{6}$$

and

$$p(\nu) = \nu s/(s+\nu). \tag{7}$$

In order to demonstrate that Eq. (5) is a sufficiently general solution for the considered half-space applications we must show that

$$F(\mu) = \int_0^1 A(\nu)\Phi_0(\nu, \mu)\, d\nu,\ \mu > 0, \tag{8}$$

can be solved, for a class of boundary conditions represented by $F(\mu)$, to yield the expansion coefficient $A(\nu)$. Pomraning and Larsen[5] have, from a physical point-of-view, suggested that for sufficiently large values of s there should be some discrete solutions in addition to the solution given by Eq. (5). On the other hand, Larsen, Pomraning and Badham[4] have, in the manner of Martin,[6] shown that Eq. (8) can be solved uniquely for some values of ω_0 and s, and Mullikin[7] has argued that $I(\tau, \mu)$ can be expressed as in Eq. (5) for all $0 < \omega_0 < 1$ and $0 < s < \infty$. However, to the authors' knowledge, proof that Eq. (8) can be solved uniquely, for $0 < \omega_0 \leq 1$ and $0 < s < \infty$, has not been reported.

If $A(\nu)$ could be found from Eq. (8), then clearly Eq. (5) would yield the radiation intensity for all τ and all μ; however, since we are concerned here with computing only the surface result $I(0, -\mu)$, $\mu > 0$, we use the orthogonality relation[1]

$$\int_{-1}^1 \mu\Phi_\tau(\nu', \mu)\Phi_\tau(\nu, -\mu)\, d\mu = 0,\ \nu, \nu' \in (0, 1) \tag{9}$$

to deduce from Eq. (5) that

$$\int_{-1}^1 \mu\Phi_0(\nu, \mu)I(0, -\mu)\, d\mu = 0,\ \nu \in (0, 1), \tag{10}$$

or

$$\int_0^1 \mu\Phi_0(\nu, \mu)I(0, -\mu)\, d\mu = \frac{1}{2}\omega_0 p(\nu)\int_0^1 \mu F(\mu)\frac{d\mu}{p(\nu)+\mu},\quad \nu \in (0, 1). \tag{11}$$

If we now let $\xi = p(\nu)$ we can write Eq. (11) as

$$\frac{2W(\xi)}{\omega_0\xi} I[0, -W(\xi)] + \xi I(0, -\xi) P\int_{-1}^1 \frac{d\mu}{\mu-\xi} - P\int_0^1 \mu I(0, -\mu)\frac{d\mu}{\mu-\xi} = \int_0^1 \mu F(\mu)\frac{d\mu}{\mu+\xi},\ \xi \in \Xi, \tag{12}$$

where $\xi \in \Xi \Rightarrow \xi \in (0, s/[s+1])$ and

$$W(\xi) = s\xi/(s-\xi). \tag{13}$$

We note that for the case of a homogeneous atmosphere, $s=\infty$, Bowden and Bullard[8] have used a direct numerical technique to solve an appropriate version of Eq. (12). A study concerning the possibility of using that method for $0<s<\infty$ is currently being carried out by Bowden.[9] We prefer to use the F_N method. Noting that Eq. (5) yields

$$I(0,-\mu)=\frac{1}{2}\omega_0\int_0^1 A(\nu)p(\nu)\frac{d\nu}{p(\nu)+\mu},\ \mu>0, \tag{14}$$

we introduce the approximation

$$I(0,-\mu)=\frac{1}{2}\omega_0\sum_{\alpha=0}^{N} a_\alpha p(\nu_\alpha)\left(\frac{1}{p(\nu_\alpha)+\mu}\right),\quad \mu>0, \tag{15}$$

where ν_α, $\alpha=0, 1, 2,\cdots, N$, are selected points contained in the interval $(0,1)$ and the constants $\{a_\alpha\}$ are to be determined. We can also write Eq. (15) as

$$I(0,-\mu)=\frac{1}{2}\omega_0\sum_{\alpha=0}^{N} a_\alpha\zeta_\alpha\left(\frac{1}{\zeta_\alpha+\mu}\right),\quad \mu>0, \tag{16}$$

where $\zeta_\alpha\in\Xi$. We can now substitute Eq. (16) into Eq. (12) and evaluate the integrals to obtain

$$\sum_{\alpha=0}^{N} a_\alpha\Gamma_\alpha(\xi)=\xi\int_0^1 \mu F(\mu)\frac{d\mu}{\mu+\xi},\quad \xi\in\Xi, \tag{17}$$

where

$$\Gamma_\alpha(\xi)=\xi\zeta_\alpha\left\{\frac{s}{s(\xi+\zeta_\alpha)-\xi\zeta_\alpha}-\frac{1}{2}\omega_0\left(\frac{1}{\xi+\zeta_\alpha}\right)\left[\xi\log(1+1/\xi)+\zeta_\alpha\log(1+1/\zeta_\alpha)\right]\right\}. \tag{18}$$

On considering Eq. (17) for $N+1$ values of ξ, say ξ_β, we generate the following $N+1$ linear algebraic equations that can readily be solved to yield the desired constants $\{a_\alpha\}$:

$$\sum_{\alpha=0}^{N} a_\alpha\Gamma_\alpha(\xi_\beta)=\xi_\beta\int_0^1 \mu F(\mu)\frac{d\mu}{\mu+\xi_\beta},\ \xi_\beta\in\Xi. \tag{19}$$

3. NUMERICAL RESULTS

In order to simplify matters we *in general* take $\zeta_\beta=\xi_\beta$, $\beta=0, 1, 2,\cdots N$; for our scheme *a* of choosing the collocation points we define

$$\xi_\beta=\frac{2\beta+1}{2(N+1)}\left(\frac{s}{s+1}\right),\quad \beta=0, 1, 2, 3,\cdots, N. \tag{20}$$

We note that we can obtain $I(0,-\mu)$, from Eq. (16), for the case $s=\infty$ if we simply let $s\to\infty$ in Eq. (18) and use the following scheme *d* of choosing the collocation points: $\xi_0=\nu_0$ and

$$\xi_\beta=\frac{2\beta-1}{2N},\quad \beta=1,2,3,\cdots, N, \tag{21}$$

where ν_0 is the discrete eigenvalue relevant to a homogeneous atmosphere, i.e. ν_0 is the positive zero of

$$\Lambda(z)=1+\frac{1}{2}\omega_0 z\int_{-1}^{1}\frac{d\mu}{\mu-z}. \tag{22}$$

Clearly once we have specified a scheme for choosing the points $\{\xi_\beta\}$ and have solved Eq. (19)

for the constants $\{a_\alpha\}$ then the exit distribution of radiation and the albedo are given, in our approximation, by

$$I(0,-\mu)=\frac{1}{2}\omega_0\sum_{\alpha=0}^{N}a_\alpha\zeta_\alpha\left(\frac{1}{\zeta_\alpha+\mu}\right),\ \mu>0, \tag{23}$$

and

$$A^*=\left[\int_0^1\mu F(\mu)\,\mathrm{d}\mu\right]^{-1}\frac{1}{2}\omega_0\sum_{\alpha=0}^{N}a_\alpha\zeta_\alpha\left[1-\zeta_\alpha\log(1+1/\zeta_\alpha)\right]. \tag{24}$$

In our study of the computational merit of our solution we have considered incident distributions of the forms

$$F(\mu)=\mu^l,\ l=0,1,\text{ and }2, \tag{25a}$$

and

$$F(\mu)=\delta(\mu-\mu_0),\quad \mu_0=l/10,\quad l=1,2,3\cdots,10. \tag{25b}$$

For $\omega_0\leq 0.7$ and $s\leq 10^6$ we have found that scheme a yields excellent numerical results; see, for example, the case $\omega_0=0.7$ listed in Tables 1 and 2. For all ω_0 and $s=\infty$ we have also found that scheme d yields excellent results. In fact, for the cases considered (apparent) convergence to seven significant figures was achieved for N typically between 4 and 15.

For $\omega_0>0.7$ and $10<s<10^6$ we have found that scheme a, though apparently convergent,

Table 1. The albedo for $F(\mu)=1$.

ω_0	s	Scheme	F_0	F_1	F_2	F_5	Converged
0.7	1	a	0.14516	0.15515	0.15524	0.15524	0.1552404
	10	a	0.21641	0.23408	0.23536	0.23542	0.2354182
	10^2	a	0.22943	0.25095	0.25368	0.25404	0.2540444
	10^3	a	0.23086	0.25289	0.25586	0.25630	0.2562999
	10^6	a	0.23102	0.25310	0.25610	0.25655	0.2565564
	∞	d	0.25430	0.25647	0.25655	0.25656	0.2565566
0.9	1	a	0.20587	0.22416	0.22431	0.22431	0.2243149
	10	a	0.33198	0.38693	0.39476	0.39544	0.3954451
	10^2	a	0.45918	0.46373	0.46417	0.46436	0.4645445
	10^3	b	0.47459	0.47592	0.47616	0.47635	0.4765388
	10^6	b	0.47634	0.47741	0.47764	0.47784	0.4780230
	∞	d	0.47720	0.47799	0.47802	0.47802	0.4780245
0.99	1	a	0.23747	0.26127	0.26147	0.26148	0.2614794
	10	a	0.39994	0.49335	0.51226	0.51465	0.5146519
	10^2	b	0.68294	0.69427	0.69581	0.69602	0.6960620
	10^3	b	0.71430	0.77229	0.77426	0.77433	0.7743447
	10^6	b	0.71792	0.78714	0.79435	0.79452	0.7945376
	∞	d	0.79448	0.79456	0.79456	0.79456	0.7945637
0.999	1	a	0.24080	0.26523	0.26544	0.26545	0.2654478
	10	a	0.40746	0.50611	0.52684	0.52957	0.5295672
	10^2	b	0.71430	0.73459	0.73681	0.73705	0.7370920
	10^3	b	0.74835	0.84690	0.85755	0.85755	0.8575570
	10^6	c	0.92905	0.92938	0.92940	0.92940	0.9294084
	∞	d	0.92970	0.92971	0.92971	0.92971	0.9297133
1.0	1	a	0.24117	0.26567	0.26588	0.26589	0.2658917
	10	a	0.40830	0.50755	0.52850	0.53127	0.5312681
	10^2	b	0.71792	0.73950	0.74180	0.74204	0.7420819
	10^3	b	0.75230	0.85695	0.86982	0.86983	0.8698366
	10^6	c	0.96747	0.98422	0.98608	0.98609	0.9860854
	∞	d	1.00000	1.00000	1.00000	1.00000	1.0000000

Table 2. The albedo for $F(\mu) = \delta(\mu - 0.9)$.

ω_0	s	Scheme	F_0	F_1	F_2	F_5	Converged
	1	*a*	0.10560	0.11858	0.11885	0.11885	0.1188532
	10	*a*	0.16949	0.19464	0.19687	0.19699	0.1969850
0.7	10^2	*a*	0.18159	0.21193	0.21620	0.21679	0.2167861
	10^3	*a*	0.18292	0.21393	0.21853	0.21924	0.2192390
	10^6	*a*	0.18306	0.21416	0.21879	0.21952	0.2195188
	∞	*d*	0.22220	0.21956	0.21952	0.21952	0.2195191
	1	*a*	0.14977	0.17216	0.17262	0.17262	0.1726166
	10	*a*	0.26001	0.32825	0.33924	0.34028	0.3402779
0.9	10^2	*a*	0.28356	0.37657	0.40471	0.41516	0.4153221
	10^3	*b*	0.43278	0.43067	0.42998	0.42936	0.4288527
	10^6	*b*	0.43438	0.43234	0.43166	0.43103	0.4305394
	∞	*d*	0.43356	0.43057	0.43055	0.43054	0.4305411
	1	*a*	0.17276	0.20113	0.20172	0.20171	0.2017128
	10	*a*	0.31324	0.42303	0.44742	0.45067	0.4506724
0.99	10^2	*b*	0.62278	0.64550	0.64955	0.65051	0.6502452
	10^3	*b*	0.65137	0.74101	0.74107	0.74077	0.7406130
	10^6	*b*	0.65468	0.75963	0.76385	0.76442	0.7642752
	∞	*d*	0.76575	0.76432	0.76431	0.76431	0.7643058
	1	*a*	0.17518	0.20423	0.20483	0.20483	0.2048259
	10	*a*	0.31913	0.43445	0.46100	0.46466	0.4646644
0.999	10^2	*b*	0.65137	0.68741	0.69262	0.69362	0.6933447
	10^3	*b*	0.68243	0.82528	0.83282	0.83288	0.8328052
	10^6	*c*	0.91856	0.91805	0.91790	0.91780	0.9173692
	∞	*d*	0.91824	0.91773	0.91773	0.91773	0.9177295
	1	*a*	0.17545	0.20457	0.20518	0.20517	0.2051743
	10	*a*	0.31979	0.43575	0.46254	0.46626	0.4662623
1.0	10^2	*b*	0.65468	0.69254	0.69789	0.69888	0.6986115
	10^3	*b*	0.68602	0.83676	0.84644	0.84663	0.8465626
	10^6	*c*	0.95654	0.98301	0.98349	0.98362	0.9835482
	∞	*d*	1.00000	1.00000	1.00000	1.00000	1.0000000

does not always yield sufficiently accurate results, and therefore we have found it convenient to use a different strategy for defining the collocation points for these values of ω_0 and s. We note that we can readily extend, by way of Eq. (5), the definition of $I(0,-\xi)$ to all $\xi \notin [-1,0]$, and therefore Eq. (12) can also be readily extended to all $\xi \notin [-1,0]$. We note that Mullikin[7] has used the integral form of Eq. (1), with Eqs. (2), and Fourier transform techniques to derive a form of Eq. (12) that is also valid for an extended domain of ξ. Thus for larger values of ω_0 and s we use our scheme *b* given by

$$\xi_\beta = \beta + 2, \quad \beta = 0, 1, 2, \cdots \min(9, N) \tag{26a}$$

plus

$$\xi_\beta = \frac{2\beta - 19}{2(N-9)}\left(\frac{s}{s+1}\right), \beta = 10, 11, \cdots, N, \tag{26b}$$

to define the collocation *and basis* points. Finally, if $\omega_0 \simeq 1$ and s is very large but bounded, say $\omega_0 = 1.0$ and $s = 10^6$, we use the following scheme *c* to define the collocation *and basis* points:

$$\xi_\beta = 10(\beta + 2), \quad \beta = 0, 1, 2, \cdots \min(9, N) \tag{27a}$$

plus

$$\xi_\beta = \frac{2\beta - 19}{2(N-9)}\left(\frac{s}{s+1}\right), \quad \beta = 10, 11, \cdots, N. \tag{27b}$$

Table 3. The exit angular distribution for $\omega_0 = 1.0$ and $F(\mu) = 1$.

μ	$s = 1$	$s = 10$	$s = 10^2$	$s = 10^3$	$s = 10^6$	$s = \infty$
0.05	0.58966	0.76085	0.87190	0.93580	0.99315	1.00000
0.1	0.53112	0.73402	0.85895	0.92949	0.99248	1.00000
0.2	0.44328	0.68637	0.83529	0.91793	0.99126	1.00000
0.3	0.38031	0.64424	0.81318	0.90701	0.99010	1.00000
0.4	0.33296	0.60654	0.79211	0.89645	0.98898	1.00000
0.5	0.29609	0.57264	0.77191	0.88613	0.98787	1.00000
0.6	0.26656	0.54205	0.75250	0.87600	0.98678	1.00000
0.7	0.24239	0.51435	0.73384	0.86604	0.98570	1.00000
0.8	0.22223	0.48919	0.71588	0.85623	0.98462	1.00000
0.9	0.20517	0.46626	0.69861	0.84656	0.98355	1.00000
1.0	0.19055	0.44530	0.68201	0.83704	0.98248	1.00000

Table 4. The exit angular distribution for $\omega_0 = 1.0$ and $F(\mu) = \delta(\mu - 0.9)$.

μ	$s = 1$	$s = 10$	$s = 10^2$	$s = 10^3$	$s = 10^6$	$s = \infty$
0.05	0.69801	0.98407	1.20114	1.33392	1.45551	1.47009
0.1	0.65651	0.99353	1.23673	1.38317	1.51670	1.53270
0.2	0.57637	0.98484	1.27462	1.44609	1.60153	1.62013
0.3	0.50955	0.96002	1.28991	1.48469	1.66083	1.68189
0.4	0.45523	0.92864	1.29262	1.50941	1.70558	1.72904
0.5	0.41078	0.89488	1.28748	1.52504	1.74078	1.76659
0.6	0.37396	0.86082	1.27715	1.53431	1.76925	1.79738
0.7	0.34304	0.82752	1.26334	1.53892	1.79272	1.82316
0.8	0.31676	0.79553	1.24715	1.54000	1.81237	1.84510
0.9	0.29416	0.76511	1.22937	1.53837	1.82901	1.86403
1.0	0.27454	0.73634	1.21056	1.53462	1.84325	1.88053

It is apparent from the results of Tables 1 and 2 (and from other cases we have considered) that the strategy defined here for selecting the collocation *and basis* points is an effective one. For essentially all cases that we considered we were able to obtain final results that appear to be accurate to seven significant figures. We note the Pomraning[10] has confirmed selected results for $\omega_0 < 1$, generally to three significant figures, with the code ANISN. However, for the interesting case of $\omega_0 = 1.0$ and $s = 10^6$ the ANISN code yielded only one significant figure. We note also that for selected values of s and $\omega_0 = 0.99$ we agree to four or five significant figures with numerical results obtained from a different basis and collocation scheme by Mullikin.[7] In Tables 3 and 4 we report some converged results for the exit distribution of radiation.

Though there are clearly many unresolved mathematical questions concerning the solution of the considered problem, we believe we have here reported an especially concise and accurate method for computing the desired solution. We note, for example, that the matrix elements

$$\Gamma_\alpha(\xi_\beta) = \xi_\alpha \xi_\beta \left\{ \frac{s}{s(\xi_\alpha + \xi_\beta) - \xi_\alpha \xi_\beta} - \frac{1}{2}\omega_0 \left(\frac{1}{\xi_\alpha + \xi_\beta} \right) \left[\xi_\alpha \log(1 + 1/\xi_\alpha) + \xi_\beta \log(1 + 1/\xi_\beta) \right] \right\} \quad (28)$$

used in Eq. (19) are very easy to evaluate.

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