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ON THE EQUATION OF TRANSFER RELEVANT TO THE SCATTERING OF POLARIZED LIGHT

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ABSTRACT

A general problem concerning the diffusion of light in a finite plane-parallel atmosphere illuminated by an arbitrarily polarized radiation field is formulated. A decomposition in the azimuthal angle is used to reduce the problem to a set of radiative transfer equations, for the Stokes parameters, based on real quantities.

Subject headings: polarization — radiative transfer

I. INTRODUCTION

In a fundamental paper, hereafter referred to as KR, Kuščer and Ribarič (1959) developed, by way of a matrix formulation, the equation of transfer that describes the diffusion of polarized light in a scattering and absorbing host medium. Kuščer and Ribarič based their formulation on an expansion of the components of the phase matrix in the generalized spherical functions discussed by Gel'fand and Šapiro (1956). Kuščer and Ribarič also noted that the desired vector solution of the developed equation of transfer could be conveniently expressed, for plane-parallel media, in a Fourier series in the azimuthal angle ϕ. This azimuthal decomposition has been used by Lenoble (1961), who proposed the use of a generalized spherical harmonics method for this problem. In a subsequent paper Herman and Lenoble (1968) used the KR formulation and developed, in terms of the Stokes parameters, the defining equations for the azimuthally symmetric component of the complete solution. In a more extensive work Herman (1968) reported the results of a study of the properties of the asymptotic part of the azimuthally symmetric component of the complete solution. We note also that Domke (1975*a*, *b*, 1976) has developed elementary solutions of the equation of transfer and extensive mathematical analysis basic to the KR formulation of the general problem. For an exhaustive review of basic work concerning the scattering of light in planetary atmospheres the reader is referred to the excellent paper by Hansen and Travis (1974).

The KR formulation allows a reasonably general scattering law and thus is an extension of the pioneering work of Chandrasekhar (1950), who, in addition to developing a mathematical formulation in terms of the Stokes parameters, established working solutions for Rayleigh-scattering atmospheres. Since we intend eventually to consider inhomogeneous atmospheres stratified in plane-parallel layers, we do not wish to assume that the light incident on a given layer is unpolarized. We also wish to have a final formulation for the general case in terms of real quantities; e.g., we use the classical Stokes parameters to characterize the radiation field. Thus here we start with the KR formulation. We then transform the resulting equations to a Stokes representation for the radiation vector, and we utilize basic properties of the established phase matrix to obtain a final formulation based on real quantities. We begin, then, with the equation of transfer:

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}_*(\tau, \mu, \phi) + \boldsymbol{I}_*(\tau, \mu, \phi) = \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \boldsymbol{P}_*(\mu, \phi; \mu', \phi') \boldsymbol{I}_*(\tau, \mu', \phi') d\mu' d\phi' , \tag{1}$$

where the four elements of the density vector $\boldsymbol{I}_*$ derived from the "circularly polarized" basis by Kuščer and Ribarič are I_2, I_0, I_{-0}, and I_{-2}. In addition, μ is the direction cosine of the propagating radiation (as measured from the positive τ axis), ϕ is the azimuthal angle required to specify the direction of propagation, τ is the optical variable, and $\omega \in (0, 1]$ is the albedo for single scattering. Further we write the phase matrix as

$$\boldsymbol{P}_*(\mu, \phi; \mu', \phi') = \sum_{s=-L}^{L} \boldsymbol{P}_*{}^s(\mu, \mu') e^{is(\phi - \phi')} , \tag{2}$$

where the elements of the Fourier coefficients are given by

$$[\boldsymbol{P}_*{}^s(\mu, \mu')]_{m,n} = (-1)^s \sum_{l=|s|}^{L} p^l_{m,n} P^l_{s,m}(\mu) P^l_{s,n}(\mu') . \tag{3}$$

Observe that we use here a labeling scheme, for the elements of the matrix $\boldsymbol{P}_*{}^s(\mu, \mu')$, that is consistent with the use of I_2, I_0, I_{-0}, and I_{-2} as the elements of $\boldsymbol{I}_*$. It is evident that in writing equations (2) and (3) we have truncated the infinite series used in the KR formulation. We consider that the matrix elements $p^l_{m,n}$, $m, n = \pm 0, \pm 2$, are given (Kuščer and Ribarič,

1959) and such that $p^l_{m,n} \equiv 0$ if $l < \sup(|m|, |n|)$, $p^l_{m,m}$ and $p^l_{m,-m}$ are real, $p^l_{2,0} = \bar{p}^l_{2,-0}$ (we use $\bar{z}$ to denote the complex conjugate of z) and

$$p^l_{m,n} = p^l_{n,m} = p^l_{-m,-n} . \tag{4}$$

To complete the definition of the equation of transfer we note that the functions $P^l_{m,n}(\mu)$ are the generalized spherical functions discussed by Gel'fand and Šapiro (1956); for our purposes here these functions are defined by

$$P^l_{m,n}(\mu) = A(1-\mu)^{-(n-m)/2}(1+\mu)^{-(n+m)/2} \frac{d^{l-n}}{d\mu^{l-n}}[(1-\mu)^{l-m}(1+\mu)^{l+m}] , \tag{5}$$

where $l \geq \sup(|m|, |n|)$ and

$$A = \frac{(-1)^{l-m}(i)^{n-m}}{2^l(l-m)!}\left|\frac{(l-m)!(l+n)!}{(l+m)!(l-n)!}\right|^{1/2} . \tag{6}$$

Gel'fand and Šapiro (1956) have shown that

$$P^l_{m,n}(\mu) = P^l_{n,m}(\mu) = P^l_{-m,-n}(\mu) . \tag{7}$$

If we now express the density vector as

$$\boldsymbol{I}_*(\tau, \mu, \phi) = \sum_{s=-L}^{L} \boldsymbol{I}_*{}^s(\tau, \mu)e^{is(\phi-\phi_r)} + \boldsymbol{R}_*(\mu, \phi)e^{-\tau/\mu} , \tag{8}$$

where ϕ_r is a reference angle, then we obtain the Fourier decomposition if

$$\int_0^{2\pi} \boldsymbol{R}_*(\mu, \phi)e^{-is\phi}d\phi = \boldsymbol{0} , \qquad -L \leq s \leq L , \tag{9}$$

and the coefficients $\boldsymbol{I}_*{}^s(\tau, \mu)$ satisfy

$$\mu\frac{\partial}{\partial\tau}\boldsymbol{I}_*{}^s(\tau, \mu) + \boldsymbol{I}_*{}^s(\tau, \mu) = \frac{\omega}{2}\int_{-1}^{1} \boldsymbol{P}_*{}^s(\mu, \mu')\boldsymbol{I}_*{}^s(\tau, \mu')d\mu' . \tag{10}$$

We include the vector $\boldsymbol{R}_*(\mu, \phi)$ in equation (8) so that $\boldsymbol{I}_*(\tau, \mu, \phi)$ can be constrained to satisfy boundary conditions involving functions that do not have a Fourier representation limited to a finite number of terms.

II. THE STOKES PARAMETERS

The formulation of Kuščer and Ribarič (1959) is, of course, based on a density vector $\boldsymbol{I}_*(\tau, \mu, \phi)$ that has complex components. We thus now prefer, since the Fourier decomposition has been established, to switch to a representation based on the Stokes parameters and to seek defining equations that are based on real quantities. If we let $\boldsymbol{I}(\tau, \mu, \phi)$ denote a density vector with the four Stokes parameters I, Q, U, and V as components, then, as observed by Kuščer and Ribarič (1959),

$$\boldsymbol{I}(\tau, \mu, \phi) = \boldsymbol{T}\boldsymbol{I}_*(\tau, \mu, \phi) , \tag{11}$$

where

$$\boldsymbol{T} = \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ i & 0 & 0 & -i \\ 0 & -1 & 1 & 0 \end{vmatrix} . \tag{12}$$

It follows that we can now multiply equations (8), (9) and (10) by $\boldsymbol{T}$ to obtain

$$\boldsymbol{I}(\tau, \mu, \phi) = \sum_{s=-L}^{L} \boldsymbol{I}^s(\tau, \mu)e^{is(\phi-\phi_r)} + \boldsymbol{R}(\mu, \phi)e^{-\tau/\mu} , \tag{13}$$

where

$$\int_0^{2\pi} \boldsymbol{R}(\mu, \phi)e^{-is\phi}d\phi = \boldsymbol{0} , \qquad -L \leq s \leq L , \tag{14}$$

$$\mu\frac{\partial}{\partial\tau}\boldsymbol{I}^s(\tau, \mu) + \boldsymbol{I}^s(\tau, \mu) = \frac{\omega}{2}\int_{-1}^{1} \boldsymbol{P}^s(\mu, \mu')\boldsymbol{I}^s(\tau, \mu')d\mu' , \tag{15}$$

and

$$P^s(\mu, \mu') = TP_*{}^s(\mu, \mu')T^{-1} . \tag{16}$$

Because we intend to use this work as the basis for a study of inhomogeneous atmospheres stratified in plane-parallel layers, we consider that the solution given by equation (13) is appropriate for $\tau \in [L, R]$, and we impose boundary conditions, for $\mu > 0$ and $\phi \in [0, 2\pi]$, of the form

$$I(L, \mu, \phi) = \pi\delta(\mu - \mu_0)\delta(\phi - \phi_0)F + F_1(\mu, \phi) \tag{17a}$$

and

$$I(R, -\mu, \phi) = F_2(\mu, \phi) + \lambda_0 \frac{1}{\pi} L \int_0^{2\pi} \int_0^1 I(R, \mu', \phi')\mu' d\mu' d\phi' . \tag{17b}$$

Here the flux constant F has elements F_I, F_Q, F_U, F_V, and λ_0 is the coefficient for Lambert reflection,

$$L = \mathrm{diag}\,\{1, 0, 0, 0\} , \tag{18}$$

and $F_1(\mu, \phi)$ and $F_2(\mu, \phi)$ are considered given real functions (not generalized functions). Thus if we write, for $\mu > 0$ and $\phi \in [0, 2\pi]$,

$$R(\mu, \phi)e^{-L/\mu} = \pi\delta(\mu - \mu_0)\delta(\phi - \phi_0)F + F_1(\mu, \phi) - \sum_{s=-L}^{L} I^s(L, \mu)e^{is(\phi - \phi_r)} \tag{19}$$

and

$$R(-\mu, \phi)e^{R/\mu} = F_2(\mu, \phi) - \sum_{s=-L}^{L} I^s(R, -\mu)e^{is(\phi - \phi_r)} + 2\lambda_0 L \int_0^1 I^0(R, \mu')\mu' d\mu' , \tag{20}$$

and impose the condition given by equation (14), we find that, if we let $\phi_r = \phi_0$, the Fourier coefficients must satisfy, for $\mu > 0$, the boundary conditions

$$I^s(L, \mu) = \tfrac{1}{2}\delta(\mu - \mu_0)F + \frac{1}{2\pi}\int_0^{2\pi} F_1(\mu, \phi)e^{-is(\phi - \phi_0)}d\phi \tag{21}$$

and

$$I^s(R, -\mu) = \frac{1}{2\pi}\int_0^{2\pi} F_2(\mu, \phi)e^{-is(\phi - \phi_0)}d\phi + 2\lambda_0\delta_{0,s}L \int_0^1 I^0(R, \mu')\mu' d\mu' . \tag{22}$$

Now since $I(\tau, \mu, \phi)$ is real, it is clear from equation (13) that

$$I^{-s}(\tau, \mu) = \bar{I}^s(\tau, \mu) , \tag{23}$$

and so we write equation (13) as

$$I(\tau, \mu, \phi) = \sum_{s=0}^{L} [I_c{}^s(\tau, \mu) \cos s(\phi - \phi_0) + I_s{}^s(\tau, \mu) \sin s(\phi - \phi_0)] + R(\mu, \phi)e^{-\tau/\mu} , \tag{24}$$

where

$$I_c{}^s(\tau, \mu) = (2 - \delta_{0,s}) \,\mathrm{Re}\, I^s(\tau, \mu) \tag{25}$$

and

$$I_s{}^s(\tau, \mu) = -2 \,\mathrm{Im}\, I^s(\tau, \mu) . \tag{26}$$

Here we have, for $\mu > 0$,

$$R(\mu, \phi)e^{-L/\mu} = \pi\delta(\mu - \mu_0)\left[\delta(\phi - \phi_0) - \sum_{s=0}^{L} \left(\frac{2 - \delta_{0,s}}{2\pi}\right) \cos s(\phi - \phi_0)\right]F + F_1(\mu, \phi) - \hat{F}_1(\mu, \phi) \tag{27a}$$

and

$$R(-\mu, \phi)e^{R/\mu} = F_2(\mu, \phi) - \hat{F}_2(\mu, \phi) , \tag{27b}$$

where

$$\hat{F}_x(\mu, \phi) = \sum_{s=0}^{L} \left[\left(\frac{2 - \delta_{0,s}}{2\pi} \int_0^{2\pi} F_x(\mu, \phi') \cos s(\phi' - \phi_0) d\phi' \right) \cos s(\phi - \phi_0) + \left(\frac{1}{\pi} \int_0^{2\pi} F_x(\mu, \phi') \sin s(\phi' - \phi_0) d\phi' \right) \sin s(\phi - \phi_0) \right] . \quad (28)$$

Now if we consider the real and imaginary parts of equations (15), (21) and (22) we see that $\boldsymbol{I}_s^0(\tau, \mu) \equiv \boldsymbol{0}$ and that $\boldsymbol{I}_c^s(\tau, \mu)$ and $\boldsymbol{I}_s^s(\tau, \mu)$ must satisfy, for $0 \le s \le L$, the coupled system of equations

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}_c^s(\tau, \mu) + \boldsymbol{I}_c^s(\tau, \mu) = \frac{\omega}{2} \int_{-1}^{1} [\boldsymbol{P}_R^s(\mu, \mu') \boldsymbol{I}_c^s(\tau, \mu') + \boldsymbol{P}_I^s(\mu, \mu') \boldsymbol{I}_s^s(\tau, \mu')] d\mu' \quad (29a)$$

and

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}_s^s(\tau, \mu) + \boldsymbol{I}_s^s(\tau, \mu) = \frac{\omega}{2} \int_{-1}^{1} [\boldsymbol{P}_R^s(\mu, \mu') \boldsymbol{I}_s^s(\tau, \mu') - \boldsymbol{P}_I^s(\mu, \mu') \boldsymbol{I}_c^s(\tau, \mu')] d\mu' \quad (29b)$$

and the boundary conditions, for $\mu > 0$,

$$\boldsymbol{I}_c^s(L, \mu) = \left(\frac{2 - \delta_{0,s}}{2} \right) \delta(\mu - \mu_0) \boldsymbol{F} + \left(\frac{2 - \delta_{0,s}}{2\pi} \right) \int_0^{2\pi} \boldsymbol{F}_1(\mu, \phi) \cos s(\phi - \phi_0) d\phi , \quad (30a)$$

$$\boldsymbol{I}_s^s(L, \mu) = \frac{1}{\pi} \int_0^{2\pi} \boldsymbol{F}_1(\mu, \phi) \sin s(\phi - \phi_0) d\phi , \quad (30b)$$

$$\boldsymbol{I}_c^s(R, -\mu) = \left(\frac{2 - \delta_{0,s}}{2\pi} \right) \int_0^{2\pi} \boldsymbol{F}_2(\mu, \phi) \cos s(\phi - \phi_0) d\phi + 2\lambda_0 \delta_{0,s} \boldsymbol{L} \int_0^1 \boldsymbol{I}_c^0(R, \mu') \mu' d\mu' \quad (30c)$$

and

$$\boldsymbol{I}_s^s(R, -\mu) = \frac{1}{\pi} \int_0^{2\pi} \boldsymbol{F}_2(\mu, \phi) \sin s(\phi - \phi_0) d\phi . \quad (30d)$$

In writing equations (29) we have used

$$\boldsymbol{P}^s(\mu, \mu') = \boldsymbol{P}_R^s(\mu, \mu') + i \boldsymbol{P}_I^s(\mu, \mu') . \quad (31)$$

Equations (29) clearly represent a system of eight coupled scalar equations; however, a reduction in the number of equations to be solved simultaneously is possible. We note, first of all, that the matrices $\boldsymbol{P}^l$, with elements $p_{m,n}^l$, that define the phase matrix $\boldsymbol{P}^s(\mu, \mu')$ are such that

$$\boldsymbol{P}^l = \boldsymbol{M}\boldsymbol{P}^l\boldsymbol{M} \quad (32)$$

and

$$\bar{\boldsymbol{P}}^l = \boldsymbol{N}\boldsymbol{P}^l\boldsymbol{N} , \quad (33)$$

where

$$\boldsymbol{M} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \quad \text{and} \quad \boldsymbol{N} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad (34a, b)$$

We can now use equations (7), (32), and (33) to show that

$$\bar{\boldsymbol{P}}^s(\mu, \mu') = \boldsymbol{P}^{-s}(\mu, \mu') = \boldsymbol{D}\boldsymbol{P}^s(\mu, \mu')\boldsymbol{D} , \quad (35)$$

where

$$\boldsymbol{D} = \text{diag}\,\{1, 1, -1, -1\} . \quad (36)$$

From equation (35) it is apparent that

$$\boldsymbol{P}_R^s(\mu, \mu') = \boldsymbol{D}\boldsymbol{P}_R^s(\mu, \mu')\boldsymbol{D} \quad (37a)$$

and

$$\boldsymbol{P}_I^s(\mu, \mu') = -\boldsymbol{D}\boldsymbol{P}_I^s(\mu, \mu')\boldsymbol{D} , \quad (37b)$$

and thus we find that equations (29) admit two types of solutions, *viz.*,

$$\boldsymbol{D}\boldsymbol{I}_c^s(\tau, \mu) = \boldsymbol{I}_c^s(\tau, \mu) \quad \text{and} \quad \boldsymbol{D}\boldsymbol{I}_s^s(\tau, \mu) = -\boldsymbol{I}_s^s(\tau, \mu) \tag{38a}$$

or

$$\boldsymbol{D}\boldsymbol{I}_c^s(\tau, \mu) = -\boldsymbol{I}_c^s(\tau, \mu) \quad \text{and} \quad \boldsymbol{D}\boldsymbol{I}_s^s(\tau, \mu) = \boldsymbol{I}_s^s(\tau, \mu)\,. \tag{38b}$$

It is evident from the boundary conditions given by equations (30) that neither of the solution-types given by equations (38) is sufficient to solve the general problem we consider; on the other hand, the desired solution can be constructed from linear combinations of these solutions. We therefore express the desired Fourier components as

$$\boldsymbol{I}_c^s(\tau, \mu) = \tfrac{1}{2}[(\boldsymbol{I} + \boldsymbol{D})\boldsymbol{\Upsilon}^s(\tau, \mu) + (\boldsymbol{I} - \boldsymbol{D})\boldsymbol{\Xi}^s(\tau, \mu)] \tag{39}$$

and

$$\boldsymbol{I}_s^s(\tau, \mu) = \tfrac{1}{2}[(\boldsymbol{I} - \boldsymbol{D})\boldsymbol{\Upsilon}^s(\tau, \mu) + (\boldsymbol{I} + \boldsymbol{D})\boldsymbol{\Xi}^s(\tau, \mu)]\,, \tag{40}$$

where $\boldsymbol{\Upsilon}^s(\tau, \mu)$ and $\boldsymbol{\Xi}^s(\tau, \mu)$ satisfy

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{\Upsilon}^s(\tau, \mu) + \boldsymbol{\Upsilon}^s(\tau, \mu) = \frac{\omega}{2} \int_{-1}^{1} \boldsymbol{A}^s(\mu, \mu')\boldsymbol{\Upsilon}^s(\tau, \mu')d\mu' \tag{41a}$$

and

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{\Xi}^s(\tau, \mu) + \boldsymbol{\Xi}^s(\tau, \mu) = \frac{\omega}{2} \int_{-1}^{1} \boldsymbol{B}^s(\mu, \mu')\boldsymbol{\Xi}^s(\tau, \mu')d\mu'\,. \tag{41b}$$

Here

$$\boldsymbol{A}^s(\mu, \mu') = \boldsymbol{P}_R^s(\mu, \mu') + \boldsymbol{D}\boldsymbol{P}_I^s(\mu, \mu') \tag{42a}$$

and

$$\boldsymbol{B}^s(\mu, \mu') = \boldsymbol{P}_R^s(\mu, \mu') - \boldsymbol{D}\boldsymbol{P}_I^s(\mu, \mu')\,. \tag{42b}$$

Equations (41) clearly are two sets of four simultaneous equations to be solved subject to the boundary conditions, for $\mu > 0$,

$$\boldsymbol{\Upsilon}^s(L, \mu) = \left(\frac{2 - \delta_{0,s}}{2}\right)\delta(\mu - \mu_0) \begin{vmatrix} F_I \\ F_Q \\ 0 \\ 0 \end{vmatrix} + \left(\frac{2 - \delta_{0,s}}{2\pi}\right) \int_0^{2\pi} \boldsymbol{\Phi}^s(\phi)\boldsymbol{F}_1(\mu, \phi)d\phi\,, \tag{43a}$$

$$\boldsymbol{\Upsilon}^s(R, -\mu) = \left(\frac{2 - \delta_{0,s}}{2\pi}\right) \int_0^{2\pi} \boldsymbol{\Phi}^s(\phi)\boldsymbol{F}_2(\mu, \phi)d\phi + 2\lambda_0 \delta_{0,s} \boldsymbol{L} \int_0^1 \mu' \boldsymbol{\Upsilon}^0(\tau, \mu')d\mu'\,, \tag{43b}$$

$$\boldsymbol{\Xi}^s(L, \mu) = \left(\frac{2 - \delta_{0,s}}{2}\right)\delta(\mu - \mu_0) \begin{vmatrix} 0 \\ 0 \\ F_U \\ F_V \end{vmatrix} + \left(\frac{2 - \delta_{0,s}}{2\pi}\right) \boldsymbol{M} \int_0^{2\pi} \boldsymbol{\Phi}^s(\phi)\boldsymbol{M}\boldsymbol{F}_1(\mu, \phi)d\phi\,, \tag{44a}$$

and

$$\boldsymbol{\Xi}^s(R, -\mu) = \left(\frac{2 - \delta_{0,s}}{2\pi}\right) \boldsymbol{M} \int_0^{2\pi} \boldsymbol{\Phi}^s(\phi)\boldsymbol{M}\boldsymbol{F}_2(\mu, \phi)d\phi\,, \tag{44b}$$

where

$$\boldsymbol{\Phi}^s(\phi) = \text{diag}\,\{\cos s(\phi - \phi_0)\,,\ \ \cos s(\phi - \phi_0)\,,\ \sin s(\phi - \phi_0)\,,\ \ \sin s(\phi - \phi_0)\}\,. \tag{45}$$

Finally the desired solution can be written, for the general case, as

$$\boldsymbol{I}(\tau, \mu, \phi) = \sum_{s=0}^{L} [\boldsymbol{\Phi}^s(\phi)\boldsymbol{\Upsilon}^s(\tau, \mu) + \boldsymbol{M}\boldsymbol{\Phi}^s(\phi)\boldsymbol{M}\boldsymbol{\Xi}^s(\tau, \mu)] + \boldsymbol{R}(\mu, \phi)e^{-\tau/\mu}\,, \tag{46}$$

where $\boldsymbol{R}(\mu, \phi)$ is given by equations (27). It is clear that for a single slab illuminated by an unpolarized solar beam, i.e., $\boldsymbol{F}_1(\mu, \phi) = \boldsymbol{F}_2(\mu, \phi) = \boldsymbol{0}$ and $F_U = F_V = 0$, we require only $\boldsymbol{\Upsilon}(\tau, \mu)$ to construct the desired solution; in fact, even for a multislab problem illuminated by an unpolarized solar beam and with interface conditions that allow the density vector

$\boldsymbol{I}(\tau, \mu, \phi)$ to be continuous, the solution can be developed from a sequence of problems based on $\boldsymbol{\Upsilon}(\tau, \mu)$. On the other hand, for polarized incident light or for multiregion problems with active interface conditions we clearly must consider $\boldsymbol{\Upsilon}(\tau, \mu)$ and $\boldsymbol{\Xi}(\tau, \mu)$.

III. THE SCATTERING KERNELS

We consider equations (41) written as

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{\Gamma}(\tau, \mu) + \boldsymbol{\Gamma}(\tau, \mu) = \frac{\omega}{2} \int_{-1}^{1} \boldsymbol{\Pi}(\mu, \mu') \boldsymbol{\Gamma}(\tau, \mu') d\mu' , \tag{47}$$

where clearly $\boldsymbol{\Gamma}(\tau, \mu)$ represents either $\boldsymbol{\Upsilon}(\tau, \mu)$ or $\boldsymbol{\Xi}(\tau, \mu)$, and $\boldsymbol{\Pi}(\mu, \mu')$ the corresponding $\boldsymbol{A}(\mu, \mu')$ or $\boldsymbol{B}(\mu, \mu')$. Now if we compute $\boldsymbol{P}^s(\mu, \mu')$ from equations (3) and (16), we find we can write

$$\boldsymbol{P}^s(\mu, \mu') = \sum_{l=s}^{L} \frac{(l-s)!}{(l+s)!} \boldsymbol{P}_l^s(\mu, \mu') , \tag{48}$$

where

$$\boldsymbol{P}_l^s(\mu, \mu') = \begin{vmatrix} \beta P(\mu)P(\mu') & \gamma P(\mu)R(\mu') & -i\gamma P(\mu)T(\mu') & 0 \\ \gamma R(\mu)P(\mu') & \alpha R(\mu)R(\mu') + \zeta T(\mu)T(\mu') & -i[\alpha R(\mu)T(\mu') + \zeta T(\mu)R(\mu')] & i\epsilon T(\mu)P(\mu') \\ i\gamma T(\mu)P(\mu') & i[\alpha T(\mu)R(\mu') + \zeta R(\mu)T(\mu')] & \alpha T(\mu)T(\mu') + \zeta R(\mu)R(\mu') & -\epsilon R(\mu)P(\mu') \\ 0 & i\epsilon P(\mu)T(\mu') & \epsilon P(\mu)R(\mu') & \delta P(\mu)P(\mu') \end{vmatrix} . \tag{49}$$

We note that in equation (49) $\{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\} \Rightarrow \{\alpha_l, \beta_l, \gamma_l, \delta_l, \epsilon_l, \zeta_l\}$,

$$P(\mu) \Rightarrow P_l^s(\mu) , \tag{50a}$$

$$R(\mu) \Rightarrow R_l^s(\mu) = -\tfrac{1}{2}(i)^s \left[\frac{(l+s)!}{(l-s)!}\right]^{1/2} [P_{s,2}^l(\mu) + P_{s,-2}^l(\mu)] , \tag{50b}$$

and

$$T(\mu) \Rightarrow T_l^s(\mu) = -\tfrac{1}{2}(i)^s \left[\frac{(l+s)!}{(l-s)!}\right]^{1/2} [P_{s,2}^l(\mu) - P_{s,-2}^l(\mu)] . \tag{50c}$$

In addition we use $P_l^s(\mu)$ to denote the associated Legendre function, i.e.,

$$P_l^s(\mu) = (1-\mu^2)^{s/2} \frac{d^s}{d\mu^s} P_l(\mu) , \tag{51}$$

where $P_l(\mu)$ is the usual Legendre polynomial of order l. The real constants appearing in equation (49) are similar to those suggested by Herman and Lenoble (1968), *viz.*,

$$\alpha_l = p_{2,2}^l + p_{2,-2}^l , \tag{52a}$$

$$\beta_l = p_{0,0}^l + p_{0,-0}^l , \tag{52b}$$

$$\gamma_l = -p_{0,2}^l - p_{0,-2}^l , \tag{52c}$$

$$\delta_l = p_{0,0}^l - p_{0,-0}^l , \tag{52d}$$

$$\epsilon_l = i(p_{0,-2}^l - p_{0,2}^l) , \tag{52e}$$

and

$$\zeta_l = p_{2,2}^l - p_{2,-2}^l . \tag{52f}$$

Now that we have equation (49), we can write $A^s(\mu, \mu')$ in a form suggested by Santer (1980); thus we find

$$A^s(\mu, \mu') = \sum_{l=s}^{L} A_l^s(\mu) \boldsymbol{B}_l^s A_l^s(\mu') , \tag{53}$$

where

$$A_l^s(\mu) = \begin{vmatrix} P_l^s(\mu) & 0 & 0 & 0 \\ 0 & R_l^s(\mu) & -T_l^s(\mu) & 0 \\ 0 & -T_l^s(\mu) & R_l^s(\mu) & 0 \\ 0 & 0 & 0 & P_l^s(\mu) \end{vmatrix} \tag{54}$$

and

$$B_l^s = \frac{(l-s)!}{(l+s)!} \begin{vmatrix} \beta_l & \gamma_l & 0 & 0 \\ \gamma_l & \alpha_l & 0 & 0 \\ 0 & 0 & \zeta_l & -\epsilon_l \\ 0 & 0 & \epsilon_l & \delta_l \end{vmatrix} . \tag{55}$$

We note that the matrices $A_l^s(\mu)$ can all be diagonalized by a single constant matrix and that this fact may be useful when we attempt to develop solutions of equation (47). Of course, since

$$B^s(\mu, \mu') = DA^s(\mu, \mu')D , \tag{56}$$

we can write

$$\Upsilon(\tau, \mu) = \Psi_1(\tau, \mu) \tag{57}$$

and

$$\Xi(\tau, \mu) = D\Psi_2(\tau, \mu) , \tag{58}$$

where $\Psi_1(\tau, \mu)$ and $\Psi_2(\tau, \mu)$ are solutions of

$$\mu \frac{\partial}{\partial \tau} \Psi(\tau, \mu) + \Psi(\tau, \mu) = \frac{\omega}{2} \sum_{l=s}^{L} A_l^s(\mu) B_l^s \int_{-1}^{1} A_l^s(\mu') \Psi(\tau, \mu') d\mu' \tag{59}$$

sufficiently general that the boundary conditions given by equations (43) and (44) can be satisfied. The matter of solving equation (59) is, naturally, the subject of continuing work on this subject.

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