10. On the Normal-Mode Expansion Technique in Spherical Geometry, *G. E. Siewert (NC State U)*

The singular eigenfunction expansion technique that was introduced by Case has been used quite successfully to develop exact solutions for many problems in neutron transport theory.[1-3] Although the class of problems for which the Case method has been used is a broad one, the major limitation appears to be the restriction to plane geometries. The purpose of this paper is, therefore, to indicate a procedure by which the angular density for a problem in spherical geometry can be obtained directly from the normal modes of the transport equation.

A solution, subject to the constraint that $r^2\psi(r,\mu)$ must vanish as r increases without bound, to the equation

$$\mu \frac{\partial}{\partial r} \psi(r,\mu) + \frac{(1-\mu^2)}{r} \frac{\partial}{\partial \mu} \psi(r,\mu) + \psi(r,\mu) = \frac{c}{2} \int_{-1}^{1} \psi(r,\mu')d\mu' + \frac{\delta(r)}{4\pi r^2} \quad (1)$$

is sought. A solution to the homogeneous equation, that also satisfies the condition at infinity, can be written in terms of the normal modes that were discussed by Mitsis, i.e.,

$$\psi(r,\mu) = \sum_{m=0}^{\infty} \frac{2m+1}{2} P_m(\mu) \{ A_+ c\nu_0 Q_m(\nu_0)\, k_m(r/\nu_0) + \int_0^1 A(\nu)\, [c\nu Q_m(\nu) + \lambda(\nu)\, P_m(\nu)]\, k_m(r/\nu)d\nu\}, \quad (2)$$

where the notation follows that of the references. The expansion coefficients A_+ and $A(\nu)$ must be determined from the boundary condition at the origin, viz.,

$$\lim_{r \to 0} 4\pi r^2 \psi(r,\mu) = \delta(1-\mu) \ . \quad (3)$$

The boundary condition given by Eq. (3) implies that for all m,

$$\lim_{r \to 0} 4\pi r^2 S_m(r) = 1 , \tag{4}$$

where

$$S_m(r) \triangleq A_+ c\nu_0 Q_m(\nu_0) k_m(r/\nu_0) + \int_0^1 A(\nu)[c\nu Q_m(\nu) + \lambda(\nu) P_m(\nu)] k_m(r/\nu) d\nu. \tag{5}$$

The technique used here is to solve for A_+ and $A(\nu)$ from Eq. (4) for the particular case of m = 1; mathematical induction is then used to prove that the coefficients so determined are such that Eq. (4) is satisfied for all m. It is found that

$$A_+ = 1/2\pi^2 \nu_0^2 N_+ \tag{6a}$$

and

$$A(\nu) = g(c,\nu)/2\pi^2 \nu^3 \quad . \tag{6b}$$

The angular density is thus determined; the density and current are found immediately by integration

$$\rho(r) = A_+ k_0(r/\nu_0) + \int_0^1 A(\nu)\, k_0\, (r/\nu) d\nu \tag{7a}$$

and

$$j(r) = A_+ \nu_0 (1-c)\, k_1\, (r/\nu_0) + \int_0^1 A(\nu)\, \nu(1-c)\, k_1\, (r/\nu) d\nu \quad . \tag{7b}$$

In the limit as c approaches zero, the identity,[4]

$$\int_0^1 k_m(r/\nu)\, P_m(\nu) \frac{d\nu}{\nu^3} = \frac{\pi}{2r^2} \exp(-r) \quad , \tag{8}$$

can be used to show that the given expression for $\psi(r,\mu)$ reduces the usual result, i.e.,

$$\psi_0(r,\mu) = \frac{\exp(-r)}{4\pi r^2}\, \delta(1-\mu) \quad . \tag{9}$$

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3. GEORGE J. MITSIS, "Transport Solutions to the Monoenergetic Critical Problems," ANL-6787, Argonne National Laboratory (1963).
4. W. J. HARRINGTON, C. E. SIEWERT, and R. L. MURRAY, "On a Bessel-Legendre Integral Identity," (submitted to *SIAM Review*).