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Multigroup Transport Theory. I. Basic Analysis

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The special case of a triangular transfer matrix relevant to multigroup neutron-transport theory is discussed. The theory developed for the slowing down of neutrons (or photons) reduces the multigroup problem to a sequence of one-group problems. Thus, in contrast to strictly numerical techniques, a method for multigroup theory that does not require increased computational time for thick slabs is made available.

I. INTRODUCTION

We consider here the multigroup neutron-transport equation written in the form[1]

$$\mu \frac{\partial}{\partial z} \boldsymbol{\Psi}(z,\mu) + \boldsymbol{\Sigma}\boldsymbol{\Psi}(z,\mu) = \frac{1}{2}\boldsymbol{\Sigma}_s \int_{-1}^{1} \boldsymbol{\Psi}(z,\mu')d\mu' \quad , \quad (1)$$

where the M-vector $\boldsymbol{\Psi}(z,\mu)$ has the group angular fluxes $\psi_1(z,\mu)$, $\psi_2(z,\mu)$, ... and $\psi_M(z,\mu)$ as elements. In addition $\boldsymbol{\Sigma} = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_M\}$ has the group total cross sections as elements, the transfer matrix $\boldsymbol{\Sigma}_s$ has elements σ_{ij}, z is the space variable, and μ is the direction cosine of the neutron-velocity vector. Since we are interested here in the "slowing down" problem, we consider the groups to be ordered such that the transfer matrix $\boldsymbol{\Sigma}_s$ is lower triangular, and thus in scalar notation Eq. (1) is

$$\mu \frac{\partial}{\partial z} \psi_i(z,\mu) + \sigma_i \psi_i(z,\mu)$$
$$= \frac{1}{2}\sigma_{ii} \int_{-1}^{1} \psi_i(z,\mu')d\mu' + \frac{1}{2}\sum_{j=1}^{i-1} \sigma_{ij}\phi_j(z) \quad , \quad (2)$$

where $i = 1, 2, \ldots, M$ and the flux for group j is

$$\phi_j(z) = \int_{-1}^{1} \psi_j(z,\mu)d\mu \quad . \quad (3)$$

We are concerned here with a subcritical, $\sigma_{ii} < \sigma_i$, slab $z \in [L,R]$, and thus we seek a solution to Eq. (1) subject to the boundary conditions

$$\boldsymbol{\Psi}(L,\mu) = \boldsymbol{L}(\mu) \quad , \quad \mu > 0 \quad , \quad (4a)$$

and

$$\boldsymbol{\Psi}(R,-\mu) = \boldsymbol{R}(\mu) \quad , \quad \mu > 0 \quad , \quad (4b)$$

where $\boldsymbol{L}(\mu)$ and $\boldsymbol{R}(\mu)$ are considered given.

II. ANALYSIS

We note that Larsen and Zweifel[2] have investigated the Wiener-Hopf factorization for the dispersion matrix relevant to the case of a triangular transfer matrix; however, here we seek to establish the desired solution by using scalar theory[3-5] to solve a sequence of one-group problems. Thus, for the first

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[1]B. DAVISON, *Neutron Transport Theory*, Oxford University Press, London (1957).

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group we write[3]

$$\psi_1(z,\mu) = A(\nu_1)\phi_1(\nu_1,\mu)\exp(-\sigma_1 z/\nu_1) + A(-\nu_1)\phi_1(-\nu_1,\mu)\exp(\sigma_1 z/\nu_1) + \int_{-1}^{1} A_1(\nu)\phi_1(\nu,\mu)\exp(-\sigma_1 z/\nu)d\nu \quad , \quad (5)$$

where the elementary solutions are, in general,

$$\phi_i(\nu_i,\mu) = \frac{1}{2}c_i\nu_i\left(\frac{1}{\nu_i-\mu}\right) \quad (6a)$$

and

$$\phi_i(\nu,\mu) = \frac{1}{2}c_i\nu P\upsilon\left(\frac{1}{\nu-\mu}\right) + (1-c_i\nu\tanh^{-1}\nu)\delta(\nu-\mu) \ . \quad (6b)$$

Here $c_i = \sigma_{ii}/\sigma_i$ and ν_i is the positive zero of

$$\Lambda_i(z) = 1 + \frac{1}{2}c_i z\int_{-1}^{1}\frac{d\mu}{\mu - z} \ . \quad (7)$$

We can now use the full-range orthogonality condition[3]

$$(\xi-\xi')\int_{-1}^{1}\mu\phi_i(\xi,\mu)\phi_i(\xi',\mu)d\mu = 0 \quad , \quad (8)$$

where $\pm\xi$, $\pm\xi' \in P_i = \nu_i \cup [0,1]$ to deduce[5] the following singular integral equations and constraints for $\psi_1(L,-\mu)$ and $\psi_1(R,\mu)$, $\mu > 0$:

$$\int_{-1}^{1}\mu\phi_1(\pm\xi,\mu)[\psi_1(L,\mu) - \exp(\pm\Delta_1/\xi)\psi_1(R,\mu)]d\mu = 0 \quad , \quad \xi\in P_1 \quad , \quad (9)$$

where, in general, $\Delta_i = \sigma_i(R-L)$. We can rewrite Eq. (9) as

$$\int_0^1 \mu\phi_1(\xi,\mu)\psi_1(L,-\mu)d\mu + \exp(-\Delta_1/\xi) \times \int_0^1 \mu\phi_1(-\xi,\mu)\psi_1(R,\mu)d\mu = U_1(\xi) \quad (10a)$$

and

$$\int_0^1 \mu\phi_1(\xi,\mu)\psi_1(R,\mu)d\mu + \exp(-\Delta_1/\xi) \times \int_0^1 \mu\phi_1(-\xi,\mu)\psi_1(L,-\mu)d\mu = V_1(\xi) \quad , \quad (10b)$$

where $\xi \in P_1$ and the known inhomogeneous terms are, in general,

$$U_i(\xi) = \int_0^1 \mu\phi_i(-\xi,\mu)L_i(\mu)d\mu + \exp(-\Delta_i/\xi)\int_0^1 \mu\phi_i(\xi,\mu)R_i(\mu)d\mu \quad (11a)$$

and

$$V_i(\xi) = \int_0^1 \mu\phi_i(-\xi,\mu)R_i(\mu)d\mu + \exp(-\Delta_i/\xi)\int_0^1 \mu\phi_i(\xi,\mu)L_i(\mu)d\mu \ . \quad (11b)$$

We consider now that Eqs. (10a) and (10b) can be solved, for example, by using the theory of Muskhelishvili[6] to obtain Fredholm integral equations or by the F_N method,[4] and thus we go on to the second group. Here we write

$$\psi_2(z,\mu) = A(\nu_2)\phi_2(\nu_2,\mu)\exp(-\sigma_2 z/\nu_2) + A(-\nu_2)\phi_2(-\nu_2,\mu)\exp(\sigma_2 z/\nu_2) + \int_{-1}^{1} A_2(\nu)\phi_2(\nu,\mu)\exp(-\sigma_2 z/\nu)d\nu + \frac{1}{2}\sigma_{21}\psi_{21}^*(z,\mu) \quad , \quad (12)$$

where $\psi_{21}^*(z,\mu)$ is used to denote a particular solution of

$$\mu\frac{\partial}{\partial z}\psi(z,\mu) + \sigma_2\psi(z,\mu) = \frac{1}{2}\sigma_{22}\int_{-1}^{1}\psi(z,\mu')d\mu' + \phi_1(z) \quad , \quad (13)$$

and $\phi_1(z)$ is the (now considered) known flux relevant to the first group. We can use the orthogonality condition given by Eq. (8) and the solution for the second group, as expressed by Eq. (12), to deduce that

$$\int_{-1}^{1}\mu\phi_2(\pm\xi,\mu)[\psi_2(L,\mu) - \exp(\pm\Delta_2/\xi)\psi_2(R,\mu)]d\mu = \frac{1}{2}\sigma_{21}W_{21}(\pm\xi) \quad , \quad (14)$$

where $\xi \in P_2 = \nu_2 \cup [0,1]$ and

$$W_{21}(\xi) = \int_{-1}^{1}\mu\phi_2(\xi,\mu)[\psi_{21}^*(L,\mu) - \exp(\Delta_2/\xi)\psi_{21}^*(R,\mu)]d\mu \ . \quad (15)$$

It is clear that $\psi_{21}^*(z,\mu)$ can be expressed in terms of the infinite-medium Green's function basic to Eq. (13). Thus we write

$$\psi_{21}^*(z,\mu) = \int_L^R G_2(z_0\to z;\mu)\phi_1(z_0)dz_0 \quad , \quad (16)$$

where,[7] in general, for $z > z_0$

$$G_i(z_0\to z;\mu) = \frac{1}{N_i(\nu_i)}\phi_i(\nu_i,\mu)\exp[-\sigma_i(z-z_0)/\nu_i] + \int_0^1 \frac{1}{N_i(\nu)}\phi_i(\nu,\mu) \times \exp[-\sigma_i(z-z_0)/\nu]d\nu \quad , \quad (17a)$$

[6]N. I. MUSKHELISHVILI, *Singular Integral Equations*, Noordhoff, Groningen, Holland (1953).

[7]K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts (1967).

and for $z < z_0$

$$G_i(z_0 \to z;\mu) = -\frac{1}{N_i(-\nu_i)}\phi_i(-\nu_i,\mu)\exp[\sigma_i(z-z_0)/\nu_i] - \int_0^1 \frac{1}{N_i(-\nu)}\phi_i(-\nu,\mu) \times \exp[\sigma_i(z-z_0)/\nu]d\nu \ . \tag{17b}$$

In addition,

$$N_i(\nu_i) = \frac{1}{2}c_i\nu_i^3\left(\frac{c_i}{\nu_i^2-1} - \frac{1}{\nu_i^2}\right) \tag{18a}$$

and

$$N_i(\nu) = \nu\left[(1-c_i\nu\tanh^{-1}\nu)^2 + \frac{1}{4}\pi^2\nu^2c_i^2\right] \ . \tag{18b}$$

We now substitute Eqs. (16) and (17) into Eq. (15) to find

$$W_{21}(\xi) = -\exp(-\sigma_2L/\xi)\int_L^R \phi_1(z)\exp(\sigma_2z/\xi)dz \ , \tag{19}$$

so that we can write Eq. (14) as

$$\int_0^1 \mu\phi_2(\xi,\mu)\psi_2(L,-\mu)d\mu + \exp(-\Delta_2/\xi)\int_0^1 \mu\phi_2(-\xi,\mu)\psi_2(R,\mu)d\mu = U_2(\xi) + \frac{1}{2}\xi\sigma_{21}I_{21}(\xi) \tag{20a}$$

and

$$\int_0^1 \mu\phi_2(\xi,\mu)\psi_2(R,\mu)d\mu + \exp(-\Delta_2/\xi)\int_0^1 \mu\phi_2(-\xi,\mu)\psi_2(L,-\mu)d\mu = V_2(\xi) + \frac{1}{2}\xi\sigma_{21}J_{21}(\xi) \ , \tag{20b}$$

where $\xi \in P_2$,

$$\xi I_{21}(\xi) = \exp(\sigma_2L/\xi)\int_L^R \phi_1(z)\exp(-\sigma_2z/\xi)dz \tag{21a}$$

and

$$\xi J_{21}(\xi) = \exp(-\sigma_2R/\xi)\int_L^R \phi_1(z)\exp(\sigma_2z/\xi)dz \ . \tag{21b}$$

It is apparent that the right sides of Eqs. (20) will be known once we have solved the considered problem for the first group. However, we seek a theory that first yields the desired solution at the boundaries, $z = L$ and $z = R$, and thus we would like to express $I_{21}(\xi)$ and $J_{21}(\xi)$ in terms of $\psi_1(L,\mu)$ and $\psi_1(R,\mu)$, $\mu \in [-1,1]$. If we write Eq. (2), for the first group, as

$$\mu\frac{\partial}{\partial z}[\psi_1(z,\mu)\exp(\sigma_1z/\mu)] = \frac{1}{2}\sigma_{11}\exp(\sigma_1z/\mu)\phi_1(z) \ , \tag{22}$$

then we can integrate Eq. (22) to find, for $\xi \in [0,\sigma_2/\sigma_1]$,

$$I_{21}(\xi) = \frac{2}{\sigma_{11}}\left(\frac{\sigma_1}{\sigma_2}\right)[\psi_1(L,-\sigma_1\xi/\sigma_2) - R_1(\sigma_1\xi/\sigma_2)\exp(-\Delta_2/\xi)] \tag{23a}$$

and

$$J_{21}(\xi) = \frac{2}{\sigma_{11}}\left(\frac{\sigma_1}{\sigma_2}\right)[\psi_1(R,\sigma_1\xi/\sigma_2) - L_1(\sigma_1\xi/\sigma_2)\exp(-\Delta_2/\xi)] \ . \tag{23b}$$

Equations (23) clearly express $I_{21}(\xi)$ and $J_{21}(\xi)$ in terms of the boundary fluxes $\psi_1(L,\mu)$ and $\psi_1(R,\mu)$; however, Eqs. (23a) and (23b) are not sufficiently general since Eqs. (20a) and (20b) require $I_{21}(\xi)$ and $J_{21}(\xi)$ for all $\xi \in P_2$. It is clear that we can formally solve Eq. (22) for $\psi_1(z,\mu)$ and then integrate to find

$$\phi_1(z) = K_1(z) + \frac{1}{2}\sigma_{11}\int_L^R \phi_1(z')E_1(\sigma_1|z-z'|)dz' \ , \tag{24}$$

where $E_1(x)$ is the exponential integral function and, in general,

$$K_j(z) = \int_0^1 \{L_j(\mu)\exp[-\sigma_j(z-L)/\mu] + R_j(\mu)\exp[-\sigma_j(R-z)/\mu]\}\,d\mu \ . \tag{25}$$

Multiplying Eq. (24) by $\exp(-z/s)$, integrating over z and using Eqs. (21) and (23), we find

$$\Lambda_1(\sigma_1\xi/\sigma_2)I_{21}(\xi) = \int_{-1}^1 \mu[\psi_1(L,\mu) - \psi_1(R,\mu)\exp(-\Delta_2/\xi)] \times \frac{d\mu}{\sigma_2\mu+\sigma_1\xi} \tag{26a}$$

and

$$\Lambda_1(\sigma_1\xi/\sigma_2)J_{21}(\xi) = \int_{-1}^1 \mu[\psi_1(R,\mu) - \psi_1(L,\mu)\exp(-\Delta_2/\xi)] \times \frac{d\mu}{\sigma_2\mu-\sigma_1\xi} \tag{26b}$$

for all $\xi \notin [-\sigma_2/\sigma_1,\ \sigma_2/\sigma_1]$. Equations (23) and (26) clearly provide the expressions sought for the right sides of Eqs. (20a) and (20b).

We now wish to extend the foregoing to the i'th group. We thus write

$$\psi_i(z,\mu) = A(\nu_i)\phi_i(\nu_i,\mu)\exp(-\sigma_i z/\nu_i) + A(-\nu_i)\phi_i(-\nu_i,\mu)\exp(\sigma_i z/\nu_i) + \int_{-1}^{1} A_i(\nu)\phi_i(\nu,\mu)\exp(-\sigma_i z/\nu)d\nu + \frac{1}{2}\sum_{j=1}^{i-1} \sigma_{ij}\psi_{ij}^*(z,\mu) \ , \quad (27)$$

where

$$\psi_{ij}^*(z,\mu) = \int_L^R G_i(z_0 \to z;\mu)\phi_j(z_0)dz_0 \ . \quad (28)$$

It is thus apparent that for the i'th group Eqs. (20a) and (20b) are replaced by

$$\int_0^1 \mu\phi_i(\xi,\mu)\psi_i(L,-\mu)d\mu + \exp(-\Delta_i/\xi) \times \int_0^1 \mu\phi_i(-\xi,\mu)\psi_i(R,\mu)d\mu = U_i(\xi) + \frac{1}{2}\xi\sum_{j=1}^{i-1}\sigma_{ij}I_{ij}(\xi) \quad (29a)$$

and

$$\int_0^1 \mu\phi_i(\xi,\mu)\psi_i(R,\mu)d\mu + \exp(-\Delta_i/\xi) \times \int_0^1 \mu\phi_i(-\xi,\mu)\psi_i(L,-\mu)d\mu = V_i(\xi) + \frac{1}{2}\xi\sum_{j=1}^{i-1}\sigma_{ij}J_{ij}(\xi) \ , \quad (29b)$$

where $\xi \in P_i$,

$$\xi I_{ij}(\xi) = \exp(\sigma_i L/\xi)\int_L^R \phi_j(z)\exp(-\sigma_i z/\xi)dz \quad (30a)$$

and

$$\xi J_{ij}(\xi) = \exp(-\sigma_i R/\xi)\int_L^R \phi_j(z)\exp(\sigma_i z/\xi)dz \ . \quad (30b)$$

Considering now the j'th group, we write

$$\mu\frac{\partial}{\partial z}[\psi_j(z,\mu)\exp(\sigma_j z/\mu)] = \frac{1}{2}\exp(\sigma_j z/\mu)\sum_{k=1}^{j}\sigma_{jk}\phi_k(z) \quad (31)$$

and integrate to find the desired generalizations of Eqs. (23) to be, for $\xi \in [0,1/s_{ij}]$,

$$I_{ij}(\xi) = \frac{1}{\sigma_{jj}}\left[X_{ij}(\xi) - \sum_{k=1}^{j-1}\sigma_{jk}I_{ik}(\xi)\right] \quad (32a)$$

and

$$J_{ij}(\xi) = \frac{1}{\sigma_{jj}}\left[Y_{ij}(\xi) - \sum_{k=1}^{j-1}\sigma_{jk}J_{ik}(\xi)\right] \ , \quad (32b)$$

where $s_{ij} = \sigma_j/\sigma_i$,

$$X_{ij}(\xi) = 2s_{ij}[\psi_j(L,-s_{ij}\xi) - R_j(s_{ij}\xi)\exp(-\Delta_i/\xi)] \ , \quad (33a)$$

and

$$Y_{ij}(\xi) = 2s_{ij}[\psi_j(R,s_{ij}\xi) - L_j(s_{ij}\xi)\exp(-\Delta_i/\xi)] \ . \quad (33b)$$

Now Eq. (31) yields

$$\phi_j(z) = K_j(z) + \frac{1}{2}\sum_{k=1}^{j}\sigma_{jk}\int_L^R \phi_k(z')E_1(\sigma_j|z-z'|)dz' \ , \quad (34)$$

which we can multiply by $\exp(-z/s)$ and integrate to find, for all $\xi \notin [-1/s_{ij}, 1/s_{ij}]$,

$$\Lambda_j(s_{ij}\xi)I_{ij}(\xi) = \int_{-1}^{1}\mu[\psi_j(L,\mu) - \psi_j(R,\mu)\exp(-\Delta_i/\xi)] \times \frac{d\mu}{\sigma_i\mu + \sigma_j\xi} - \frac{1}{\sigma_j}\Delta(s_{ij}\xi)\sum_{k=1}^{j-1}\sigma_{jk}I_{ik}(\xi) \quad (35a)$$

and

$$\Lambda_j(s_{ij}\xi)J_{ij}(\xi) = \int_{-1}^{1}\mu[\psi_j(R,\mu) - \psi_j(L,\mu)\exp(-\Delta_i/\xi)] \times \frac{d\mu}{\sigma_i\mu - \sigma_j\xi} - \frac{1}{\sigma_j}\Delta(s_{ij}\xi)\sum_{k=1}^{j-1}\sigma_{jk}J_{ik}(\xi) \ , \quad (35b)$$

where $\Lambda_j(z)$ is given by Eq. (7) and

$$\Delta(z) = \frac{1}{2}z\int_{-1}^{1}\frac{d\mu}{\mu - z} \ . \quad (36)$$

It is clear that Eqs. (32) and (35), along with Eqs. (11a) and (11b), provide the right sides of Eqs. (29). It follows that to find $\mathbf{\Psi}(L,-\mu)$ and $\mathbf{\Psi}(R,\mu)$, $\mu > 0$, we need only consider a sequence of one-group problems involving angular fluxes at the boundaries. In the following paper,[8] the F_N method[4,5] is used to develop concise approximate solutions to Eqs. (29a) and (29b) and to establish numerical results for two basic problems.

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