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Multigroup Transport Theory. II. Numerical Results

R. D. M. Garcia and C. E. Siewert

North Carolina State University, Departments of Nuclear Engineering and Mathematics
Raleigh, North Carolina 27650

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The F_N method is used to establish particularly accurate solutions, at modest cost, for the emerging angular fluxes basic to a class of multigroup particle-transport problems. A study of the fundamental computational aspects of the established solution is reported, and numerical results are given, accurate to five significant figures, for the reflected and transmitted angular fluxes relevant to a 16-group albedo problem and to a 19-group albedo problem.

I. INTRODUCTION

It is apparent that the multigroup neutron-transport problem defined by

$$\mu \frac{\partial}{\partial z} \mathbf{\Psi}(z,\mu) + \mathbf{\Sigma}\mathbf{\Psi}(z,\mu) = \frac{1}{2}\mathbf{\Sigma}_s \int_{-1}^{1} \mathbf{\Psi}(z,\mu')d\mu' \quad , \qquad (1)$$

$$\mathbf{\Psi}(L,\mu) = \boldsymbol{L}(\mu) \quad , \quad \mu > 0 \quad , \qquad (2a)$$

and

$$\mathbf{\Psi}(R,-\mu) = \boldsymbol{R}(\mu) \quad , \quad \mu > 0 \quad , \qquad (2b)$$

can, for the case of a triangular transfer matrix ("slowing down"), be solved as a sequence of one-group problems. However, it is also clear that to solve Eq. (1) for all z, as a sequence of one-group problems, would require a major computational effort. In Part I of this work,[1] which appears immediately preceding this paper, exact analysis was used to reduce the problem of finding the surface angular distributions relevant to Eqs. (1) and (2) to a system of one-group problems, each of which is based only on surface results. We thus now wish to demonstrate that the F_N method[2,3] can be used to establish accurate numerical results for the considered multigroup model and especially for the challenging deep-penetration problem. To avoid excessive repetition we use liberally the equations reported in Ref. 1 and denoted here by the inclusion of I with the equation number.

[1]C. E. SIEWERT and P. BENOIST, *Nucl. Sci. Eng.*, **78**, 311 (1981).

[2]C. E. SIEWERT and P. BENOIST, *Nucl. Sci. Eng.*, **69**, 156 (1979).

[3]P. GRANDJEAN and C. E. SIEWERT, *Nucl. Sci. Eng.*, **69**, 161 (1979).

II. THE F_N METHOD

It is evident that Chandrasekhar's H function,[4] for example, can be used to convert Eqs. (I-10a) and (I-10b) to a system of Fredholm integral equations, which can, of course, be solved by an iterative numerical method to yield $\psi_1(L,-\mu)$ and $\psi_1(R,\mu)$, $\mu > 0$. It follows that Eqs. (I-29) can, in a like manner, be solved for $i = 2$, then $i = 3$ and so on. Rather than pursue this exact analysis, we prefer here to use the F_N method[2,3] to develop a concise approximate solution. We therefore introduce, for the i'th group and $\mu > 0$,

$$\psi_i(L,-\mu) = R_i(\mu)\exp(-\Delta_i/\mu) + \sum_{\alpha=0}^{N} a_{i,\alpha}P_\alpha(2\mu - 1) \qquad (3a)$$

[4]S. CHANDRASEKHAR, *Radiative Transfer*, Oxford University Press, London (1950).

and

$$\psi_i(R,\mu) = L_i(\mu)\exp(-\Delta_i/\mu) + \sum_{\alpha=0}^{N} b_{i,\alpha}P_\alpha(2\mu-1) \tag{3b}$$

into Eqs. (I-29) to find

$$\sum_{\alpha=0}^{N} [a_{i,\alpha}B_{i,\alpha}(\xi) + c_i\exp(-\Delta_i/\xi)b_{i,\alpha}A_\alpha(\xi)] = c_iI_i(\xi) + \sum_{j=1}^{i-1}\sigma_{ij}I_{ij}(\xi) \tag{4a}$$

and

$$\sum_{\alpha=0}^{N} [b_{i,\alpha}B_{i,\alpha}(\xi) + c_i\exp(-\Delta_i/\xi)a_{i,\alpha}A_\alpha(\xi)] = c_iJ_i(\xi) + \sum_{j=1}^{i-1}\sigma_{ij}J_{ij}(\xi) \tag{4b}$$

for all $\xi \in P_i$. Here, for $\alpha \geqslant 1$,

$$B_{i,\alpha}(\xi) = \left(\frac{2\alpha-1}{\alpha}\right)(2\xi-1)B_{i,\alpha-1}(\xi) - \left(\frac{\alpha-1}{\alpha}\right)B_{i,\alpha-2}(\xi) - \frac{1}{2}c_i\delta_{\alpha,2} - c_i\delta_{\alpha,1}\ , \tag{5a}$$

with

$$B_{i,0}(\xi) = 2 - c_i[1 + \xi\ln(1+1/\xi)] \tag{5b}$$

and

$$A_\alpha(\xi) = -\left(\frac{2\alpha-1}{\alpha}\right)(2\xi+1)A_{\alpha-1}(\xi) - \left(\frac{\alpha-1}{\alpha}\right)A_{\alpha-2}(\xi) + \frac{1}{2}\delta_{\alpha,2} + \delta_{\alpha,1}\ , \tag{6a}$$

with

$$A_0(\xi) = 1 - \xi\ln|1+1/\xi|\ . \tag{6b}$$

In Eqs. (3a) and (3b) we use a Legendre basis $P_\alpha(2\mu - 1)$ that is orthogonal on the half range $\mu \in [0, 1]$ in order to avoid, in subsequent systems of linear algebraic equations, the inversion of ill-conditioned matrices that have been encountered, for large N, with the use[2,3] of the simple basis functions μ^α. The functions $I_i(\xi)$ and $J_i(\xi)$ required in Eqs. (4a) and (4b) are given in terms of the boundary data for the i'th group, i.e.,

$$I_i(\xi) = \int_0^1 \mu[L_i(\mu)S_i(\mu,\xi) + R_i(\mu)C_i(\mu,\xi)]\,d\mu \tag{7a}$$

and

$$J_i(\xi) = \int_0^1 \mu[L_i(\mu)C_i(\mu,\xi) + R_i(\mu)S_i(\mu,\xi)]\,d\mu\ , \tag{7b}$$

where

$$S_i(\mu,\xi) = \frac{1-\exp[-\Delta_i(1/\mu+1/\xi)]}{\mu+\xi} \tag{8a}$$

and

$$C_i(\mu,\xi) = \left[\frac{\exp(-\Delta_i/\mu) - \exp(-\Delta_i/\xi)}{\mu-\xi}\right]\ . \tag{8b}$$

The additional known terms in Eqs. (4a) and (4b) represent slowing down contributions to the i'th group. Thus, for $\xi \in [0, 1/s_{ij}]$ we write Eqs. (I-32a) and (I-32b) as

$$\sigma_{ij}I_{ij}(\xi) = 2s_{ij}\sum_{\alpha=0}^{N} a_{j,\alpha}P_\alpha(2s_{ij}\xi - 1) - \sum_{k=1}^{j-1}\sigma_{jk}I_{ik}(\xi) \tag{9a}$$

and

$$\sigma_{ij}J_{ij}(\xi) = 2s_{ij}\sum_{\alpha=0}^{N} b_{j,\alpha}P_\alpha(2s_{ij}\xi - 1) - \sum_{k=1}^{j-1}\sigma_{jk}J_{ik}(\xi)\ . \tag{9b}$$

For $\xi \notin [0, 1/s_{ij}]$ we deduce from Eqs. (I-35a) and (I-35b) that

$$\Lambda_j(s_{ij}\xi)\sigma_iI_{ij}(\xi) = \Upsilon_j(s_{ij}\xi) + \xi\tanh^{-1}\left(\frac{1}{s_{ij}\xi}\right)\sum_{k=1}^{j-1}\sigma_{jk}I_{ik}(\xi) \tag{10a}$$

and

$$\Lambda_j(s_{ij}\xi)\sigma_iJ_{ij}(\xi) = \Xi_j(s_{ij}\xi) + \xi\tanh^{-1}\left(\frac{1}{s_{ij}\xi}\right)\sum_{k=1}^{j-1}\sigma_{jk}J_{ik}(\xi)\ , \tag{10b}$$

where

$$\Upsilon_j(\xi) = I_j(\xi) + \sum_{\alpha=0}^{N}[a_{j,\alpha}A_\alpha(-\xi) - \exp(-\Delta_j/\xi)b_{j,\alpha}A_\alpha(\xi)] \tag{11a}$$

and

$$\Xi_j(\xi) = J_j(\xi) + \sum_{\alpha=0}^{N}[b_{j,\alpha}A_\alpha(-\xi) - \exp(-\Delta_j/\xi)a_{j,\alpha}A_\alpha(\xi)]\ . \tag{11b}$$

It is clear that Eqs. (9) and (10) establish the required $I_{ij}(\xi)$ and $J_{ij}(\xi)$, except when $\sigma_{jj} = 0$ and thus alternatives to Eqs. (9) are required. For this case, we find that Eqs. (9) and (10) yield, for $\xi \in [0, 1/s_{ij}]$,

$$\sigma_i I_{ij}(\xi) = I_j(s_{ij}\xi) + [1 + s_{ij}\xi \ln(1 + s_{ji}/\xi)] \times \sum_{\alpha=0}^{N} a_{j,\alpha} P_\alpha(2s_{ij}\xi - 1) + \sum_{\alpha=0}^{N} [a_{j,\alpha} G_\alpha(s_{ij}\xi) - \exp(-\Delta_i/\xi) b_{j,\alpha} A_\alpha(s_{ij}\xi)] \quad (12a)$$

and

$$\sigma_i J_{ij}(\xi) = J_j(s_{ij}\xi) + [1 + s_{ij}\xi \ln(1 + s_{ji}/\xi)] \times \sum_{\alpha=0}^{N} b_{j,\alpha} P_\alpha(2s_{ij}\xi - 1) + \sum_{\alpha=0}^{N} [b_{j,\alpha} G_\alpha(s_{ij}\xi) - \exp(-\Delta_i/\xi) a_{j,\alpha} A_\alpha(s_{ij}\xi)] \ , \quad (12b)$$

where $G_0(\xi) = 0$ and, for $\alpha \geqslant 1$,

$$G_\alpha(\xi) = \left(\frac{2\alpha - 1}{\alpha}\right)(2\xi - 1) G_{\alpha-1}(\xi) - \left(\frac{\alpha - 1}{\alpha}\right) G_{\alpha-2}(\xi) + \frac{1}{2}\delta_{\alpha,2} + \delta_{\alpha,1} \ . \quad (13)$$

We also note that alternatives to Eqs. (10) are required if $s_{ij}\xi = \nu_j$. Appendix A is devoted to a discussion of this matter.

If the constants $\{a_{j,\alpha}\}$ and $\{b_{j,\alpha}\}$ have been established for all $j < i$, then clearly the right sides of Eqs. (4a) and (4b) are known. Thus, on considering Eqs. (4a) and (4b) at $(N + 1)$ values of $\xi \in P_i$, say $\xi_{i,\beta}$, we generate $2(N + 1)$ linear algebraic equations to be solved for the $2(N + 1)$ unknowns $a_{i,\alpha}$ and $b_{i,\alpha}$, $\alpha = 0, 1, 2, \ldots, N$.

III. NUMERICAL RESULTS

To establish the constants $\{a_{i,\alpha}\}$ and $\{b_{i,\alpha}\}$ required in Eqs. (3a) and (3b), we first must define a strategy for selecting the collocation points $\xi_{i,\beta}$. We then must compute accurately the known matrix elements and inhomogeneous terms in the system of linear algebraic equations

$$\sum_{\alpha=0}^{N} [a_{i,\alpha} B_{i,\alpha}(\xi_{i,\beta}) + c_i \exp(-\Delta_i/\xi_{i,\beta}) b_{i,\alpha} A_\alpha(\xi_{i,\beta})] = c_i I_i(\xi_{i,\beta}) + \sum_{j=1}^{i-1} \sigma_{ij} I_{ij}(\xi_{i,\beta}) \quad (14a)$$

and

$$\sum_{\alpha=0}^{N} [b_{i,\alpha} B_{i,\alpha}(\xi_{i,\beta}) + c_i \exp(-\Delta_i/\xi_{i,\beta}) a_{i,\alpha} A_\alpha(\xi_{i,\beta})] = c_i J_i(\xi_{i,\beta}) + \sum_{j=1}^{i-1} \sigma_{ij} J_{ij}(\xi_{i,\beta}) \ . \quad (14b)$$

In this paper, we have used, for various orders of the F_N approximation, the collocation scheme given by

$$\xi_{i,0} = \nu_i \ , \quad \text{all } N \ , \quad (15a)$$

and

$$\xi_{i,\beta} = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\beta - 1}{2N}\pi\right) \ , \quad \beta = 1, 2, \ldots, N \ , \quad N \neq 0 \ . \quad (15b)$$

The points given by Eq. (15b) are the zeros of the Chebyshev polynomial of the first kind $T_N(2x - 1)$. The use of these points to define a collocation strategy was suggested by the work of Sloan and Burn.[5] In Appendix B we discuss the various methods we have used to compute the functions $A_\alpha(\xi)$, $B_{i,\alpha}(\xi)$, and $G_\alpha(\xi)$ required to define Eqs. (14a) and (14b).

To demonstrate the computational merit of the foregoing solution, we now consider a special 16-group albedo problem. A 1-cm-thick slab has an isotropic incident distribution of neutrons only in the first group and only on the surface at $z = L$, i.e., for $\mu > 0$,

$$L_i(\mu) = \delta_{i,1} \quad (16a)$$

and

$$R_i(\mu) = 0 \ . \quad (16b)$$

The macroscopic total cross sections (cm^{-1}) are given by $\sigma_1 = 11$, $\sigma_i = 10 + i/14$, $i = 2, 3, \ldots, 14$, $\sigma_{15} = 10^4$ and $\sigma_{16} = 20$. In addition, the macroscopic transfer cross sections are given by $\sigma_{i+k,i} = 3/(k + 1)$, $i = 1, 2, \ldots, 14$ and $k = 0, 1, \ldots, (16 - i)$; $\sigma_{15,15} = 0$, $\sigma_{16,15} = 10^{-4}$, and $\sigma_{16,16} = 5$. This sample problem was designed to be a severe test of the established solution. Note, for example, the very strong absorption in group 15; in fact, the in-group scattering is zero for this group. For this data set we also have a degenerate case, in other words, $\sigma_{14}\nu_1 = \sigma_1\nu_{14}$, so that we must use the alternatives to Eqs. (10a) and (10b) that are discussed in Appendix A. We seek the exit distributions $\boldsymbol{\Psi}(L, -\mu)$ and $\boldsymbol{\Psi}(R, \mu)$, $\mu > 0$, the group albedos

$$A_i^* = 2\int_0^1 \mu\psi_i(L, -\mu)d\mu \quad (17a)$$

and the group transmission factors

$$B_i^* = 2\int_0^1 \mu\psi_i(R, \mu)d\mu \ . \quad (17b)$$

[5] I. H. SLOAN and B. J. BURN, *J. Integral Equations*, **1**, 77 (1979).

If we use the solutions given by Eqs. (3a) and (3b) in Eqs. (17a) and (17b), we find

$$A_i^* = a_{i,0} + \frac{1}{3} a_{i,1} \tag{18a}$$

and

$$B_i^* = 2\delta_{i,1} E_3(\Delta_i) + b_{i,0} + \frac{1}{3} b_{i,1} \ , \tag{18b}$$

where $E_3(x)$ denotes one of the exponential integral functions. In Tables I and II we list what we believe to be converged results for the exit distributions. In Table III we list our final results for the group albedos and transmission factors. Also, in Table III we compare our F_N results to a calculation of Renken[6] who used DTF69, a discrete ordinates code,[7] with 100 space points and eight directions for each of the half ranges of μ. The F_N results given in Tables I, II, and III are, we believe, correct to within ±1 in the fifth significant figure. We note that, in general, the albedos computed by Renken agree to four or five significant figures with our results. The transmission factors, however, agree only to two or three significant figures. An exception is found in group 15 where the DTF69 results clearly show a loss of accuracy. Finally we note that the collocation strategy defined by Eqs. (15a) and (15b) yielded results that represented a significant improvement to the ones initially deduced from either of the equally spaced schemes used previously.[3,8]

We now consider a second problem suggested and solved by Renken.[6] Here an iron (N = 8.466 × 10^{22} atom/cm^3) slab, 10 cm thick, has an isotropic source of gamma rays incident at $z = L$ in the first of 19 groups that span the energy interval 50 keV to 1 MeV. The cross-section set shown in Table IV was generated by Renken[6] and is based on the photoelectric effect and the P_0 component of the Klein-Nishina differential scattering cross section for photons.

In Tables V and VI we list our converged results for the exit distributions and in Table VII we report our final results for A_i^* and B_i^* along with those found by Renken with DTF69 (again with 100 space points and eight discrete directions for each half range of μ). Here we also believe our F_N results are correct to within ±1 in the fifth significant figure.

[6] J. H. RENKEN, Private Communication (1981).

[7] J. H. RENKEN and K. G. ADAMS, *An Improved Capability for Solution of Photon Transport Problems by the Method of Discrete Ordinates*, SC-RR-69-739, Sandia National Laboratories (1969).

[8] C. E. SIEWERT, J. R. MAIORINO, and M. N. ÖZIŞIK, *J. Quant. Spectros. Radiat. Transfer*, **23**, 565 (1980).

TABLE I

The Exit Angular Fluxes $\psi_i(L,-\mu)$ for the 16-Group Problem

	μ										
i	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1	1.2813(−1)[a]	1.1660(−1)	1.0041(−1)	8.8866(−2)	7.9986(−2)	7.2858(−2)	6.6973(−2)	6.2013(−2)	5.7767(−2)	5.4084(−2)	5.0857(−2)
2	7.7362(−2)	7.1154(−2)	6.2123(−2)	5.5490(−2)	5.0281(−2)	4.6037(−2)	4.2493(−2)	3.9480(−2)	3.6881(−2)	3.4613(−2)	3.2615(−2)
3	5.6265(−2)	5.2400(−2)	4.6539(−2)	4.2071(−2)	3.8471(−2)	3.5481(−2)	3.2946(−2)	3.0763(−2)	2.8861(−2)	2.7186(−2)	2.5699(−2)
4	4.5317(−2)	4.2624(−2)	3.8381(−2)	3.5035(−2)	3.2276(−2)	2.9946(−2)	2.7943(−2)	2.6199(−2)	2.4666(−2)	2.3306(−2)	2.2090(−2)
5	3.8517(−2)	3.6527(−2)	3.3271(−2)	3.0621(−2)	2.8389(−2)	2.6473(−2)	2.4806(−2)	2.3341(−2)	2.2042(−2)	2.0881(−2)	1.9838(−2)
6	3.3837(−2)	3.2315(−2)	2.9729(−2)	2.7556(−2)	2.5689(−2)	2.4062(−2)	2.2631(−2)	2.1361(−2)	2.0226(−2)	1.9206(−2)	1.8284(−2)
7	3.0396(−2)	2.9207(−2)	2.7105(−2)	2.5284(−2)	2.3687(−2)	2.2276(−2)	2.1021(−2)	1.9897(−2)	1.8886(−2)	1.7971(−2)	1.7140(−2)
8	2.7746(−2)	2.6806(−2)	2.5072(−2)	2.3521(−2)	2.2134(−2)	2.0891(−2)	1.9773(−2)	1.8764(−2)	1.7850(−2)	1.7019(−2)	1.6260(−2)
9	2.5634(−2)	2.4886(−2)	2.3442(−2)	2.2106(−2)	2.0887(−2)	1.9779(−2)	1.8773(−2)	1.7858(−2)	1.7023(−2)	1.6259(−2)	1.5558(−2)
10	2.3904(−2)	2.3310(−2)	2.2099(−2)	2.0939(−2)	1.9859(−2)	1.8864(−2)	1.7951(−2)	1.7113(−2)	1.6344(−2)	1.5637(−2)	1.4985(−2)
11	2.2458(−2)	2.1988(−2)	2.0970(−2)	1.9957(−2)	1.8994(−2)	1.8094(−2)	1.7260(−2)	1.6488(−2)	1.5775(−2)	1.5116(−2)	1.4506(−2)
12	2.1228(−2)	2.0861(−2)	2.0004(−2)	1.9116(−2)	1.8253(−2)	1.7435(−2)	1.6669(−2)	1.5954(−2)	1.5290(−2)	1.4673(−2)	1.4100(−2)
13	2.0166(−2)	1.9885(−2)	1.9166(−2)	1.8385(−2)	1.7609(−2)	1.6862(−2)	1.6156(−2)	1.5492(−2)	1.4871(−2)	1.4291(−2)	1.3749(−2)
14	1.9238(−2)	1.9031(−2)	1.8430(−2)	1.7743(−2)	1.7042(−2)	1.6359(−2)	1.5705(−2)	1.5086(−2)	1.4503(−2)	1.3956(−2)	1.3442(−2)
15	1.7897(−5)	1.7897(−5)	1.7897(−5)	1.7896(−5)	1.7896(−5)	1.7895(−5)	1.7895(−5)	1.7895(−5)	1.7894(−5)	1.7894(−5)	1.7893(−5)
16	9.1192(−3)	9.1021(−3)	8.9941(−3)	8.8398(−3)	8.6638(−3)	8.4777(−3)	8.2880(−3)	8.0982(−3)	7.9110(−3)	7.7276(−3)	7.5490(−3)

[a] Read as 1.2813 × 10^{-1}.

TABLE II

The Exit Angular Fluxes $\psi_i(R,\mu)$ for the 16-Group Problem

i	μ										
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1	4.5701(−7)	4.8998(−7)	5.6471(−7)	6.5894(−7)	7.8516(−7)	9.6666(−7)	1.2644(−6)	1.9188(−6)	3.8488(−6)	9.6935(−6)	2.5162(−5)
2	7.3831(−7)	7.9750(−7)	9.2983(−7)	1.0958(−6)	1.3185(−6)	1.6409(−6)	2.1567(−6)	3.0745(−6)	4.7969(−6)	7.9539(−6)	1.3354(−5)
3	9.6077(−7)	1.0392(−6)	1.2115(−6)	1.4230(−6)	1.7003(−6)	2.0893(−6)	2.6827(−6)	3.6719(−6)	5.4072(−6)	8.4198(−6)	1.3377(−5)
4	1.1809(−6)	1.2788(−6)	1.4910(−6)	1.7477(−6)	2.0786(−6)	2.5330(−6)	3.2046(−6)	4.2769(−6)	6.0732(−6)	9.0719(−6)	1.3866(−5)
5	1.4023(−6)	1.5200(−6)	1.7726(−6)	2.0744(−6)	2.4585(−6)	2.9771(−6)	3.7257(−6)	4.8834(−6)	6.7563(−6)	9.7887(−6)	1.4525(−5)
6	1.6263(−6)	1.7641(−6)	2.0575(−6)	2.4046(−6)	2.8416(−6)	3.4235(−6)	4.2475(−6)	5.4902(−6)	7.4451(−6)	1.0532(−5)	1.5262(−5)
7	1.8529(−6)	2.0112(−6)	2.3460(−6)	2.7385(−6)	3.2281(−6)	3.8724(−6)	4.7704(−6)	6.0965(−6)	8.1350(−6)	1.1287(−5)	1.6036(−5)
8	2.0822(−6)	2.2612(−6)	2.6377(−6)	3.0759(−6)	3.6177(−6)	4.3236(−6)	5.2938(−6)	6.7014(−6)	8.8230(−6)	1.2045(−5)	1.6828(−5)
9	2.3135(−6)	2.5137(−6)	2.9322(−6)	3.4160(−6)	4.0098(−6)	4.7762(−6)	5.8170(−6)	7.3039(−6)	9.5070(−6)	1.2799(−5)	1.7625(−5)
10	2.5465(−6)	2.7679(−6)	3.2287(−6)	3.7581(−6)	4.4034(−6)	5.2293(−6)	6.3390(−6)	7.9030(−6)	1.0185(−5)	1.3548(−5)	1.8421(−5)
11	2.7806(−6)	3.0234(−6)	3.5265(−6)	4.1013(−6)	4.7976(−6)	5.6820(−6)	6.8589(−6)	8.4974(−6)	1.0857(−5)	1.4288(−5)	1.9209(−5)
12	3.0152(−6)	3.2794(−6)	3.8250(−6)	4.4449(−6)	5.1916(−6)	6.1333(−6)	7.3754(−6)	9.0859(−6)	1.1519(−5)	1.5018(−5)	1.9985(−5)
13	3.2497(−6)	3.5354(−6)	4.1232(−6)	4.7880(−6)	5.5843(−6)	6.5821(−6)	7.8876(−6)	9.6676(−6)	1.2172(−5)	1.5734(−5)	2.0749(−5)
14	3.4835(−6)	3.7908(−6)	4.4206(−6)	5.1297(−6)	5.9749(−6)	7.0275(−6)	8.3944(−6)	1.0241(−5)	1.2813(−5)	1.6437(−5)	2.1495(−5)
15	2.3397(−9)	2.3401(−9)	2.3409(−9)	2.3417(−9)	2.3425(−9)	2.3433(−9)	2.3441(−9)	2.3449(−9)	2.3457(−9)	2.3465(−9)	2.3473(−9)
16	1.2655(−6)	1.3423(−6)	1.4865(−6)	1.6302(−6)	1.7799(−6)	1.9399(−6)	2.1142(−6)	2.3070(−6)	2.5235(−6)	2.7700(−6)	3.0545(−6)

TABLE III

A_i^* and B_i^* for the 16-Group Problem

i	Present Work		DTF69	
	A_i^*	B_i^*	A_i^*	B_i^*
1	6.6351(−2)	5.1058(−6)	6.6339(−2)	5.0413(−6)
2	4.2002(−2)	4.4781(−6)	4.1996(−2)	4.4362(−6)
3	3.2483(−2)	4.9308(−6)	3.2480(−2)	4.8877(−6)
4	2.7501(−2)	5.4649(−6)	2.7499(−2)	5.4197(−6)
5	2.4382(−2)	6.0283(−6)	2.4380(−2)	5.9806(−6)
6	2.2221(−2)	6.6050(−6)	2.2220(−2)	6.5547(−6)
7	2.0624(−2)	7.1879(−6)	2.0623(−2)	7.1349(−6)
8	1.9388(−2)	7.7730(−6)	1.9387(−2)	7.7175(−6)
9	1.8399(−2)	8.3577(−6)	1.8398(−2)	8.2996(−6)
10	1.7586(−2)	8.9399(−6)	1.7586(−2)	8.8792(−6)
11	1.6904(−2)	9.5178(−6)	1.6903(−2)	9.4547(−6)
12	1.6321(−2)	1.0090(−5)	1.6321(−2)	1.0025(−5)
13	1.5816(−2)	1.0655(−5)	1.5816(−2)	1.0587(−5)
14	1.5373(−2)	1.1212(−5)	1.5373(−2)	1.1140(−5)
15	1.7895(−5)	2.3446(−9)	1.7702(−5)	0.0
16	8.1585(−3)	2.2968(−6)	8.1578(−3)	2.2805(−6)

Further, the degree of agreement with the results of Renken is essentially the same as for the 16-group problem. Finally we note what we believe to be a slight deterioration in the DTF69 results when there is strong absorption.

We conclude from our studies that the F_N method yields consistently accurate results for the multigroup model discussed in detail in Part I of this work. Clearly for most situations the results deduced from the method of discrete ordinates are satisfactory—especially when we consider the uncertainties involved in the input data. However, when the absorption is increased and/or optically thicker slabs are considered, it is clear that strictly numerical methods will require more spatial points and/or discrete directions to achieve a desired degree of accuracy for the emerging distributions. This, of course, implies ultimately an increased computer-time requirement—a characteristic not shared by the F_N method. Thus the extension of this analysis to include the effects of anisotropic scattering is the subject of continuing work.

APPENDIX A

DEGENERATE EIGENVALUES

As mentioned in Sec. II, alternatives to Eqs. (10a) and (10b) are required when $s_{ij}\xi = \nu_j$. Clearly, if $\xi = \nu_j/s_{ij}$ happens to be a collocation point in the continuum [0, 1], we can avoid the difficulty simply

TABLE IV

The Microscopic Cross Sections (b) for the 19-Group Problem

i	σ_i	$\sigma_{i,i-k}$												
		$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$	$k=12$
1	5.66595	0.349021												
2	5.98344	0.432535	0.697148											
3	6.30367	0.405878	0.650119	0.498285										
4	6.62108	0.495359	0.802850	0.616279	0.480356									
5	6.98960	0.617669	0.972882	0.750006	0.586852	0.467569								
6	7.42746	0.790788	1.20027	0.893268	0.704703	0.566211	0.464098							
7	7.86375	0.685390	1.03224	0.729747	0.558096	0.452450	0.374217	0.315655						
8	8.28480	0.845642	1.31243	0.936166	0.682514	0.538569	0.449321	0.382203	0.330762					
9	8.79791	1.06806	1.59307	1.15715	0.855683	0.653044	0.537084	0.463710	0.406700	0.361105				
10	9.45453	1.38839	1.96463	1.36165	1.03608	0.812512	0.660727	0.570859	0.510934	0.461162	0.418387			
11	10.3618	1.87091	2.46469	1.61853	1.21904	1.00614	0.857002	0.748946	0.677485	0.623840	0.573984	0.526957		
12	11.7843	2.63912	3.15555	1.98536	1.54239	1.33667	1.21805	1.11815	1.02577	0.948740	0.881381	0.812446	0.718514	
13	13.7225	2.41159	2.70799	1.58709	1.30912	1.20715	1.14193	1.08337	0.888170	0.573596	0.354887	0.200400	0.0749019	0.00195872
14	16.4215	3.23475	3.83202	2.38131	2.08104	1.75524	1.04417	0.537960	0.180691	0.00208717				
15	21.9028	4.53244	5.08371	3.94491	2.44230	0.791035	0.0182330							
16	32.0657	4.71438	5.26866	3.10717	0.643745									
17	52.0624	6.66798	7.69813	2.32437										
18	85.1539	5.42663	5.70834	0.397560										
19	132.806	14.3846	8.56087	1.07226										

TABLE V

The Exit Angular Fluxes $\psi_i(L,-\mu)$ for the 19-Group Problem

i	μ										
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1	2.6676(−2)	2.3995(−2)	2.0336(−2)	1.7801(−2)	1.5889(−2)	1.4378(−2)	1.3145(−2)	1.2117(−2)	1.1244(−2)	1.0492(−2)	9.8369(−3)
2	5.2111(−2)	4.7223(−2)	4.0437(−2)	3.5657(−2)	3.2006(−2)	2.9093(−2)	2.6698(−2)	2.4687(−2)	2.2970(−2)	2.1485(−2)	2.0185(−2)
3	3.7432(−2)	3.4273(−2)	2.9767(−2)	2.6511(−2)	2.3980(−2)	2.1931(−2)	2.0228(−2)	1.8784(−2)	1.7541(−2)	1.6459(−2)	1.5507(−2)
4	3.6376(−2)	3.3630(−2)	2.9604(−2)	2.6618(−2)	2.4253(−2)	2.2312(−2)	2.0679(−2)	1.9282(−2)	1.8071(−2)	1.7008(−2)	1.6067(−2)
5	3.5821(−2)	3.3465(−2)	2.9892(−2)	2.7159(−2)	2.4947(−2)	2.3100(−2)	2.1526(−2)	2.0165(−2)	1.8972(−2)	1.7918(−2)	1.6979(−2)
6	3.6063(−2)	3.4070(−2)	3.0916(−2)	2.8411(−2)	2.6329(−2)	2.4556(−2)	2.3021(−2)	2.1676(−2)	2.0485(−2)	1.9422(−2)	1.8467(−2)
7	2.4796(−2)	2.3650(−2)	2.1756(−2)	2.0194(−2)	1.8861(−2)	1.7704(−2)	1.6688(−2)	1.5785(−2)	1.4978(−2)	1.4251(−2)	1.3592(−2)
8	2.6132(−2)	2.5122(−2)	2.3376(−2)	2.1882(−2)	2.0576(−2)	1.9421(−2)	1.8392(−2)	1.7468(−2)	1.6633(−2)	1.5874(−2)	1.5183(−2)
9	2.8500(−2)	2.7614(−2)	2.5995(−2)	2.4547(−2)	2.3245(−2)	2.2069(−2)	2.1004(−2)	2.0035(−2)	1.9150(−2)	1.8338(−2)	1.7592(−2)
10	3.2597(−2)	3.1830(−2)	3.0318(−2)	2.8889(−2)	2.7560(−2)	2.6330(−2)	2.5194(−2)	2.4144(−2)	2.3173(−2)	2.2273(−2)	2.1437(−2)
11	3.9795(−2)	3.9166(−2)	3.7765(−2)	3.6333(−2)	3.4940(−2)	3.3610(−2)	3.2352(−2)	3.1167(−2)	3.0054(−2)	2.9009(−2)	2.8027(−2)
12	5.1910(−2)	5.1584(−2)	5.0494(−2)	4.9168(−2)	4.7762(−2)	4.6346(−2)	4.4954(−2)	4.3603(−2)	4.2303(−2)	4.1057(−2)	3.9867(−2)
13	1.5782(−2)	1.6779(−2)	1.7971(−2)	1.8609(−2)	1.8935(−2)	1.9064(−2)	1.9060(−2)	1.8967(−2)	1.8810(−2)	1.8608(−2)	1.8376(−2)
14	1.3336(−2)	1.4244(−2)	1.5458(−2)	1.6229(−2)	1.6731(−2)	1.7049(−2)	1.7237(−2)	1.7329(−2)	1.7348(−2)	1.7312(−2)	1.7235(−2)
15	8.8738(−3)	9.4910(−3)	1.0406(−2)	1.1073(−2)	1.1575(−2)	1.1959(−2)	1.2252(−2)	1.2473(−2)	1.2636(−2)	1.2753(−2)	1.2832(−2)
16	2.3930(−3)	2.5592(−3)	2.8282(−3)	3.0459(−3)	3.2278(−3)	3.3821(−3)	3.5139(−3)	3.6267(−3)	3.7236(−3)	3.8067(−3)	3.8779(−3)
17	6.0379(−4)	6.4190(−4)	7.0558(−4)	7.5958(−4)	8.0700(−4)	8.4932(−4)	8.8743(−4)	9.2194(−4)	9.5330(−4)	9.8188(−4)	1.0080(−3)
18	4.0768(−5)	4.3372(−5)	4.7756(−5)	5.1532(−5)	5.4913(−5)	5.7999(−5)	6.0844(−5)	6.3487(−5)	6.5953(−5)	6.8263(−5)	7.0432(−5)
19	6.1142(−6)	6.4726(−6)	7.0782(−6)	7.6032(−6)	8.0771(−6)	8.5134(−6)	8.9197(−6)	9.3012(−6)	9.6613(−6)	1.0003(−5)	1.0327(−5)

TABLE VI

The Exit Angular Fluxes $\psi_i(R,\mu)$ for the 19-Group Problem

	μ										
i	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1	4.6286(-5)	4.9431(-5)	5.7083(-5)	6.7649(-5)	8.9146(-5)	1.7505(-4)	4.8010(-4)	1.2490(-3)	2.7431(-3)	5.1717(-3)	8.6617(-3)
2	9.8793(-5)	1.0537(-4)	1.2091(-4)	1.4138(-4)	1.7025(-4)	2.1330(-4)	2.7680(-4)	3.6438(-4)	4.7562(-4)	6.0693(-4)	7.5306(-4)
3	8.3317(-5)	8.8923(-5)	1.0170(-4)	1.1787(-4)	1.3969(-4)	1.7073(-4)	2.1516(-4)	2.7581(-4)	3.5305(-4)	4.4501(-4)	5.4851(-4)
4	9.3154(-5)	9.9491(-5)	1.1351(-4)	1.3070(-4)	1.5305(-4)	1.8364(-4)	2.2618(-4)	2.8350(-4)	3.5647(-4)	4.4389(-4)	5.4319(-4)
5	1.0571(-4)	1.1298(-4)	1.2863(-4)	1.4722(-4)	1.7059(-4)	2.0138(-4)	2.4288(-4)	2.9791(-4)	3.6772(-4)	4.5175(-4)	5.4802(-4)
6	1.2265(-4)	1.3119(-4)	1.4907(-4)	1.6967(-4)	1.9472(-4)	2.2656(-4)	2.6810(-4)	3.2208(-4)	3.9012(-4)	4.7223(-4)	5.6704(-4)
7	9.4412(-5)	1.0103(-4)	1.1460(-4)	1.2983(-4)	1.4790(-4)	1.7019(-4)	1.9848(-4)	2.3455(-4)	2.7964(-4)	3.3408(-4)	3.9731(-4)
8	1.0859(-4)	1.1624(-4)	1.3167(-4)	1.4865(-4)	1.6836(-4)	1.9212(-4)	2.2158(-4)	2.5850(-4)	3.0422(-4)	3.5935(-4)	4.2364(-4)
9	1.2831(-4)	1.3739(-4)	1.5538(-4)	1.7478(-4)	1.9680(-4)	2.2274(-4)	2.5414(-4)	2.9272(-4)	3.3996(-4)	3.9671(-4)	4.6309(-4)
10	1.5735(-4)	1.6847(-4)	1.9015(-4)	2.1301(-4)	2.3840(-4)	2.6758(-4)	3.0202(-4)	3.4340(-4)	3.9329(-4)	4.5287(-4)	5.2261(-4)
11	2.0326(-4)	2.1750(-4)	2.4478(-4)	2.7287(-4)	3.0332(-4)	3.3740(-4)	3.7652(-4)	4.2228(-4)	4.7635(-4)	5.4016(-4)	6.1467(-4)
12	2.8111(-4)	3.0041(-4)	3.3661(-4)	3.7286(-4)	4.1102(-4)	4.5244(-4)	4.9844(-4)	5.5048(-4)	6.1015(-4)	6.7904(-4)	7.5855(-4)
13	1.6780(-4)	1.8023(-4)	2.0290(-4)	2.2480(-4)	2.4706(-4)	2.7036(-4)	2.9526(-4)	3.2232(-4)	3.5212(-4)	3.8526(-4)	4.2228(-4)
14	1.5666(-4)	1.6825(-4)	1.8907(-4)	2.0873(-4)	2.2820(-4)	2.4801(-4)	2.6857(-4)	2.9022(-4)	3.1330(-4)	3.3815(-4)	3.6512(-4)
15	1.1581(-4)	1.2400(-4)	1.3847(-4)	1.5181(-4)	1.6465(-4)	1.7733(-4)	1.9006(-4)	2.0300(-4)	2.1628(-4)	2.3004(-4)	2.4440(-4)
16	3.6635(-5)	3.9089(-5)	4.3366(-5)	4.7232(-5)	5.0873(-5)	5.4381(-5)	5.7808(-5)	6.1193(-5)	6.4563(-5)	6.7942(-5)	7.1349(-5)
17	9.5846(-6)	1.0174(-5)	1.1189(-5)	1.2093(-5)	1.2930(-5)	1.3722(-5)	1.4483(-5)	1.5219(-5)	1.5937(-5)	1.6641(-5)	1.7335(-5)
18	6.6623(-7)	7.0824(-7)	7.7968(-7)	8.4231(-7)	8.9961(-7)	9.5318(-7)	1.0039(-6)	1.0525(-6)	1.0992(-6)	1.1445(-6)	1.1885(-6)
19	1.0023(-7)	1.0606(-7)	1.1596(-7)	1.2461(-7)	1.3249(-7)	1.3983(-7)	1.4675(-7)	1.5333(-7)	1.5965(-7)	1.6573(-7)	1.7161(-7)

by choosing a different point. However, if $\xi = \nu_i = \nu_j/s_{ij}$ such a simple remedy is not possible. We note that, from the point of view of a matrix formulation of this multigroup model, the phenomenon $\nu_i = \nu_j/s_{ij}$ appears as a degenerate eigenvalue, i.e., a double zero of the dispersion function. On the other hand, if we view the problem as a sequence of one-group problems, this phenomenon is clearly equivalent to seeking a particular solution corresponding to an inhomogeneous source of the form $\exp(-x/\eta_0)$, where η_0 is the eigenvalue for the homogeneous equation. To find the desired particular solution requires, as noted previously, special attention.[9] For our purpose here we find we can use l'Hospital's rule to obtain the following alternatives to Eqs. (10) for $\xi = \nu_j/s_{ij}$:

$$\sigma_i I_{ij}(\xi) = \frac{1}{2} c_j (s_{ij}\xi)^2 N_j^{-1}(s_{ij}\xi) \times \left[\Gamma_j(s_{ij}\xi) + \frac{\sigma_i}{s_{ij}\sigma_{jj}} \sum_{k=1}^{j-1} \sigma_{jk} I'_{ik}(\xi)\right] - \frac{\sigma_i}{\sigma_{jj}} \sum_{k=1}^{j-1} \sigma_{jk} I_{ik}(\xi) \tag{A.1a}$$

and

$$\sigma_i J_{ij}(\xi) = \frac{1}{2} c_j (s_{ij}\xi)^2 N_j^{-1}(s_{ij}\xi) \times \left[\Delta_j(s_{ij}\xi) + \frac{\sigma_i}{s_{ij}\sigma_{jj}} \sum_{k=1}^{j-1} \sigma_{jk} J'_{ik}(\xi)\right] - \frac{\sigma_i}{\sigma_{jj}} \sum_{k=1}^{j-1} \sigma_{jk} J_{ik}(\xi) \ . \tag{A.1b}$$

Here,

$$\Gamma_j(\xi) = I'_j(\xi) + \sum_{\alpha=0}^{N} a_{j,\alpha} F_\alpha(-\xi) + \exp(-\Delta_j/\xi) \times \sum_{\alpha=0}^{N} b_{j,\alpha} \left[F_\alpha(\xi) - \frac{\Delta_j}{\xi^2} A_\alpha(\xi)\right] \tag{A.2a}$$

and

[9] C. E. SIEWERT, *J. Quant. Spectros. Radiat. Transfer*, **15**, 851 (1975).

TABLE VII

A_i^* and B_i^* for the 19-Group Problem

i	Present Work		DTF69	
	A_i^*	B_i^*	A_i^*	B_i^*
1	1.3060(−2)	2.4188(−3)	1.3059(−2)	2.4154(−3)
2	2.6476(−2)	3.9163(−4)	2.6475(−2)	3.9124(−4)
3	2.0013(−2)	2.9446(−4)	2.0013(−2)	2.9415(−4)
4	2.0420(−2)	3.0057(−4)	2.0420(−2)	3.0026(−4)
5	2.1216(−2)	3.1339(−4)	2.1216(−2)	3.1306(−4)
6	2.2650(−2)	3.3596(−4)	2.2650(−2)	3.3560(−4)
7	1.6399(−2)	2.4287(−4)	1.6398(−2)	2.4260(−4)
8	1.8059(−2)	2.6601(−4)	1.8059(−2)	2.6572(−4)
9	2.0613(−2)	2.9930(−4)	2.0612(−2)	2.9896(−4)
10	2.4717(−2)	3.4873(−4)	2.4717(−2)	3.4834(−4)
11	3.1745(−2)	4.2569(−4)	3.1744(−2)	4.2519(−4)
12	4.4141(−2)	5.5031(−4)	4.4140(−2)	5.4965(−4)
13	1.8729(−2)	3.1938(−4)	1.8729(−2)	3.1901(−4)
14	1.7023(−2)	2.8604(−4)	1.7024(−2)	2.8570(−4)
15	1.2201(−2)	1.9924(−4)	1.2201(−2)	1.9901(−4)
16	3.5378(−3)	5.9937(−5)	3.5379(−3)	5.9867(−5)
17	9.0059(−4)	1.4905(−5)	9.0062(−4)	1.4888(−5)
18	6.2046(−5)	1.0301(−6)	6.2049(−5)	1.0289(−6)
19	9.1048(−6)	1.5020(−7)	9.1123(−6)	1.5013(−7)

$$\Delta_j(\xi) = J_j'(\xi) + \sum_{\alpha=0}^{N} b_{j,\alpha} F_\alpha(-\xi) + \exp(-\Delta_j/\xi) \times \sum_{\alpha=0}^{N} a_{j,\alpha} \left[F_\alpha(\xi) - \frac{\Delta_j}{\xi^2} A_\alpha(\xi) \right] . \tag{A.2b}$$

We have found that the functions $F_\alpha(\xi) = -A'_\alpha(\xi)$ appearing in Eqs. (A.2a) and (A.2b) can be expressed as

$$F_\alpha(\xi) = \frac{1}{2\xi(\xi+1)} \{-[2(\xi+1) + \alpha(2\xi+1)]A_\alpha(\xi) - \alpha A_{\alpha-1}(\xi) + 2\delta_{\alpha,0} + \delta_{\alpha,1}\} , \tag{A.3}$$

and thus no additional recursive relations are required to deduce $F_\alpha(\xi)$. We note that Eqs. (A.1a) and (A.1b) require, for $j > 1$, the derivatives of $I_{ik}(\xi)$ and $J_{ik}(\xi)$ for all $k < j$. These derivatives were not required for the degenerate case found in the 16-group problem discussed in Sec. II since there j was equal to one.

Clearly there exist possibilities for higher order degeneracies; however, from a practical point of view, the possibility of even a first-order degeneracy is slight. Nevertheless, a given data set should be reviewed with regard to this matter before an F_N calculation is initiated.

APPENDIX B

RECURSIVE RELATIONS

The functions $A_\alpha(\xi)$ and $B_{i,\alpha}(\xi)$ appearing in Eqs. (4a) and (4b) are defined by

$$A_\alpha(\xi) = \frac{2}{c_i \xi} \int_0^1 \mu P_\alpha(2\mu - 1)\phi_i(-\xi, \mu) d\mu \tag{B.1a}$$

and

$$B_{i,\alpha}(\xi) = \frac{2}{\xi} \int_0^1 \mu P_\alpha(2\mu - 1)\phi_i(\xi, \mu) d\mu . \tag{B.1b}$$

We note from Sec. II that the functions $A_\alpha(\xi)$ are required for all $\xi \notin [-1, 0)$ and $B_{i,\alpha}(\xi)$ only for $\xi \in P_i$. Equations (5) and (6) clearly are recursive relations that are easier to use, from a computational standpoint, than the definitions given by Eqs. (B.1a) and (B.1b). However, some care must be taken to avoid a loss of accuracy when using the recursive Eqs. (5) and (6). Here we note the strategy we use to compute (working in double precision with an IBM 370/165 machine) the required functions accurate to at least 13 significant figures for α up to 40. For the functions $A_\alpha(\xi)$ we have found that forward recursion is stable only for $\xi \in (-1, 0]$ and that backward recursion is stable for all $\xi \notin [-1, 0]$. For the functions $B_{i,\alpha}(\xi)$, forward recursion is stable for $\xi \in [0, 1]$, and thus backward recursion must be used when $\xi = \nu_i$. In practice the use of backward recursion, in the manner of Miller,[10] can be very time-consuming for ξ close to the transition points, i.e., points that define the regions of forward and backward stability. For this reason we do not always use the defined regions of stability, e.g., in computing $A_\alpha(\xi)$ we have actually used forward recursion for $\xi \in [0, 0.001]$ without losing too many significant figures.

The polynomials $G_\alpha(\xi)$ introduced in Eq. (13) and required for $\xi \in [0, 1]$ are defined by

$$G_\alpha(\xi) = \int_0^1 \mu [P_\alpha(2\mu - 1) - P_\alpha(2\xi - 1)] \frac{d\mu}{\mu - \xi} . \tag{B.2}$$

These polynomials satisfy the same recursive relation as the Legendre polynomials and thus we use forward recursion to obtain accurate results.

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[10]J. C. P. MILLER, in *British Association for the Advancement of Science, Bessel Functions, Part II. Functions of Integer Order, Mathematical Tables*, Vol. X, Cambridge University Press, Cambridge (1952).

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