J. Quant. Spectrosc. Radiat. Transfer Vol. 26, No. 3, pp. 223–227, 1981
Printed in Great Britain.

0022–4073/81/090223–05$02.00/0
Pergamon Press Ltd.

PARTICULAR SOLUTIONS OF THE EQUATION OF TRANSFER

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(*Received* 5 *December* 1980)

Abstract—Particular solutions corresponding to various forms for the inhomogeneous source term are established for the equation of transfer with Lth order anisotropic scattering.

1. INTRODUCTION

Some years ago particular solutions relevant to several forms for the inhomogeneous source term were reported[1] for a model of the equation of transfer that was limited to linear anisotropic scattering. Here we consider the general case of Lth order anisotropic scattering, and thus we seek particular solutions of

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu)+I(\tau,\mu)=\frac{1}{2}\omega\sum_{l=0}^{L}(2l+1)f_lP_l(\mu)\int_{-1}^{1}P_l(\mu')I(\tau,\mu')\,d\mu'+S(\tau) \tag{1}$$

for $1-\omega f_l\neq 0$ and various prescribed source terms $S(\tau)$.

2. POLYNOMIAL SOURCE

We consider first source terms of the form

$$S_\alpha(\tau)=\tau^\alpha \tag{2}$$

and note that Devaux and Siewert[2] have expressed the desired solution, for $\alpha=0$, 1, 2, and 3, in the form

$$I_\alpha(\tau,\mu)=\left(\frac{1}{1-\omega}\right)\sum_{l=0}^{\alpha}(2l+1)T_l^\alpha(\tau)P_l(\mu), \tag{3}$$

where the polynomials $T_l^\alpha(\tau)$, of degree $\alpha-l$, are listed in Table 1. We use the definition

$$h_l=(2l+1)(1-\omega f_l). \tag{4}$$

Table 1. The polynomials $T_l^\alpha(\tau)$.

l	$\alpha=0$	$\alpha=1$	$\alpha=2$	$\alpha=3$
0	1	τ	$\frac{2}{h_0h_1}+\tau^2$	$\frac{6}{h_0h_1}\tau+\tau^3$
1		$-\frac{1}{h_1}$	$-\frac{2}{h_1}\tau$	$-6\left(\frac{4h_0+h_2}{h_0h_1^2h_2}\right)-\frac{3}{h_1}\tau^2$
2			$\frac{4}{h_1h_2}$	$\frac{12}{h_1h_2}\tau$
3				$\frac{-36}{h_1h_2h_3}$

Since

$$S_{\alpha+1}(\tau) = (\alpha+1)\int_0^\tau S_\alpha(t)\,\mathrm{d}t, \tag{5}$$

we can readily deduce that Eq. (3) is a general form and further that

$$T^{\alpha+1}_{\alpha+1}(\tau) = K^{\alpha+1}_{\alpha+1} \tag{6}$$

and, for $l = 0, 1, 2, \ldots, \alpha$, that

$$T^{\alpha+1}_l(\tau) = K^{\alpha+1}_l + (\alpha+1)\int_0^\tau T^\alpha_l(t)\,\mathrm{d}t, \tag{7}$$

where the constants $K^{\alpha+1}_l$, $l = 0, 1, 2, \ldots, \alpha+1$, are to be determined. It is apparent that $K^{\alpha+1}_\alpha$, $K^{\alpha+1}_{\alpha-3}$, $K^{\alpha+1}_{\alpha-5}, \ldots$ are all zero. Thus if we assume that $I_{\alpha-1}(\tau, \mu)$ has been obtained, then $I_\alpha(\tau, \mu)$ will be available from Eq. (3) and

$$T^\alpha_l(\tau) = K^\alpha_l + \alpha(1-\delta_{\alpha,l})\int_0^\tau T^{\alpha-1}_l(t)\,\mathrm{d}t \tag{8}$$

once we establish the $[1+\alpha/2]$ constants $K_\alpha{}^\alpha, K^\alpha_{\alpha-2}, K^\alpha_{\alpha-4}\ldots$. If we now substitute Eq. (3) into Eq. (1), with $S(\tau) = S_\alpha(\tau)$, set $\tau = 0$ in the resulting equation and use Eq. (8), we find the constants required in Eq. (8) to be

$$K^\alpha_\alpha = -\frac{\alpha^2}{h_\alpha}K^{\alpha-1}_{\alpha-1} \tag{9a}$$

and, for $j = 1, 2, 3\ldots\left[\dfrac{\alpha}{2}\right]$,

$$K^\alpha_{\alpha-2j} = -\frac{\alpha}{h_{\alpha-2j}}[(\alpha-2j)K^{\alpha-1}_{\alpha-1-2j} + (\alpha-2j+1)K^{\alpha-1}_{\alpha+1-2j}]. \tag{9b}$$

To conclude this section, we note that

$$I(\tau, \mu) = \sum_{\alpha=0}^N S_\alpha I_\alpha(\tau, \mu) \tag{10}$$

is the desired particular solution corresponding to

$$S(\tau) = \sum_{\alpha=0}^N S_\alpha \tau^\alpha. \tag{11}$$

3. EXPONENTIAL SOURCE

We consider now a source term of the form

$$S(\tau:\nu) = \mathrm{e}^{-\tau/\nu},\ \nu \in (-1, 1), \tag{12}$$

and, upon substituting

$$I(\tau, \mu:\nu) = F(\nu, \mu)\,\mathrm{e}^{-\tau/\nu} \tag{13}$$

into Eq. (1), with $S(\tau) = S(\tau:\nu)$, we find

$$(\nu-\mu)F(\nu, \mu) = \nu\left[1 + \frac{\omega}{2}\sum_{l=0}^L \beta_l P_l(\mu)F_l(\nu)\right], \tag{14}$$

where $\beta_l = (2l+1)f_l$ and

$$F_l(\nu) = \int_{-1}^{1} P_l(\mu) F(\nu, \mu)\, d\mu \tag{15}$$

can, from Eq. (14), readily be seen to satisfy, for $l \geq 0$,

$$\nu h_l F_l(\nu) = (l+1) F_{l+1}(\nu) + l F_{l-1}(\nu) + 2\nu \delta_{0,l}. \tag{16}$$

We can therefore solve Eq. (14) to find

$$F(\nu, \mu) = \nu P v \left(\frac{1}{\nu - \mu}\right) \left[1 + \frac{\omega}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) F_l(\nu)\right] + \omega(\nu) \delta(\nu - \mu), \tag{17}$$

where

$$\omega(\nu) = F_0(\nu) + \nu P \int_{-1}^{1} \left[1 + \frac{\omega}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) F_l(\nu)\right] \frac{d\mu}{\mu - \nu}. \tag{18}$$

If we let

$$R_l(\nu, \mu) = \left(\frac{1}{\nu - \mu}\right) [P_l(\nu) - P_l(\mu)], \tag{19}$$

we may write Eq. (17) as

$$F(\nu, \mu) = -\frac{\omega \nu}{2} \sum_{l=1}^{L} \beta_l F_l(\nu) R_l(\nu, \mu) + \omega(\nu) \delta(\nu - \mu), \tag{20}$$

where $F_0(\nu)$ is to be fixed so that

$$1 + \frac{\omega}{2} \sum_{l=0}^{L} \beta_l P_l(\nu) F_l(\nu) = 0. \tag{21}$$

We let

$$F_l(\nu) = -2\pi_l(\nu) - \frac{2}{\omega} M(\nu) g_l(\nu) \tag{22}$$

with $\pi_0(\nu) = 0$. Here the polynomials satisfy, for $l \geq 0$,

$$\nu h_l g_l(\nu) = (l+1) g_{l+1}(\nu) + l g_{l-1}(\nu) \tag{23}$$

with $g_0(\nu) = 1$. On substituting Eq. (22) into Eq. (21), we find

$$M(\nu) = \frac{1}{R(\nu)} \left[1 - \omega \sum_{l=1}^{L} \beta_l P_l(\nu) \pi_l(\nu)\right], \tag{24}$$

where

$$R(\nu) = \sum_{l=0}^{L} \beta_l P_l(\nu) g_l(\nu). \tag{25}$$

Entering Eq. (22) into Eq. (16), we find, for $l \geq 1$,

$$\nu h_l \pi_l(\nu) = (l+1) \pi_{l+1}(\nu) + l \pi_{l-1}(\nu) \tag{26}$$

and $\pi_1(\nu) = \nu$. Thus the desired particular solution can be expressed as

$$I(\tau, \mu : \nu) = F(\nu, \mu)\, \mathrm{e}^{-\tau/\nu} \tag{27}$$

with

$$F(\nu, \mu) = \nu \sum_{l=1}^{L} \beta_l[\omega\pi_l(\nu) + M(\nu)g_l(\nu)]R_l(\nu, \mu) + \omega(\nu)\delta(\nu - \mu), \tag{28}$$

where

$$\omega(\nu) = -\frac{2}{\omega}M(\nu) - 2\nu \sum_{l=1}^{L} \beta_l[\omega\pi_l(\nu) + M(\nu)g_l(\nu)]\Gamma_l(\nu). \tag{29}$$

Here the polynomials $\Gamma_l(\nu)$ satisfy, for $l \geq 1$,

$$(2l + 1)\nu\Gamma_l(\nu) = (l + 1)\Gamma_{l+1}(\nu) + l\Gamma_{l-1}(\nu) \tag{30}$$

with $\Gamma_1(\nu) = 1$.

Having found the desired solution corresponding to the exponential source given by Eq. (12), we now consider

$$S(\tau : \xi) = \mathrm{e}^{-\tau/\xi},\ \xi \notin [-1, 1]. \tag{31}$$

Here

$$\Lambda(\xi) \neq 0, \tag{32}$$

where the dispersion function, the zeroes of which are the discrete eigenvalues relevant to the homogeneous version of Eq. (1), is given by

$$\Lambda(z) = 1 + z \int_{-1}^{1} \Psi(\mu) \frac{\mathrm{d}\mu}{\mu - z} \tag{33}$$

with

$$\Psi(\mu) = \frac{\omega}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) g_l(\mu). \tag{34}$$

On writing

$$I(\tau, \mu : \xi) = F(\xi, \mu)\, \mathrm{e}^{-\tau/\xi}, \tag{35}$$

we find that we can express $F(\xi, \mu)$ as

$$F(\xi, \mu) = \frac{\xi\Delta(\xi)}{\Lambda(\xi)} \left[\frac{1}{\xi - \mu} - \frac{1}{R(\xi)} \sum_{l=1}^{L} \beta_l g_l(\xi) R_l(\xi, \mu)\right] + \xi \sum_{l=1}^{L} \beta_l[\omega\pi_l(\xi) + M(\xi)g_l(\xi)]R_l(\xi, \mu), \tag{36}$$

where

$$\Delta(\xi) = \left[1 + \omega\xi \sum_{l=1}^{L} \beta_l g_l(\xi)\Gamma_l(\xi)\right]\left[1 - \omega \sum_{l=1}^{L} \beta_l P_l(\xi)\pi_l(\xi)\right] + \omega^2\xi \sum_{l=0}^{L} \beta_l P_l(\xi) g_l(\xi) \sum_{l=1}^{L} \beta_l \pi_l(\xi)\Gamma_l(\xi). \tag{37}$$

It is apparent that Eq. (36) is not valid if $\Lambda(\xi) = 0$ and thus, for that case, a modification as discussed previously[3] for the case $L = 1$, would be required. In concluding this section, we note also that Eq. (28) is not valid if $R(\nu) = 0$; nor is Eq. (36) valid if $R(\xi) = 0$.

4. ADDITONAL RESULTS

It is clear that we can take linear combinations of the solutions given by Eqs. (13) and (35) to establish particular solutions corresponding to source terms of the form

$$S_1(\tau, \zeta) = \begin{Bmatrix} \sinh \tau/\zeta \\ \cosh \tau/\zeta \end{Bmatrix} \tag{38}$$

for all real ζ such that $\Lambda(\zeta) \neq 0$ and $R(\zeta) \neq 0$ and

$$S_2(\tau, \zeta) = \begin{Bmatrix} \sin \tau/\zeta \\ \cos \tau/\zeta \end{Bmatrix}, \tag{39}$$

for all real ζ such that $\Lambda(i\zeta) \neq 0$ and $R(i\zeta) \neq 0$. It is also apparent that a particular solution corresponding to a source term of the form

$$S(\tau) = \int_{-1}^{1} A(\nu)\, e^{-\tau/\nu}\, d\nu \tag{40}$$

is readily available, when $R(\nu) \neq 0$, from Eqs. (7) and (28); we find

$$I_p(\tau, \mu) = A(\mu)\omega(\mu)\, e^{-\tau/\mu} + \sum_{l=1}^{L} \beta_l \int_{-1}^{1} \nu A(\nu)[\omega\pi_l(\nu) + M(\nu)g_l(\nu)]R_l(\nu, \mu)\, e^{-\tau/\nu}\, d\nu. \tag{41}$$

In conclusion we note that the infinite-medium Green's function,[4] or perhaps the method discussed by Mendelson and Congdon,[5] could be used to generate particular solutions appropriate to source terms not considered here.

Acknowledgement—This work was supported in part by the National Science Foundation through grants INT 8000517 and CPE 8016775.

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