Journal of Applied Mathematics and Physics (ZAMP)
Vol. 32, 1981

0044-2275/81/004/421/13/$ 4.10/0

Strong evaporation into a half space

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I. Introduction

The fact that the kinetic theory of vapors close to interphase boundaries has attracted considerable attention from theorists in recent years reflects the importance of this area of physics to basic engineering problems (in addition to the intrinsic mathematical interest). Most early work on this subject focused on near-equilibrium processes [1 – 5] which can be approached *via* linearized analysis. More recently the problem of arbitrarily strong evaporation into a low-pressure region has been of concern. This is basically a non-linear problem as reflected in the work of Kogan and Makashev [6], Murakami and Oshima [7] and Yen and Akai [8], all of whom computed numerical solutions of the Boltzmann equation or appropriate non-linear model equations. Ytrehus [9] has recently studied the problem by solving the four moment equations of Lees and Liu [10]. A common prediction of the non-linear studies is the existence of a limiting pressure ratio or downstream speed ratio beyond which no steady-state solution exists.

It has, however, been pointed out by Cercignani [11] that the strong evaporation problem can be approached through linear analysis simply by linearizing the distribution function about a downstream Maxwell distribution containing a drift speed v_∞. In particular, Cercignani [11] conjectured that the limiting speed ratio discovered in the non-linear analysis would be manifested in the linear case through a failure of the singular eigenmodes [12, 13] of the linearized equation to possess the usual completeness properties. This conjecture was subsequently investigated by Arthur and Cercignani [14] who used the resolvent integration method [15] to study the linearized BGK equation [16] with one degree of freedom [17].

In the present work the strong evaporation problem is solved, for the linear case, by the method of elementary solutions introduced for neutron-transport theory by Case [12] and subsequently used by Cercignani [13] for problems in the kinetic theory of gases.

We consider a liquid evaporating at the plane $x = 0$ into a vacuum which occupies the region $x > 0$. The state of the gas is described by the BGK model, with one degree of freedom, which we write as

$$\xi \frac{\partial}{\partial x} f(x, \xi) = \nu[\Phi(x, \xi) - f(x, \xi)], \tag{1}$$

where $f(x, \xi)$ is the distribution function, ξ is the molecular velocity in the x direction, ν is an appropriate collision frequency and $\Phi(x, \xi)$ is a local Maxwell distribution

$$\Phi(x, \xi) = \frac{\varrho(x)}{\sqrt{2\pi RT(x)}} \exp\left\{-\frac{[\xi - v(x)]^2}{2RT(x)}\right\}. \tag{2}$$

We note that Weitzner [17] has studied sound-wave propagation on the basis of this one-dimensional model. Here the density, mass velocity and temperature are defined by

$$\varrho(x) = \int_{-\infty}^{\infty} f(x, \xi)\, \mathrm{d}\xi \tag{3a}$$

$$\varrho(x)\, v(x) = \int_{-\infty}^{\infty} \xi f(x, \xi)\, \mathrm{d}\xi \tag{3b}$$

and

$$\varrho(x)\, RT(x) = \int_{-\infty}^{\infty} [\xi - v(x)]^2 f(x, \xi)\, \mathrm{d}\xi. \tag{3c}$$

We assume that far downstream the gas relaxes to an equilibrium distribution characterized by steady drift velocity v_∞, density ϱ_∞ and temperature T_∞; i.e.

$$\lim_{x\to\infty} \Phi(x, \xi) = f_\infty(\xi) = \frac{\varrho_\infty}{\sqrt{2\pi RT_\infty}} \exp\left\{-\frac{(\xi - v_\infty)^2}{2RT_\infty}\right\}. \tag{4}$$

We now follow Arthur and Cercignani [14] and linearize $f(x, \xi)$ and $\Phi(x, \xi)$ about $f_\infty(\xi)$. Introducing the shifted variable

$$c = \xi - v_\infty, \tag{5}$$

we write, with an obvious change in notation concerning the dependent variable,

$$f(x, c) = f_\infty(c)[1 + h(x, c)] \tag{6a}$$

where

$$f_\infty(c) = \frac{\varrho_\infty}{\sqrt{2\pi RT_\infty}} \exp\left\{-\frac{c^2}{2RT_\infty}\right\}. \tag{6b}$$

After substituting Eq. (6a) into Eq. (1) and linearizing $\Phi(x, \xi)$ about $f_\infty(\xi)$, we find

$$(\bar{c} + u)\frac{\partial}{\partial \bar{x}} h(\bar{x}, \bar{c}) + h(\bar{x}, \bar{c}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} [1 + 2(\bar{c}^2 - \tfrac{1}{2})(\mu^2 - \tfrac{1}{2}) + 2\bar{c}\mu]\, \mathrm{e}^{-\mu^2} h(\bar{x}, \mu)\, \mathrm{d}\mu \tag{7}$$

where

$$\bar{x} = \nu x (2RT_\infty)^{-1/2}, \tag{8a}$$

$$\bar{c} = c (2RT_\infty)^{-1/2}, \tag{8b}$$

and

$$u = v_\infty (2RT_\infty)^{-1/2}. \tag{8c}$$

Note that u is the normalized downstream drift velocity.

At $\bar{x} = 0$ we equate the distribution function to some specified distribution

$$f(0, \bar{c}) = f_0(\bar{c}), \qquad \bar{c} > -u, \tag{9a}$$

which means

$$h(0, \bar{c}) = \frac{f_0(\bar{c}) - f_\infty(\bar{c})}{f_\infty(\bar{c})}, \qquad \bar{c} > -u. \tag{9b}$$

As $\bar{x} \to \infty$, $f(\bar{x}, \bar{c})$ relaxes to $f_\infty(\bar{c})$ and thus

$$\lim_{\bar{x} \to \infty} h(\bar{x}, \bar{c}) = 0. \tag{10}$$

Equations (7), (9b) and (10) constitute the mathematical formulation of the problem we consider.

II. Elementary solutions

We now suppress the overbar notation and seek to establish, for $u > 0$, the elementary solutions of the linearized BGK equation

$$(c+u)\frac{\partial}{\partial x} h(x, c) + h(x, c) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[1 + 2\left(c^2 - \frac{1}{2}\right)\left(\mu^2 - \frac{1}{2}\right) + 2c\mu\right] e^{-\mu^2} h(x, \mu)\, d\mu. \tag{11}$$

Thus if we substitute

$$h(x, c : \eta) = \varphi(\eta, c)\, e^{-x/(\eta+u)} \tag{12}$$

into Eq. (11) we find

$$(\eta - c)\, \varphi(\eta, c) = \frac{1}{\sqrt{\pi}} (\eta + u)\, q(c)\, e^{-\eta^2} \tag{13}$$

where

$$q(c) = 1 - 2uc + 2\left(u^2 - \frac{1}{2}\right)\left(c^2 - \frac{1}{2}\right). \tag{14}$$

Here we have normalized $\varphi(\eta, c)$ such that

$$\int_{-\infty}^{\infty} \varphi(\eta, c)\, \mathrm{e}^{-c^2}\, \mathrm{d}c = \mathrm{e}^{-\eta^2} \tag{15a}$$

with the consequence that

$$\int_{-\infty}^{\infty} \varphi(\eta, c)\, c\, \mathrm{e}^{-c^2}\, \mathrm{d}c = -u\, \mathrm{e}^{-\eta^2} \tag{15b}$$

and

$$\int_{-\infty}^{\infty} \varphi(\eta, c)\, (c^2 - \tfrac{1}{2})\, \mathrm{e}^{-c^2}\, \mathrm{d}c = (u^2 - \tfrac{1}{2})\, \mathrm{e}^{-\eta^2}. \tag{15c}$$

For $\eta \in (-\infty, \infty)$ we express the solution to Eq. (13), subject to the normalization of Eq. (15a), as

$$\varphi(\eta, c) = \frac{1}{\sqrt{\pi}} (\eta + u)\, q(c)\, P v\left(\frac{1}{\eta - c}\right) \mathrm{e}^{-\eta^2} + \lambda(\eta)\, \delta(\eta - c) \tag{16}$$

where Pv denotes the principal-value operator and

$$\lambda(\eta) = 1 + (\eta + u)\, P \int_{-\infty}^{\infty} \psi(\mu) \frac{\mathrm{d}\mu}{\mu - \eta} \tag{17}$$

with

$$\psi(\mu) = \frac{1}{\sqrt{\pi}}\, q(\mu)\, \mathrm{e}^{-\mu^2}. \tag{18}$$

For $\eta = \zeta \notin (-\infty, \infty)$ Eqs. (13) and (15a) can be solved to yield

$$\varphi(\zeta, c) = \frac{1}{\sqrt{\pi}} (\zeta + u)\, q(c) \left(\frac{1}{\zeta - c}\right) \mathrm{e}^{-\zeta^2} \tag{19}$$

where ζ must be a zero of

$$\Lambda(z) = 1 + (z + u) \int_{-\infty}^{\infty} \psi(\mu) \frac{\mathrm{d}\mu}{\mu - z}. \tag{20}$$

We can use the argument principle [18] to show that $\Lambda(z)$ has no zeros in the finite plane. However, since, as $|z| \to \infty$,

$$\Lambda(z) \to -\frac{1}{z^3}\, u \left(u^2 - \frac{3}{2}\right) - \frac{1}{z^4} \frac{3}{2} \left(u^2 - \frac{1}{2}\right) + \dots \tag{21}$$

we find that Eq. (11) admits the solutions

$$h_\alpha(x, c) = c^\alpha, \qquad \alpha = 0, 1 \text{ and } 2, \tag{22}$$

plus, if $u^2 = 3/2$,

$$h_3(x, c) = (x - u - c)\, q(c). \tag{23}$$

Having found the desired elementary solutions, we now express the general solution to Eq. (11) as

$$h(x, c) = \sum_{\alpha=0}^{\varkappa} A_\alpha h_\alpha(x, c) + \int_{-\infty}^{\infty} A(\eta)\, \varphi(\eta, c)\, \mathrm{e}^{-x/(\eta+u)}\, \mathrm{d}\eta\,, \tag{24}$$

where $\varkappa = 2$ for $u^2 \neq 3/2$ and $\varkappa = 3$ if $u^2 = 3/2$. In addition, the constants A_α, $\alpha = 0, 1, \ldots, \varkappa$, and $A(\eta)$ are expansion coefficients to be determined from the boundary conditions imposed on $h(x, c)$.

III. Boundary-value problem

As discussed in Section II, we now seek a solution of Eq. (11) subject to the boundary conditions

$$\lim_{x \to \infty} h(x, c) = 0 \tag{25a}$$

and, for $c > -u$,

$$h(0, c) = \mathscr{F}(c) = \frac{f_0(c)}{f_\infty(c)} - 1\,, \tag{25b}$$

where $f_0(c)$ is considered given and

$$f_\infty(c) = \frac{\varrho_\infty}{\sqrt{2\pi RT_\infty}}\, \mathrm{e}^{-c^2}\,. \tag{26}$$

In order that $h(x, c)$ vanish at infinity, as prescribed by Eq. (25a), we take $A_\alpha = 0$, $\alpha = 0, 1, \ldots, \varkappa$, and $A(\eta) = 0$, $\eta < -u$, in Eq. (24) and therefore write, for all $x \geqq 0$ and $-\infty < c < \infty$,

$$h(x, c) = \int_{-u}^{\infty} A(\eta)\, \varphi(\eta, c)\, \mathrm{e}^{-x/(\eta+u)}\, \mathrm{d}\eta\,. \tag{27}$$

To satisfy Eq. (25b) we clearly must determine $A(\eta)$ such that, for $c > -u$,

$$\mathscr{F}(c) = \int_{-u}^{\infty} A(\eta)\, \varphi(\eta, c)\, \mathrm{d}\eta\,. \tag{28}$$

We consider for the moment that $\mathscr{F}(c)$ is known and we write Eq. (28) as

$$\mathscr{F}(c) = A(c)\, \lambda(c) + \frac{1}{\sqrt{\pi}}\, q(c)\, P \int_{-u}^{\infty} A(\eta)\,(\eta + u)\, \mathrm{e}^{-\eta^2} \frac{\mathrm{d}\eta}{\eta - c}\,. \tag{29}$$

Thus, following Muskhelishvili [19], we introduce the sectionally analytic function

$$N(z) = \frac{1}{2\pi i} \int_{-u}^{\infty} A(\eta)\,(\eta + u)\, \mathrm{e}^{-\eta^2} \frac{\mathrm{d}\eta}{\eta - z}\,, \tag{30}$$

with limiting values, for $c \in (-u, \infty)$,

$$N^{\pm}(c) = \frac{1}{2\pi i} P \int_{-u}^{\infty} A(\eta)(\eta + u) \, e^{-\eta^2} \frac{d\eta}{\eta - c} \pm \frac{1}{2} A(c)(c+u) \, e^{-c^2}, \tag{31}$$

and use

$$\Lambda^{\pm}(c) = \lambda(c) \pm \pi i (c+u) \, \psi(c) \tag{32}$$

to rewrite Eq. (29) in the form of a Riemann-Hilbert problem, i.e.

$$N^{+}(c) = G^{-1}(c) \, N^{-}(c) + \frac{(c+u)}{\Lambda^{+}(c)} e^{-c^2} \mathscr{F}(c), \tag{33}$$

where

$$G(c) = \frac{\Lambda^{+}(c)}{\Lambda^{-}(c)}. \tag{34}$$

If we now let

$$\Theta(\tau) = \arg \Lambda^{+}(\tau), \tag{35}$$

with $\Theta(-u) = 0$, we can write the solution of Eq. (33) as [19]

$$N(z) = \frac{1}{2\pi i X(z)} \int_{-u}^{\infty} K(c) \mathscr{F}(c) \frac{dc}{c - z}, \tag{36}$$

where

$$K(c) = (c+u) \frac{X^{+}(c)}{\Lambda^{+}(c)} e^{-c^2} \tag{37}$$

and, for $u^2 < 3/2$,

$$X(z) = \frac{1}{(z+u)^2} \exp\left\{\frac{1}{\pi} \int_{-u}^{\infty} [\Theta(\tau) - 2\pi] \frac{d\tau}{\tau - z}\right\} \tag{38a}$$

and, for $u^2 \geqq 3/2$,

$$X(z) = \frac{1}{(z+u)^3} \exp\left\{\frac{1}{\pi} \int_{-u}^{\infty} [\Theta(\tau) - 3\pi] \frac{d\tau}{\tau - z}\right\}. \tag{38b}$$

From Eq. (30) we see that $N(z)$ vanishes as $1/z$ as $|z| \to \infty$, and thus in order for Eq. (36) to yield the correct $N(z)$ we impose on $\mathscr{F}(c)$ the conditions

$$\int_{-u}^{\infty} K(c) \mathscr{F}(c) \, c^{\alpha} \, dc = 0, \tag{39}$$

where $\alpha = 0$ and 1 for $u^2 < 3/2$ and $\alpha = 0, 1$ and 2 for $u^2 \geqq 3/2$. We can use Eq. (25b) to rewrite Eq. (39) as

$$\int_{-u}^{\infty} K(c) \left(\frac{f_0(c)}{f_{\infty}(c)}\right) c^{\alpha} \, dc = \int_{-u}^{\infty} K(c) \, c^{\alpha} \, dc, \tag{40}$$

where again $\alpha = 0$ and 1 for $u^2 < 3/2$ and $\alpha = 0, 1$ and 2 for $u^2 \geqq 3/2$.

IV. Factorization of $\Lambda(z)$

We now wish to factor the dispersion function $\Lambda(z)$ in the manner

$$\Lambda(z) = k\, X(z)\, Y(z)\,, \tag{41}$$

where k is a constant and the sectionally analytic function $Y(z)$ is analytic in the plane cut from $-\infty$ to $-u$ along the real axis. We can readily deduce from Eq. (41) that the limiting values of $Y(z)$ satisfy, for $\tau \in (-\infty, -u)$,

$$Y^+(\tau) = G(\tau)\, Y^-(\tau) \tag{42}$$

and thus we can write the canonical solution [19], for $u^2 \leqq 3/2$, as

$$Y(z) = \frac{1}{z+u} \exp\left\{\frac{1}{\pi} \int_{-\infty}^{-u} [\Theta(\tau) + \pi] \frac{d\tau}{\tau - z}\right\} \tag{43a}$$

and, for $u^2 > 3/2$, as

$$Y(z) = \exp\left\{\frac{1}{\pi} \int_{-\infty}^{-u} \Theta(\tau) \frac{d\tau}{\tau - z}\right\}. \tag{43b}$$

We can now let $|z| \to \infty$ in Eq. (41) and use Eqs. (21), (38) and (43) to establish that

$$k = -\tfrac{3}{2}, \qquad u^2 = 3/2\,, \tag{44a}$$

and

$$k = -u(u^2 - \tfrac{3}{2})\,, \qquad u^2 \neq 3/2\,. \tag{44b}$$

Since $X(z)$ vanishes at infinity and is analytic in the plane cut along the real axis from $-u$ to ∞ we can use Cauchy's integral formula to write

$$X(z) = \frac{1}{2\pi i} \int_{-u}^{\infty} [X^+(c) - X^-(c)] \frac{dc}{c - z} \tag{45}$$

or, after we use Eqs. (32) and (41),

$$X(z) = \frac{1}{k} \int_{-u}^{\infty} \frac{(c+u)\, \psi(c)}{Y(c)} \frac{dc}{c - z}\,. \tag{46}$$

In a similar manner we can deduce that

$$Y(z) = Y(\infty) + \frac{1}{k} \int_{-\infty}^{-u} \frac{(\tau+u)\, \psi(\tau)}{X(\tau)} \frac{d\tau}{\tau - z} \tag{47}$$

where $Y(\infty) = 0$ if $u^2 \leqq 3/2$ and $Y(\infty) = 1$ for $u^2 > 3/2$. From Eqs. (37) and (41) we see that, for $c \in (-u, \infty)$,

$$K(c) = \frac{(c+u)}{k\, Y(c)} e^{-c^2} \tag{48}$$

and thus if we let

$$\Phi_1(c) = \frac{1}{\sqrt{k}\ Y(c)}, \quad c \in (-u, \infty), \tag{49a}$$

and

$$\Phi_2(\tau) = \frac{1}{\sqrt{k}\ X(\tau)}, \quad \tau \in (-\infty, -u), \tag{49b}$$

then we can write Eqs. (46) and (47) as

$$\frac{1}{\Phi_2(\tau)} = \int_{-u}^{\infty} \psi(c)\, \Phi_1(c)\, \mathrm{d}c + (\tau + u) \int_{-u}^{\infty} \psi(c)\, \Phi_1(c) \frac{\mathrm{d}c}{c - \tau}, \tag{50a}$$

for $\tau \in (-\infty, -u)$, and

$$\frac{1}{\Phi_1(c)} = \sqrt{k}\ Y(\infty) + \int_{-\infty}^{-u} \psi(\tau)\, \Phi_2(\tau)\, \mathrm{d}\tau + (c + u) \int_{-\infty}^{-u} \psi(\tau)\, \Phi_2(\tau) \frac{\mathrm{d}\tau}{\tau - c}, \tag{50b}$$

for $c \in (-u, \infty)$. It is clear that $\Phi_1(-u)\, \Phi_2(-u) = 1$ and that

$$\frac{1}{\Phi_2(-u)} = \int_{-u}^{\infty} \psi(c)\, \Phi_1(c)\, \mathrm{d}c \tag{51a}$$

and

$$\frac{1}{\Phi_1(-u)} = \sqrt{k}\ Y(\infty) + \int_{-\infty}^{-u} \psi(\tau)\, \Phi_2(\tau)\, \mathrm{d}\tau. \tag{51b}$$

We therefore define

$$H_1(c) = \Phi_1(c)\, \Phi_2(-u) \tag{52a}$$

and

$$H_2(\tau) = \Phi_2(\tau)\, \Phi_1(-u) \tag{52b}$$

and write Eqs. (50) as

$$H_2(-\tau) = 1 + (\tau - u)\, H_2(-\tau) \int_{-u}^{\infty} \psi(c)\, H_1(c) \frac{\mathrm{d}c}{c + \tau} \tag{53a}$$

for $\tau \in (u, \infty)$, and

$$H_1(c) = 1 + (c + u)\, H_1(c) \int_{u}^{\infty} \psi(-\tau)\, H_2(-\tau) \frac{\mathrm{d}\tau}{\tau + c} \tag{53b}$$

for $c \in (-u, \infty)$. If we now extend the domain of $H_1(c)$ and $H_2(\tau)$ by, for example, Eqs. (49) or (53), we can write Eq. (41) as

$$\Lambda(z)\, H_1(z)\, H_2(z) = 1. \tag{54}$$

To conclude this section we note that Eqs. (51) yield the identities

$$\int_{-u}^{\infty} \psi(c)\, H_1(c)\, \mathrm{d}c = 1 \tag{55}$$

and

$$\int_{u}^{\infty} \psi(-\tau)\, H_2(-\tau)\, \mathrm{d}\tau = 1, \qquad u^2 \leqq 3/2\,, \tag{56}$$

while from Eqs. (38) and (46) we can deduce that

$$\int_{-u}^{\infty} (c+u)\, \psi(c)\, H_1(c)\, \mathrm{d}c = 0 \tag{57}$$

and

$$\int_{-u}^{\infty} c(c+u)\, \psi(c)\, H_1(c)\, \mathrm{d}c = 0, \qquad u^2 \geqq 3/2\,. \tag{58}$$

V. Final results

Returning now to the boundary-value problem considered in section III, we note that

$$h(x, c) = \int_{-u}^{\infty} A(\eta)\, \varphi(\eta, c)\, \mathrm{e}^{-x/(\eta+u)}\, \mathrm{d}\eta\,, \tag{59}$$

where $A(\eta)$ can be found from Eqs. (30) and (36) once the conditions prescribed by Eqs. (40) are satisfied. We rewrite these conditions here as

$$\int_{-u}^{\infty} \left\{ \frac{f_0(c)}{f_\infty(c)} - 1 \right\} (c+u)\, H_1(c)\, \mathrm{e}^{-c^2}\, c^\alpha \mathrm{d}c = 0 \tag{60}$$

where $\alpha = 0$ and 1 for $u^2 < 3/2$ and $\alpha = 0$, 1 and 2 for $u^2 \geqq 3/2$. Now if we consider $f_0(\xi)$ to be a Maxwellian distribution and linearize about $f_\infty(\xi)$ we find that

$$\frac{f_0(c)}{f_\infty(c)} = 1 + \Delta\varrho + 2c(u_0 - u) + \left(c^2 - \frac{1}{2}\right) \Delta T, \tag{61}$$

where

$$\Delta\varrho = \frac{\varrho_0 - \varrho_\infty}{\varrho_\infty}, \tag{62a}$$

$$\Delta T = \frac{T_0 - T_\infty}{T_\infty} \tag{62b}$$

and u_0 is the drift speed at the boundary. It is apparent that for $u^2 < 3/2$ we can enter Eq. (61) into Eqs. (60), for $\alpha = 0$ and 1, and determine $\Delta\varrho$ and ΔT for prescribed values of $u_0 - u$. In this way we deduce that

$$\begin{aligned}(H_{1,1} + u\, H_{1,0})\, \Delta\varrho + [H_{1,3} - \tfrac{1}{2} H_{1,1} + u(H_{1,2} - \tfrac{1}{2} H_{1,0})]\, \Delta T \\ = 2(u - u_0)(H_{1,2} + u\, H_{1,1})\end{aligned} \tag{63a}$$

and

$$(H_{1,2}+u\,H_{1,1})\,\Delta\varrho+[H_{1,4}-\tfrac{1}{2}H_{1,2}+u(H_{1,3}-\tfrac{1}{2}H_{1,1})]\,\Delta T = 2(u-u_0)(H_{1,3}+u\,H_{1,2}) \tag{63b}$$

where we have used

$$H_{1,\alpha}=\int_{-u}^{\infty} H_1(c)\,\mathrm{e}^{-c^2}c^\alpha\,\mathrm{d}c \tag{64}$$

to denote moments of $H_1(c)$. Assuming that Eqs. (63) are linearly independent, we can readily solve them to find $\Delta\varrho$ and ΔT. We note that Eq. (31) yields

$$(\eta+u)\,\mathrm{e}^{-\eta^2}A(\eta)=N^+(\eta)-N^-(\eta) \tag{65}$$

and thus we can deduce from Eq. (36) that

$$A(\eta)=\frac{1}{\Lambda^+(\eta)\,\Lambda^-(\eta)\,H_1(\eta)}\int_{-u}^{\infty}\left[\Delta\varrho+2c(u_0-u)+\left(c^2-\frac{1}{2}\right)\Delta T\right]\cdot H_1(c)\,\varphi^\dagger(\eta,c)\,\mathrm{d}c \tag{66}$$

where

$$\varphi^\dagger(\eta,c)=(c+u)\,\psi(c)\,P\,v\left(\frac{1}{\eta-c}\right)+\lambda(\eta)\,\delta(\eta-c)\,. \tag{67}$$

From Eqs. (53) and (54) we can establish that, for $\eta\in(-u,\infty)$,

$$\lambda(\eta)\,H_1(\eta)=P\int_{-u}^{\infty}(c+u)\,\psi(c)\,H_1(c)\frac{\mathrm{d}c}{c-\eta} \tag{68}$$

so that Eq. (66) can be written as

$$A(\eta)=-\frac{\Delta T}{\Lambda^+(\eta)\,\Lambda^-(\eta)\,H_1(\eta)}\int_{-u}^{\infty}c(c+u)\,\psi(c)\,H_1(c)\,\mathrm{d}c\,. \tag{69}$$

Finally to compute the density, temperature and speed perturbations we use Eqs. (3), (6) and (27) to deduce that

$$\Delta T(x)=2(u^2-\tfrac{1}{2})\,\Delta\varrho(x) \tag{70a}$$

and

$$\Delta v(x)=-\Delta\varrho(x) \tag{70b}$$

where

$$\Delta\varrho(x)=\frac{1}{\sqrt{\pi}}\int_{-u}^{\infty}A(\eta)\,\mathrm{e}^{-\eta^2-x/(\eta+u)}\,\mathrm{d}\eta\,. \tag{71}$$

Thus for $u^2<3/2$ Eqs. (63), (69), (70) and (71) yield the desired solution. We have used the L functions discussed in the Appendix to compute, for selected values of $u^2\in(0,3/2)$, $H_1(c)$ and $H_2(-\tau)$. In Table 1 we compare

Table 1. Density and temperature perturbations

u	ϱ_∞/ϱ_0		T_∞/T_0	
	Present work	Ytrehus [9]	Present work	Ytrehus [9]
0.0	1.0	1.0	1.0	1.0
0.1	0.8812	0.8494	0.9195	0.9567
0.2	0.7961	0.7283	0.8421	0.9152
0.3	0.7322	0.6303	0.7695	0.8756
0.4	0.6824	0.5501	0.7024	0.8378
0.5	0.6427	0.4841	0.6407	0.8016
0.6	0.6104	0.4292	0.5845	0.7671
0.7	0.5838	0.3834	0.5334	0.7342
0.8	0.5616	0.3447	0.4870	0.7028
0.9	0.5429	0.3120	0.4449	0.6729
1.0	0.5271		0.4068	
1.1	0.5138		0.3722	
1.2	0.5025		0.3407	

our numerical results (for $u_0 = 0$) for the density and temperature perturbations with those based on a non-linear model and computed by Ytrehus [9]. It is clear that the considered linearized one-dimensional model yields only qualitative agreement with Ytrehus. The extension of this analysis to the three-dimensional BGK model is thus the subject of continuing work on this problem.

For $u^2 \geqq 3/2$ the considered problem has no meaningful solution. Note, for example, that Eqs. (60) would represent three linear algebraic equations for the two considered unknowns $\Delta\varrho$ and ΔT. We also observe that Eqs. (58) and (69) require, for $u^2 \geqq 3/2$, $A(\eta) \equiv 0$.

Acknowledgements

The authors wish to express their gratitude to C. Cercignani, who suggested this work, S. K. Loyalka and T. S. Storvick for several helpful discussions concerning the formulation of the problem solved here. This work was supported in part by the National Science Foundation through grant CPE-8016775.

Appendix: The *L* equations

In trying to solve Eqs. (53), written here for $c \in (-u, \infty)$ and $\tau \in (u, \infty)$ as

$$\frac{1}{H_2(-\tau)} = 1 - (\tau - u) \int_{-u}^{\infty} \psi(c)\, H_1(c) \frac{\mathrm{d}c}{c + \tau} \tag{A.1 a}$$

and

$$\frac{1}{H_1(c)} = 1 - (c+u) \int_u^\infty \psi(-\tau)\, H_2(-\tau) \frac{\mathrm{d}\tau}{\tau + c}, \tag{A.1b}$$

numerically we have found that an iterative procedure based on Eqs. (A.1) converges very slowly, and thus we now wish to extend our previous studies [20, 21], concerning H-function calculations in order to establish a rapidly converging method for computing $H_1(c)$ and $H_2(-\tau)$. We introduce, for $u^2 < 3/2$,

$$H_1(c) = (c + 1 + u)\, L_1(c) \tag{A.2a}$$

and

$$H_2(-\tau) = (\tau + 1 - u)^2\, L_2(-\tau) \tag{A.2b}$$

and substitute Eq. (A.2a) into Eq. (A.1b) to find, after we use Eq. (56),

$$\frac{1}{L_1(c)} = 1 + (c+u) \int_u^\infty \psi(-\tau)(\tau - 1 - u)\, H_2(-\tau) \frac{\mathrm{d}\tau}{\tau + c}. \tag{A.3}$$

In a similar manner we substitute Eq. (A.2b) into Eq. (A.1a) and use Eqs. (55) and (57) to obtain

$$\frac{1}{L_2(-\tau)} = 1 - (\tau - u) \int_{-u}^\infty \psi(c)(c - 1 + u)^2\, H_1(c) \frac{\mathrm{d}c}{c + \tau}. \tag{A.4}$$

If we now use Eqs. (A.2) in Eqs. (A.3) and (A.4) we find the desired L equations:

$$\frac{1}{L_1(c)} = 1 + (c+u) \int_u^\infty \psi(-\tau)(\tau - 1 - u)(\tau + 1 - u)^2\, L_2(-\tau) \frac{\mathrm{d}\tau}{\tau + c} \tag{A.5a}$$

and

$$\frac{1}{L_2(-\tau)} = 1 - (\tau - u) \int_{-u}^\infty \psi(c)(c - 1 + u)^2 (c + 1 + u)\, L_1(c) \frac{\mathrm{d}c}{c + \tau}. \tag{A.5b}$$

We have found that a numerical solution of Eqs. (A.5), by an iterative procedure, converges rapidly, and thus we were able to establish $L_1(c)$ and $L_2(-\tau)$ and subsequently, by way of Eqs. (A.2), also to deduce numerical results for $H_1(c)$ and $H_2(-\tau)$. To establish confidence in our computed values of $H_1(c)$ and $H_2(-\tau)$ we verified Eqs. (55), (56) and (57) and the identity

$$\left(\int_u^\infty \psi(-\tau)\, H_2(-\tau)(\tau - u)\, \mathrm{d}\tau\right)\left(\int_{-u}^\infty \psi(c)\, H_1(c)(c+u)^2\, \mathrm{d}c\right) = u(u^2 - 3/2) \tag{A.6}$$

to at least ten significant figures.

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Abstract

Evaporation of a liquid into a vacuum occupying a half space is investigated on the basis of the one-dimensional BGK model linearized about a drifting Maxwellian distribution. A unique solution is shown to exist if the downstream speed remains subsonic. Exact analysis is used, and numerical results are given.

Zusammenfassung

Auf der Basis eines eindimensionalen BGK-Modells, das über eine driftende Maxwell-Verteilung linearisiert ist, wird die Verdampfung einer Flüssigkeit in ein einen Halbraum ausfüllendes Vakuum untersucht. Es wird gezeigt, daß nur eine einzige Lösung existiert, wenn die Geschwindigkeit im Unterschallbereich bleibt. Exakte Analysis wird verwendet und numerische Resultate angegeben.

(Received: February 20, 1981)