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An Analytical Expression for the **H** Matrix Relevant to Rayleigh Scattering*

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The matrix Riemann–Hilbert problem relevant to the scattering of polarized light is solved up to the resolution of two existence questions. The solution obtained is used to establish an explicit expression for the related **H** matrix, and evidence that the mentioned existence questions can be answered in the affirmative is provided by a numerical evaluation of the final result.

I. INTRODUCTION

Basic to exact analysis [1] of the equation of transfer

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \frac{1}{2} \omega \mathbf{Q}(\mu) \int_{-1}^{1} \mathbf{Q}^T(\mu') \mathbf{I}(\tau, \mu') \, d\mu' \tag{1}$$

formulated by Chandrasekhar [2] to describe the scattering of polarized light is the solution to the Riemann–Hilbert problem defined by

$$\mathbf{\Phi}^+(\mu) = \mathbf{G}(\mu) \mathbf{\Phi}^-(\mu), \qquad \mu \in (0, 1). \tag{2}$$

Here, for Rayleigh scattering,

$$\mathbf{Q}(\mu) = \frac{\sqrt{3}}{2} \begin{bmatrix} \mu^2 & \sqrt{2}(1-\mu^2) \\ 1 & 0 \end{bmatrix} \tag{3}$$

and we use $\mathbf{I}(\tau, \mu)$, with components $I_l(\tau, \mu)$ and $I_r(\tau, \mu)$, to denote the intensity vector. Also, $\omega \in (0, 1)$ is the albedo for single scattering, τ is the optical variable and μ is the direction cosine of the propagating radiation.

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Given [1] that the $\mathbf{G}$ matrix in Eq. (2) can be expressed as

$$\mathbf{G}(\mu) = \mathbf{\Lambda}^{+}(\mu)[\mathbf{\Lambda}^{-}(\mu)]^{-1}, \tag{4}$$

where

$$\mathbf{\Lambda}(z) = \mathbf{I} + z \int_{-1}^{1} \mathbf{\Psi}(\mu) \frac{d\mu}{\mu - z} \tag{5}$$

and

$$\mathbf{\Psi}(\mu) = \tfrac{1}{2}\omega \mathbf{Q}^{T}(\mu)\, \mathbf{Q}(\mu), \tag{6}$$

we seek a 2×2 matrix $\mathbf{\Phi}(z)$ that is analytic in the complex plane cut from 0 to 1 along the real axis such that $\det \mathbf{\Phi}(z) \neq 0$ and such that the limiting values of $\mathbf{\Phi}(z)$, say $\mathbf{\Phi}^{\pm}(\mu)$, as z approaches the cut from above (+) and below (−) satisfy Eq. (2).

II. Analysis

We note first of all that $\mathbf{\Lambda}(z)$ can be expressed as

$$\mathbf{\Lambda}(z) = \mathbf{\Pi}(z) + f(z)\, \mathbf{\Psi}(z), \tag{7}$$

where

$$f(z) = z \int_{-1}^{1} \frac{d\mu}{\mu - z} \tag{8}$$

and

$$\mathbf{\Pi}(z) = \mathbf{I} + \frac{1}{4}\omega z^{2} \begin{bmatrix} 1 + 3z^{2} & \sqrt{2}(2 - 3z^{2}) \\ \sqrt{2}(2 - 3z^{2}) & 6z^{2} - 10 \end{bmatrix}. \tag{9}$$

It is clear that

$$(1 - z^{2})^{2}\, \mathbf{\Lambda}(z)\, \mathbf{\Psi}^{-1}(z) = (1 - z^{2})^{2} f(z)\mathbf{I} + (1 - z^{2})^{2}\, \mathbf{\Pi}(z)\, \mathbf{\Psi}^{-1}(z) \tag{10}$$

can be diagonalized by a similarity transformation involving at worst $R(z) = \sqrt{q}(z)$, where $q(z)$ is a polynomial. This idea of diagonalizing a matrix Riemann–Hilbert problem with nonanalytic functions was used by Darrozès [3] whose solution, for a problem in rarefied gasdynamics, unfortunately is not of the correct form at infinity. Cercignani [4] was able to correct the solution of Darrozès, and Siewert and Kelley [5] used a solution similar to that reported by Cercignani to develop a *canonical* solution and to compute the associated **H** matrix. We note that Cercignani [6] has also discussed this diagonalization idea in the context of neutron transport theory.

We find that

$$\mathbf{S}(z) = \begin{bmatrix} \sqrt{2}(1-\omega)\,z^2 & \dfrac{\frac{1}{2}[R(z)+p(z)]}{1-z^2} \\ -\sqrt{2}(1-z^2) & \dfrac{\frac{1}{2}[R(z)-p(z)]}{(1-\omega)\,z^2} \end{bmatrix}, \tag{11}$$

where

$$p(z) = (1-\omega)\,z^4 + (3\omega-4)\,z^2 + 1 \tag{12}$$

and

$$q(z) = 9(1-\omega)^2\,z^8 + 6(1-\omega)(3\omega-4)\,z^6 + (13\omega^2 - 38\omega + 26)\,z^4 + 2(3\omega-4)\,z^2 + 1, \tag{13}$$

is such that

$$\mathbf{S}(z)\,3(1-z^2)^2\,\mathbf{\Lambda}(z)\,\mathbf{Q}^{-1}(z)\,\mathbf{Q}^{-T}(z)\,\mathbf{S}^{-1}(z) = \mathbf{\Omega}(z). \tag{14}$$

Here

$$\mathbf{\Omega}(z) = \operatorname{diag}[\Omega_1(z), \Omega_2(z)], \tag{15}$$

$$\Omega_\alpha(z) = 3(1-z^2)^2\,\Lambda_0(z) + 2(1-\omega)\,z^2 + (-1)^\alpha\,R(z) \tag{16}$$

and

$$\Lambda_0(z) = 1 + \frac{\omega}{2} z \int_{-1}^{1} \frac{d\mu}{\mu - z}. \tag{17}$$

We note that $q(z)$ has no real zeros, and thus we write

$$q(z) = 9(1-\omega)^2 \prod_{\alpha=1}^{4} (z - z_\alpha)(z - \bar{z}_\alpha) \tag{18}$$

and let Γ_α denote the straight-line path connecting $-\bar{z}_\alpha$ and z_α. Then we consider that branch of $R(z)$ that is analytic in the complex plane cut along $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$. We now express the desired solution as

$$\mathbf{\Phi}(z) = \mathbf{S}^{-1}(z)\,\mathbf{U}(z)\,\mathbf{S}(z) \tag{19}$$

and require that

$$\mathbf{\Phi}^+(\mu) = \mathbf{G}(\mu)\,\mathbf{\Phi}^-(\mu), \qquad \mu \in (0, 1), \tag{20}$$

and

$$\mathbf{\Phi}^{+}(\tau)=\mathbf{\Phi}^{-}(\tau), \qquad \tau \in \Gamma. \tag{21}$$

Thus the sectionally analytic $\mathbf{U}(z)$ must satisfy

$$\mathbf{U}^{+}(\mu)=\mathbf{G}_0(\mu)\,\mathbf{U}^{-}(\mu), \qquad \mu \in (0,1), \tag{22}$$

and

$$\mathbf{U}^{+}(\tau)\,\mathbf{T}(\tau)=\mathbf{T}(\tau)\,\mathbf{U}^{-}(\tau), \qquad \tau \in \Gamma. \tag{23}$$

Here

$$\mathbf{G}_0(\mu)=\mathbf{\Omega}^{+}(\mu)[\mathbf{\Omega}^{-}(\mu)]^{-1} \tag{24}$$

and

$$\mathbf{T}(\tau)=-\mathbf{S}^{+}(\tau)[\mathbf{S}^{-}(\tau)]^{-1}=\begin{bmatrix} 0 & \dfrac{(1-\omega)\,\tau^2}{1-\tau^2} \\ \dfrac{1-\tau^2}{(1-\omega)\,\tau^2} & 0 \end{bmatrix}. \tag{25}$$

As before [5], we write

$$\mathbf{U}(z)=\operatorname{diag}[U_1(z), U_2(z)], \tag{26}$$

let

$$\gamma_\alpha(\mu)=\Omega_\alpha^{+}(\mu)/\Omega_\alpha^{-}(\mu), \qquad \mu \in (0,1),\ \alpha=1 \text{ or } 2, \tag{27}$$

$$A(\mu)=\gamma_1(\mu)/\gamma_2(\mu), \tag{28}$$

$$B(\mu)=\gamma_1(\mu)\,\gamma_2(\mu), \tag{29}$$

and consider

$$U_\alpha^*(z)=\exp\left(\frac{1}{4\pi}\int_0^1\left\{\arg B(x)-(-1)^\alpha\frac{R(z)}{R(x)}\left[\arg A(x)\right.\right.\right.$$
$$\left.\left.\left.-4\pi\sum_{j=1}^{3}k_j\Delta_j(x)\right]\right\}\frac{dx}{x-z}\right). \tag{30}$$

Here the k_j are integers and

$$\Delta_j(x)=1, \qquad x \in (x_{j,0}, x_{j,1}),$$
$$=0 \qquad \text{otherwise}. \tag{31}$$

We use here continuous values of $\arg A(x)$ and $\arg B(x)$, with $\arg A(0)=$

arg $B(0) = 0$. Now since $R(z) \to z^4$ as $|z| \to \infty$, it is apparent that $U_\alpha^*(z)$ will not have finite degree at infinity unless we impose the conditions

$$\int_0^1 \left[\arg A(x) - 4\pi \sum_{j=1}^{3} k_j \Delta_j(x)\right] x^{\beta-1} \frac{dx}{R(x)} = 0, \qquad \beta = 1, 2 \text{ and } 3. \tag{32}$$

We write

$$\gamma_\alpha(x) = e^{2i\vartheta_\alpha(x)}, \tag{33}$$

with

$$\vartheta_\alpha(x) = \tan^{-1}\left(\frac{3(1-x^2)^2\,\pi\omega x}{6(1-x^2)^2\,\lambda_0(x) + 4(1-\omega)\,x^2 + 2(-1)^\alpha\,R(x)}\right) \tag{34}$$

and

$$\lambda_0(x) = 1 - \omega x \tanh^{-1} x. \tag{35}$$

Thus

$$\arg A(x) = 2[\vartheta_1(x) - \vartheta_2(x)] = 2\theta(x), \tag{36}$$

$$\arg B(x) = 2[\vartheta_1(x) + \vartheta_2(x)] = 2\phi(x) \tag{37}$$

and Eq. (32) can be written as

$$\sum_{j=1}^{3} k_j \int_0^1 \Delta_j(x) \frac{x^{\beta-1}}{R(x)}\,dx = \frac{1}{2\pi}\int_0^1 \theta(x) \frac{x^{\beta-1}}{R(x)}\,dx, \qquad \beta = 1, 2 \text{ and } 3. \tag{38}$$

We note that $\theta(x) \in [0, \pi]$ and $\phi(x) \in [0, 2\pi]$ for $x \in [0, 1]$ and that we can now write Eq. (30) as

$$U_\alpha^*(z) = \exp\left(\frac{1}{2\pi}\int_0^1 \left\{\phi(x) - (-1)^\alpha \frac{R(z)}{R(x)}\left(\frac{x}{z}\right)^3\left[\theta(x) - 2\pi \sum_{j=1}^{3} k_j \Delta_j(x)\right]\right\}\frac{dx}{x-z}\right). \tag{39}$$

Now, to be specific, we take $k_j = (-1)^{j+1}$, $x_{j,0} = x_j$ and $x_{j,1} = 1$, for $j = 1, 2$ and 3, and thus we write Eq. (38) as

$$\mathbf{F}(x_1, x_2, x_3) = \mathbf{0} \tag{40}$$

where

$$\mathbf{F}(x_1, x_2, x_3) = \int_{x_1}^{1} \frac{1}{R(x)} \mathbf{W}(x)\, dx - \int_{x_2}^{x_3} \frac{1}{R(x)} \mathbf{W}(x)\, dx - \mathbf{Y}, \tag{41}$$

$$\mathbf{W}(x) = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}, \tag{42}$$

and

$$\mathbf{Y} = \frac{1}{2\pi} \int_0^1 \frac{\theta(x)}{R(x)} \mathbf{W}(x)\, dx. \tag{43}$$

We have found that the Newton–Raphson method can be used to generate, in a few iterations, numerical solutions of Eq. (40), and thus we list in Table I some typical results. We now write Eq. (39) as

$$U_\alpha^*(z) = \exp\left(\frac{1}{2\pi} \int_0^1 \left[\phi(x) - (-1)^\alpha \frac{R(z)}{R(x)} \left(\frac{x}{z}\right)^3 \theta(x)\right] \frac{dx}{x-z} \right.$$
$$\left. + (-1)^\alpha \frac{R(z)}{z^3} \left(\int_{x_1}^{1} - \int_{x_2}^{x_3}\right) \frac{x^3}{R(x)} \frac{dx}{x-z}\right), \tag{44}$$

and to remove the singularities at $z = x_1$, x_2 and x_3 we write our final results as

$$U_\alpha(z) = (z - x_1)(z - x_2)(z - x_3)\, U_\alpha^*(z). \tag{45}$$

We note that

$$\det \mathbf{\Phi}(z) = U_1(z)\, U_2(z) \tag{46}$$

TABLE I

Computed Values of x_1, x_2 and x_3

ω	x_1	x_2	x_3
0.7	0.55291413	0.59939422	0.95359687
0.8	0.56344797	0.62574877	0.95366355
0.9	0.57739222	0.66130656	0.95908892
0.99	0.60303665	0.71781370	0.98376961

or

$$\det \mathbf{\Phi}(z) = (1 - z)(z - x_1)^2(z - x_2)^2(z - x_3)^2\, X(z) \tag{47}$$

where

$$X(z) = \frac{1}{1 - z} \exp\left(\frac{1}{\pi}\int_0^1 \phi(x)\frac{dx}{x - z}\right) \tag{48}$$

is a canonical [7] solution of

$$\det \mathbf{\Phi}^{+}(\mu) = \det \mathbf{G}(\mu) \det \mathbf{\Phi}^{-}(\mu), \qquad \mu \in (0, 1). \tag{49}$$

It is thus apparent that

$$\mathbf{\Phi}(z) = \mathbf{S}^{-1}(z)\, \mathbf{U}(z)\, \mathbf{S}(z) \tag{50}$$

is not a canonical solution of Eq. (2). If we let $\mathbf{\Phi}_0(z)$ denote such a canonical solution (with normal form at infinity), then [7]

$$\mathbf{\Phi}(z) = \mathbf{\Phi}_0(z)\, \mathbf{P}_*(z), \tag{51}$$

where $\mathbf{P}_*(z)$ is a matrix of polynomials with

$$\det \mathbf{P}_*(z) \propto (1 - z)(z - x_1)^2(z - x_2)^2(z - x_3)^2. \tag{52}$$

Since we wish to establish the $\mathbf{H}$ matrix [1], i.e.,

$$\mathbf{H}(z) = \mathbf{\Phi}_0^{-T}(-z)\, \mathbf{D}^{-1}(-z)\, \mathbf{\Phi}_0^{T}(0), \tag{53}$$

where

$$\mathbf{D}(z) = \begin{bmatrix} 1 & 0 \\ 0 & \eta_0^{-1}(\eta_0 - z) \end{bmatrix} \tag{54}$$

and $\eta_0 > 1$ is the positive zero of

$$\Lambda(z) = \det \mathbf{\Lambda}(z), \tag{55}$$

we write Eq. (51) as

$$\begin{aligned} &\mathbf{\Phi}_0(z)\, \mathbf{D}(z) \begin{bmatrix} \delta & -\beta \\ \gamma\eta_0 & -\alpha\eta_0 \end{bmatrix} \left(\frac{1}{\alpha\delta - \beta\gamma}\right) \\ &\quad = [(1 - z)(z - x_1)^2(z - x_2)^2(z - x_3)^2]^{-1}\, \mathbf{\Phi}(z)\, \mathbf{P}(z), \end{aligned} \tag{56}$$

where $\mathbf{P}(z)$ is a matrix of polynomials. We note that Siewert and Burniston [1] have shown that

$$\mathbf{\Phi}_0(z) \sim \begin{bmatrix} \alpha + \cdots & \dfrac{\beta}{z} + \cdots \\ \gamma + \cdots & \dfrac{\delta}{z} + \cdots \end{bmatrix}, \qquad |z| \to \infty, \alpha\delta \neq \beta\gamma, \tag{57}$$

so that from Eq. (56) we can establish that

$$\mathbf{P}(z) = \mathbf{A} + \mathbf{B}z + \mathbf{C}z^2 + \mathbf{D}z^3 + \mathbf{E}z^4, \tag{58}$$

where

$$\mathbf{E} = -\frac{1}{3}\begin{bmatrix} U_2^*(\infty) + 2U_1^*(\infty) & \sqrt{2}[U_1^*(\infty) - U_2^*(\infty)] \\ \sqrt{2}[U_1^*(\infty) - U_2^*(\infty)] & U_1^*(\infty) + 2U_2^*(\infty) \end{bmatrix}. \tag{59}$$

We can now use the fact [1] that

$$\mathbf{H}^T(z)\,\mathbf{\Lambda}(z)\,\mathbf{H}(-z) = \mathbf{I} \tag{60}$$

to deduce that

$$\mathbf{M}^T(\eta_0)\,\mathbf{\Phi}_0(\eta_0)\,\mathbf{D}(\eta_0) = \mathbf{0}, \tag{61}$$

where $\mathbf{M}(\eta_0)$ is a null-vector of $\mathbf{\Lambda}(\eta_0)$, i.e.,

$$\mathbf{\Lambda}(\eta_0)\,\mathbf{M}(\eta_0) = \mathbf{0}, \tag{62}$$

and thus the required constants $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ can be determined from

$$\mathbf{M}^T(\eta_0)\,\mathbf{\Phi}(\eta_0)\,\mathbf{P}(\eta_0) = \mathbf{0}, \tag{63a}$$

$$\mathbf{\Phi}(\xi)\,\mathbf{P}(\xi) = \mathbf{0}, \qquad \xi = x_1, x_2, x_3 \text{ and } 1, \tag{63b}$$

and

$$\frac{d}{d\xi}[\mathbf{\Phi}(\xi)\,\mathbf{P}(\xi)] = \mathbf{0}, \qquad \xi = x_1, x_2 \text{ and } x_3. \tag{63c}$$

Equations (63) clearly are 16 linear algebraic equations to be solved for the 16 required elements of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$. We find, however, that the inversion of a single 8×8 matrix $\mathbf{\Delta}$ is sufficient to develop the solution to Eqs. (63).

Finally we write the desired $\mathbf{H}$ matrix as

$$\mathbf{H}(z) = \frac{1}{x_1^2 x_2^2 x_3^2(\eta_0 + z)}\,\mathbf{S}^T(z)\,\mathbf{U}^{-1}(-z)\,\mathbf{S}^{-T}(z)\,\mathbf{\Xi}(z)\,\mathbf{\Phi}^T(0), \tag{64}$$

where

$$\Xi(z) = (\eta_0 + z)(1 + z)(z + x_1)^2(z + x_2)^2(z + x_3)^2\, \mathbf{P}^{-T}(-z)\, \mathbf{A}^T. \tag{65}$$

Since we can readily show that

$$U_\alpha^*(\infty) = \exp\left((-1)^\alpha\, 3(1-\omega)\left[\frac{1}{2\pi}\int_0^1 \frac{x^3}{R(x)}\,\theta(x)\,dx - \left(\int_{x_1}^1 - \int_{x_2}^{x_3}\right)\frac{x^3}{R(x)}\,dx\right]\right), \tag{66}$$

$$U_\alpha^*(0) = \frac{1}{\eta_0^{1/2}}\,\frac{1}{[(1-\omega)(1-\frac{7}{10}\omega)]^{1/4}}\exp\left(\frac{(-1)^{\alpha+1}}{2\pi}\int_0^1 \frac{\theta(x)}{R(x)}\frac{dx}{x} + (-1)^\alpha\left(\int_{x_1}^1 - \int_{x_2}^{x_3}\right)\frac{1}{R(x)}\frac{dx}{x}\right), \tag{67}$$

and

$$\mathbf{\Phi}(0) = -x_1x_2x_3\begin{bmatrix} U_2^*(0) & 0 \\ 0 & U_1^*(0)\end{bmatrix}, \tag{68}$$

we can solve the mentioned linear algebraic equations to find **A**, **B**, **C** and **D**, and thus we can compute $\mathbf{H}(\mu)$, $\mu \in [0, 1]$, from Eq. (64). To establish confidence in our final result, we have, for the cases shown in Table I, evaluated $\mathbf{H}(\mu)$ from Eq. (64) and obtained results that agree to nine significant figures with a numerical solution, by iteration, of the nonlinear **H** equation [1].

Though the mentioned numerical verification is evidence that our solution for the **H** matrix is correct, there are two matters that deserve further attention. First of all, proof of the existence of a solution to Eq. (40) is desired. We note, in fact, that although we were able to establish a solution numerically for the choice $k_j = (-1)^{j+1}$, we were able to demonstrate, again numerically, that there is no solution for $k_1 = k_2 = k_3 = 1$. Clearly then the existence of a solution to Eq. (40) will inherently depend on the particular choice of k_1, k_2 and k_3. Second, although we encountered no numerical difficulties in computing the constants **A**, **B**, **C** and **D**, to be sure that we can solve Eqs. (63) proof that det $\mathbf{\Delta} \neq 0$ is required.

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