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An approximate solution concerning strong evaporation into a half space

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I. Introduction

In a recent paper [1], hereafter referred to as I, the method of elementary solutions was used to solve "exactly" the strong evaporation problem formulated and solved initially by Arthur and Cercignani [2]. In this brief communication we use the F_N method [3, 4, 5] to solve approximately, though concisely and accurately, the same problem. We assume that I is available, and thus we do not reproduce here the formulation of the problem.

II. Analysis

We seek a solution, for $x \in [0, \infty)$ and $c \in (-\infty, \infty)$, to

$$(c+u)\frac{\partial}{\partial x} h(x,c) + h(x,c) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} [1 + 2(c^2 - 1/2)(\mu^2 - 1/2) + 2c\mu]\, e^{-\mu^2} h(x,\mu)\, d\mu \tag{1}$$

subject to the boundary conditions

$$\lim_{x \to \infty} h(x,c) = 0 \tag{2a}$$

and

$$h(0,c) = 2c(u_0 - u) + \Delta\varrho + (c^2 - 1/2)\Delta T, \qquad c > -u\,. \tag{2b}$$

In Refs. [1] and [2] the considered problem was shown to have a unique solution only for subsonic downstream conditions, i.e. $u^2 < 3/2$, and particular values of the unknown constants $\Delta\varrho$ and ΔT. Following I, we write, for $u^2 < 3/2$, the desired solution as

$$h(x,c) = \int_{-u}^{\infty} A(\eta)\, \varphi(\eta, c)\, e^{-x/(\eta+u)}\, d\eta\,, \tag{3}$$

where $A(\eta)$ is an expansion coefficient to be determined and

$$\varphi(\eta, c) = \frac{1}{\sqrt{\pi}} (\eta + u)\, q(c)\, P v \left(\frac{1}{\eta - c}\right) e^{-\eta^2} + \lambda(\eta)\, \delta(\eta - c) \tag{4}$$

with

$$\lambda(\eta) = 1 + (\eta + u)\, P \int_{-\infty}^{\infty} \psi(\mu) \frac{d\mu}{\mu - \eta}, \tag{5}$$

$$q(c) = 1 - 2uc + 2(u^2 - 1/2)(c^2 - 1/2) \tag{6}$$

and

$$\psi(\mu) = \frac{1}{\sqrt{\pi}} q(\mu)\, e^{-\mu^2}. \tag{7}$$

For $x = 0$, Eq. (3) yields, for $c \in (-\infty, \infty)$,

$$h(0, c) = \int_{-u}^{\infty} A(\eta)\, \varphi(\eta, c)\, d\eta. \tag{8}$$

Thus if we assume for the moment that $h(0, c)$ is known for all c we can use the theory of Muskhelishvili [6] to show that the singular integral equation given by Eq. (8) has a unique solution provided

$$\int_{-\infty}^{\infty} c^\beta h(0, c)(c + u)\, e^{-c^2} dc = 0, \quad \beta = 0, 1 \text{ and } 2, \tag{9a}$$

and

$$\int_{-\infty}^{\infty} h(0, c)\, \varphi^+(-\eta, c)\, dc = 0, \quad \eta \in (u, \infty), \tag{9b}$$

where

$$\varphi^+(\eta, c) = (c + u)\, \psi(c)\, P v \left(\frac{1}{\eta - c}\right) + \lambda(\eta)\, \delta(\eta - c). \tag{10}$$

III. Approximate results

We now use the F_N method to construct an approximate solution to Eqs. (9). If we note Eq. (8) and substitute the approximation

$$h(0, -c) = q(-c) \sum_{\alpha=0}^{N} \frac{a_\alpha}{\nu_\alpha + c}, \quad c > u \quad \text{and} \quad \nu_\alpha \in [-u, \infty), \tag{11}$$

into Eqs. (9) we find, for $\beta = 0, 1$ and 2,

$$\sum_{\alpha=0}^{N} a_\alpha [-(-1)^\beta I_\beta(\nu_\alpha)] + \Delta\varrho\, M_\beta + \Delta T (M_{\beta+2} - \tfrac{1}{2} M_\beta) = 2(u - u_0) M_{\beta+1} \tag{12a}$$

and, for $\eta \in (u, \infty)$,

$$\sum_{\alpha=0}^{N} a_\alpha \Gamma_\alpha(\eta) + \Delta\varrho\, A_0(\eta) + \Delta T [A_2(\eta) - \tfrac{1}{2} A_0(\eta)] = 2(u - u_0) A_1(\eta). \tag{12b}$$

Here

$$M_\beta = \frac{1}{\sqrt{\pi}} \int_{-u}^{\infty} c^\beta (c + u)\, e^{-c^2} dc, \tag{13}$$

$$I_\beta(z) = \int_{u}^{\infty} c^\beta \psi(-c)(c - u) \frac{dc}{c + z}, \tag{14}$$

$$A_\alpha(\eta) = -\int_{-u}^{\infty} (c+u)\,\psi(c)\,c^\alpha \frac{\mathrm{d}c}{c+\eta} \tag{15}$$

and

$$\Gamma_\alpha(\eta) = (1-2u^2)\int_u^\infty \psi(c)\,(c-u)\,\mathrm{d}c - \left(\frac{1}{\nu_\alpha+\eta}\right)[A_0(\eta)\,q(-\eta) - q(\nu_\alpha)\,I_0(\nu_\alpha)]. \tag{16}$$

If we now consider Eq. (12b) at N selected values of $\eta \in (u, \infty)$, say $\eta_j, j = 1, 2, \ldots N$, then we can solve the $N+3$ linear algebraic equations

$$\sum_{\alpha=0}^{N} a_\alpha[-(-1)^\beta I_\beta(\nu_\alpha)] + \Delta\varrho\, M_\beta + \Delta T\,(M_{\beta+2} - \tfrac{1}{2} M_\beta) = 2\,(u-u_0)\,M_{\beta+1} \tag{17a}$$

and

$$\sum_{\alpha=0}^{N} a_\alpha\,\Gamma_\alpha(\eta_j) + \Delta\varrho\,A_0(\eta_j) + \Delta T\,[A_2(\eta_j) - \tfrac{1}{2}A_0(\eta_j)] = 2\,(u-u_0)\,A_1(\eta_j), \tag{17b}$$

where $\beta = 0$, 1 and 2 and $j = 1, 2, \ldots, N$, to find the desired constants $\Delta\varrho$, ΔT and a_α, $\alpha = 0, 1, 2, \ldots, N$. To have a simple and effective scheme we take $\eta_j = \tau_j + u$ and $\nu_\alpha = \tau_\alpha - u$, where $\{\tau_j\}$ are the positive zeros of the Hermite polynomial $H_{2N}(\xi)$. We have solved Eqs. (17) for $u_0 = 0$ to obtain $\Delta\varrho$ and ΔT in agreement to four significant figures with the "exact" values reported in I. To achieve such accuracy we found that N in the range $1-3$ was sufficient. We thus conclude that the F_N method yields here a particularly concise and accurate result for the considered problem.

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Abstract

Evaporation of a liquid into a vacuum occupying a half space is investigated on the basis of the one-dimensional BGK model linearized about a drifting Maxwellian distribution. The F_N method is used to deduce accurate numerical results for the perturbations in the density and temperature.

Zusammenfassung

Auf der Basis eines eindimensionalen BGK-Modells, das über eine driftende Maxwell-Verteilung linearisiert ist, wird die Verdampfung einer Flüssigkeit in ein einen Halbraum ausfüllendes Vakuum untersucht. Die F_N-Methode wird verwendet zur Ableitung von genauen numerischen Ergebnissen für die Störungen in Dichte und Temperatur.

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