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THE COMPLETE SOLUTION FOR RADIATIVE TRANSFER PROBLEMS WITH REFLECTING BOUNDARIES AND INTERNAL SOURCES

C. DEVAUX, C. E. SIEWERT, AND Y. L. YUAN
Laboratoire d'Optique Atmosphérique, Université des Sciences et Techniques de Lille, Villeneuve d'Ascq, France; and
Departments of Mathematics and Nuclear Engineering, North Carolina State University, Raleigh
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ABSTRACT

The F_N method is used to establish the complete solution for the radiation field basic to radiative transfer problems, in plane-parallel media, with reflecting boundaries and internal sources. An Lth order Legendre expansion of the phase function is used, azimuthal symmetry is not required, and numerical results are reported.

Subject heading: radiative transfer

I. INTRODUCTION

There is a vast literature concerning light scattering in planetary atmospheres which essentially begins with the classic work of Chandrasekhar (1950) and which has been reviewed recently by van de Hulst (1980).

In a recent paper (Devaux and Siewert 1980, hereafter referred to as I) we reported the formalism required to establish by the F_N method (Siewert 1978; Siewert and Benoist 1979) the exit distributions of radiation $I(L,-\mu,\phi)$ and $I(R,\mu,\phi)$, $\mu>0$ and $\phi\in[0,2\pi]$, relevant to radiative transfer problems defined by the equation of transfer (Chandrasekhar 1950)

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu,\phi)+I(\tau,\mu,\phi)=\frac{\omega}{4\pi}\int_0^{2\pi}\int_{-1}^{1}p(\cos\Theta)I(\tau,\mu',\phi')\,d\mu'\,d\phi' \tag{1}$$

and the boundary conditions

$$I(L,\mu,\phi)=F_1(\mu,\phi),\qquad \mu>0\quad\text{and}\quad \phi\in[0,2\pi], \tag{2a}$$

and

$$I(R,-\mu,\phi)=F_2(\mu,\phi),\qquad \mu>0\quad\text{and}\quad \phi\in[0,2\pi]. \tag{2b}$$

Here ω is the single-scattering albedo, μ is the direction cosine, as measured from the *positive* τ axis, of the propagating radiation, ϕ is the azimuthal angle measured with respect to a reference angle ϕ_r, and $\tau\in[L,R]$ is the optical variable. In addition, Θ is the scattering angle, and we consider phase functions that have a Legendre expansion of the form

$$p(\cos\Theta)=\sum_{l=0}^{L}(2l+1)f_lP_l(\cos\Theta),\qquad f_0=1. \tag{3}$$

We note that the F_N solution for $I(L,-\mu,\phi)$ and $I(R,\mu,\phi)$, $\mu>0$ and $\phi\in[0,2\pi]$, reported in I, was tractable and that the readily established numerical results were generally accurate to four significant figures. Similar observations have been reported by McCormick and Sanchez (1981). We therefore wish to develop here the analysis required to allow the inclusion of internal sources and reflecting boundaries in our model. Also, because of current interest in the sounding of planetary atmospheres, we extend the F_N method here to find the radiation field $I(\tau,\mu,\phi)$ for all $\tau\in(L,R)$.

In order to avoid excessive repetition of mathematical equations, we assume that the dedicated reader will have I available.

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II. PRELIMINARY ANALYSIS

We consider now the equation of transfer

$$\mu \frac{\partial}{\partial \tau} I(\tau,\mu,\phi) + I(\tau,\mu,\phi) = \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} p(\cos\Theta) I(\tau,\mu',\phi') d\mu' d\phi' + S(\tau), \tag{4}$$

where $S(\tau)$ represents an internal source, and the boundary conditions, for $\mu > 0$ and $\phi \in [0, 2\pi]$,

$$I(L,\mu,\phi) = I_1(\mu,\phi) + \Delta_1 \delta(\mu-\mu_0)\delta(\phi-\phi_0) + \rho_1^s I(L,-\mu,\phi) + \frac{1}{\pi}\rho_1^d \int_0^{2\pi}\int_0^1 I(L,-\mu',\phi')\mu' d\mu' d\phi' \tag{5a}$$

and

$$I(R,-\mu,\phi) = I_2(\mu,\phi) + \Delta_2 \delta(\mu-\mu_0)\delta(\phi-\phi_0) + \rho_2^s I(R,\mu,\phi) + \frac{1}{\pi}\rho_2^d \int_0^{2\pi}\int_0^1 I(R,\mu',\phi')\mu' d\mu' d\phi'. \tag{5b}$$

Here ρ_α^s and ρ_α^d, $\alpha = 1$ and 2, are the coefficients for specular and diffuse reflection, Δ_1 and Δ_2 are scaling constants, and $I_1(\mu,\phi)$ and $I_2(\mu,\phi)$ are assumed to be given. By considering the boundary conditions as given by equations (5), we clearly allow the possibility of solar illumination of a single layer $\tau \in [L, R]$; however, by keeping the terms $I_1(\mu,\phi)$, $I_2(\mu,\phi)$, and $\Delta_2\delta(\mu-\mu_0)\delta(\phi-\phi_0)$ in equations (5) we can also use this development to solve multilayer problems by iteration.

We now express the intensity as

$$I(\tau,\mu,\phi) = \sum_{m=0}^{L} \left[I_c^m(\tau,\mu)\cos m(\phi-\phi_r) + I_s^m(\tau,\mu)\sin m(\phi-\phi_r) \right] + R(\mu,\phi)e^{-\tau/\mu} + I_p(\tau,\mu), \tag{6}$$

where

$$\int_0^{2\pi} R(\mu',\phi') p(\cos\Theta) d\phi' = 0, \qquad \mu,\mu' \in [-1,1] \quad \text{and} \quad \phi \in [0,2\pi], \tag{7}$$

and $I_p(\tau,\mu)$ is a particular solution of

$$\mu \frac{\partial}{\partial \tau} I(\tau,\mu) + I(\tau,\mu) = \frac{\omega}{2} \sum_{l=0}^{L} (2l+1) f_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I(\tau,\mu') d\mu' + S(\tau). \tag{8}$$

In addition the components $I_c^m(\tau,\mu)$ and $I_s^m(\tau,\mu)$ satisfy the equation of transfer

$$\mu \frac{\partial}{\partial \tau} I(\tau,\mu) + I(\tau,\mu) = \frac{\omega}{2} \sum_{l=m}^{L} (2l+1) f_l^m P_l^m(\mu) \int_{-1}^{1} P_l^m(\mu') I(\tau,\mu') d\mu', \tag{9}$$

where

$$f_l^m = f_l \frac{(l-m)!}{(l+m)!} \tag{10}$$

and the associated Legendre functions are

$$P_l^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} P_l(\mu). \tag{11}$$

We note that the form used in equation (6) is convenient for separating the complete problem for $I(\tau,\mu,\phi)$ into a set of problems for the ϕ-independent components $I_c^m(\tau,\mu)$ and $I_s^m(\tau,\mu)$. Since particular solutions appropriate to several specific representations of the internal source $S(\tau)$ have been recently reported (Siewert and Oruma 1981), we consider here that $I_p(\tau,\mu)$ is available. Of course, in establishing equation (9) we have used the addition theorem for the

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Legendre polynomials to express the phase function as

$$p(\cos\Theta)=\sum_{m=0}^{L}(2-\delta_{0,m})\sum_{l=m}^{L}(2l+1)f_l^m P_l^m(\mu)P_l^m(\mu')\cos m(\phi-\phi'). \tag{12}$$

If we now substitute equation (6) into equations (5), and make use of equation (7), we find that the ϕ-independent components must satisfy the boundary conditions, for $\mu>0$,

$$I^m(L,\mu)=\left(\frac{2-\delta_{0,m}}{2\pi}\right)\int_0^{2\pi}K_1(\mu,\phi)\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}m(\phi-\phi_r)\,d\phi+\rho_1^s I^m(L,-\mu)+2\rho_1^d\delta_{0,m}\int_0^1 I^0(L,-\mu')\mu'\,d\mu' \tag{13a}$$

and

$$I^m(R,-\mu)=\left(\frac{2-\delta_{0,m}}{2\pi}\right)\int_0^{2\pi}K_2(\mu,\phi)\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}m(\phi-\phi_r)\,d\phi+\rho_2^s I^m(R,\mu)+2\rho_2^d\delta_{0,m}\int_0^1 I^0(R,\mu')\mu'\,d\mu'. \tag{13b}$$

Here, for $\mu>0$ and $\phi\in[0,2\pi]$, we let

$$K_1(\mu,\phi)=I_1(\mu,\phi)+\Delta_1\delta(\mu-\mu_0)\delta(\phi-\phi_0)-I_p(L,\mu)+\rho_1^s I_p(L,-\mu)+2\rho_1^d\int_0^1 I_p(L,-\mu')\mu'\,d\mu' \tag{14a}$$

and

$$K_2(\mu,\phi)=I_2(\mu,\phi)+\Delta_2\delta(\mu-\mu_0)\delta(\phi-\phi_0)-I_p(R,-\mu)+\rho_2^s I_p(R,\mu)+2\rho_2^d\int_0^1 I_p(R,\mu')\mu'\,d\mu', \tag{14b}$$

and we have found that the required $R(\mu,\phi)$ is given, for $\mu>0$, by

$$R(\mu,\phi)e^{-L/\mu}=\left(1-\rho_1^s\rho_2^s e^{-2\Delta/\mu}\right)^{-1}\left[L_1(\mu,\phi)+\rho_1^s L_2(\mu,\phi)e^{-\Delta/\mu}\right] \tag{15a}$$

and

$$R(-\mu,\phi)e^{R/\mu}=\left(1-\rho_1^s\rho_2^s e^{-2\Delta/\mu}\right)^{-1}\left[L_2(\mu,\phi)+\rho_2^s L_1(\mu,\phi)e^{-\Delta/\mu}\right], \tag{15b}$$

where $\Delta=R-L$. We let $\phi_r=\phi_0$ to deduce that

$$L_1(\mu,\phi)=I_1(\mu,\phi)-I_1^*(\mu,\phi)+\Delta_1\delta(\mu-\mu_0)\left[\delta(\phi-\phi_0)-\frac{1}{2\pi}\sum_{m=0}^{L}(2-\delta_{0,m})\cos m(\phi-\phi_0)\right] \tag{16a}$$

and

$$L_2(\mu,\phi)=I_2(\mu,\phi)-I_2^*(\mu,\phi)+\Delta_2\delta(\mu-\mu_0)\left[\delta(\phi-\phi_0)-\frac{1}{2\pi}\sum_{m=0}^{L}(2-\delta_{0,m})\cos m(\phi-\phi_0)\right] \tag{16b}$$

where, for $\alpha=1$ or 2,

$$I_\alpha^*(\mu,\phi)=\sum_{m=0}^{l}\left\{\left[\frac{2-\delta_{0,m}}{2\pi}\int_0^{2\pi}I_\alpha(\mu,\phi')\cos m(\phi'-\phi_0)\,d\phi'\right]\cos m(\phi-\phi_0)\right.$$
$$\left.+\left[\frac{1}{\pi}\int_0^{2\pi}I_\alpha(\mu,\phi')\sin m(\phi'-\phi_0)\,d\phi'\right]\sin m(\phi-\phi_0)\right\}. \tag{17}$$

The decomposition of the original problem to a set of ϕ-independent problems is thus achieved.

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III. ANALYSIS

It is clear that to establish the complete solution for $I(\tau, \mu, \phi)$ we must solve, for $0 \leq m \leq L$, a class of problems defined by the equation of transfer

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\omega}{2} \sum_{l=m}^{L} (2l+1) f_l^m P_l^m(\mu) \int_{-1}^{1} P_l^m(\mu') I(\tau, \mu')\, d\mu' \tag{18}$$

and the boundary conditions, for $\mu > 0$,

$$I(L, \mu) = I_1(\mu) + M_1 \delta(\mu - \mu_0) + \rho_1^s I(L, -\mu) + 2\delta_{0,m} \rho_1^d \int_0^1 I(L, -\mu') \mu'\, d\mu' \tag{19a}$$

and

$$I(R, -\mu) = I_2(\mu) + M_2 \delta(\mu - \mu_0) + \rho_2^s I(R, \mu) + 2\delta_{0,m} \rho_2^d \int_0^1 I(R, \mu') \mu'\, d\mu', \tag{19b}$$

where $I_1(\mu)$ and $I_2(\mu)$ are prescribed functions and M_1 and M_2 are given constants. In I the elementary solutions found for equation (18) by McCormick and Kuščer (1966) were used to develop the following system of singular integral equations and constraints for the surface quantities $I(L, \mu)$ and $I(R, \mu)$:

$$\int_{-1}^{1} \mu \phi(\xi, \mu) I(L, -\mu)\, d\mu + e^{-\Delta/\xi} \int_{-1}^{1} \mu \phi(-\xi, \mu) I(R, \mu)\, d\mu = 0 \tag{20a}$$

and

$$\int_{-1}^{1} \mu \phi(\xi, \mu) I(R, \mu)\, d\mu + e^{-\Delta/\xi} \int_{-1}^{1} \mu \phi(-\xi, \mu) I(L, -\mu)\, d\mu = 0. \tag{20b}$$

Here $\xi \in P$ with $P = \{\nu_\beta\} \cup [0,1]$; and ν_β, $\beta = 0, 1, 2, \ldots, \kappa - 1$, are the discrete eigenvalues. We wish to use the F_N method to find approximate results for the angular distributions $I(L, -\mu)$ and $I(R, \mu)$, $\mu > 0$, and in order to approach exact results as $\omega \to 0$ we write (Garcia and Siewert 1981), for $\mu > 0$,

$$I(L, -\mu) = I_0(L, -\mu) + \frac{\omega}{2} (1 - \mu^2)^{m/2} \sum_{\alpha=0}^{N} a_\alpha P_\alpha(2\mu - 1) \tag{21a}$$

and

$$I(R, \mu) = I_0(R, \mu) + \frac{\omega}{2} (1 - \mu^2)^{m/2} \sum_{\alpha=0}^{N} b_\alpha P_\alpha(2\mu - 1), \tag{21b}$$

where $I_0(\tau, \mu)$ is used to denote the exact solution for $\omega = 0$. We find

$$I_0(\tau, \mu) = \Phi(\mu) e^{-\tau/\mu}, \tag{22}$$

where, for $\mu > 0$,

$$\Phi(\mu) e^{-L/\mu} = \left(1 - \rho_1^s \rho_2^s e^{-2\Delta/\mu}\right)^{-1} \left[K_1(\mu) + \rho_1^s K_2(\mu) e^{-\Delta/\mu} + 2\delta_{0,m}\left(\rho_1^d \Phi_1 + \rho_1^s \rho_2^d \Phi_2 e^{-\Delta/\mu}\right)\right] \tag{23a}$$

and

$$\Phi(-\mu) e^{R/\mu} = \left(1 - \rho_1^s \rho_2^s e^{-2\Delta/\mu}\right)^{-1} \left[K_2(\mu) + \rho_2^s K_1(\mu) e^{-\Delta/\mu} + 2\delta_{0,m}\left(\rho_2^d \Phi_2 + \rho_2^s \rho_1^d \Phi_1 e^{-\Delta/\mu}\right)\right]. \tag{23b}$$

In addition,

$$K_1(\mu) = I_1(\mu) + M_1 \delta(\mu - \mu_0) \tag{24a}$$

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and

$$K_2(\mu)=I_2(\mu)+M_2\delta(\mu-\mu_0); \tag{24b}$$

and the constants

$$\Phi_1=\int_0^1\Phi(-\mu)e^{L/\mu}\mu\,d\mu \tag{25a}$$

and

$$\Phi_2=\int_0^1\Phi(\mu)e^{-R/\mu}\mu\,d\mu \tag{25b}$$

clearly can be found analytically after we multiply equations (23) by $\mu\exp(-\Delta/\mu)$ and integrate over μ. If we now substitute equations (21) into equations (20), we find

$$\sum_{\alpha=0}^{N}\{a_\alpha[B_\alpha(\xi)-\omega\rho_1^s A_\alpha(\xi)-2\omega T_{\alpha,0}^0\rho_1^d\delta_{0,m}A_0(\xi)]$$
$$+e^{-\Delta/\xi}b_\alpha[\omega A_\alpha(\xi)-\rho_2^s B_\alpha(\xi)-2T_{\alpha,0}^0\rho_2^d\delta_{0,m}B_0(\xi)]\}=2\Upsilon_1(L,\xi) \tag{26a}$$

and

$$\sum_{\alpha=0}^{N}\{b_\alpha[B_\alpha(\xi)-\omega\rho_2^s A_\alpha(\xi)-2\omega T_{\alpha,0}^0\rho_2^d\delta_{0,m}A_0(\xi)]$$
$$+e^{-\Delta/\xi}a_\alpha[\omega A_\alpha(\xi)-\rho_1^s B_\alpha(\xi)-2T_{\alpha,0}^0\rho_1^d\delta_{0,m}B_0(\xi)]\}=2\Upsilon_2(R,\xi), \tag{26b}$$

where $T_{\alpha,0}^0$ and the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ are discussed in the Appendix. Also the known right-hand sides of equations (26) are available from

$$\Upsilon_1(\tau,\xi)=\frac{2}{\omega\xi}\int_0^1\{\phi(-\xi,\mu)\Xi_1(\mu)e^{-(\tau-L)/\mu}[1-e^{-(R-\tau)/\xi}e^{-(R-\tau)/\mu}]$$
$$+\phi(\xi,\mu)\Xi_2(\mu)[e^{-(R-\tau)/\xi}-e^{-(R-\tau)/\mu}]\}\mu\,d\mu \tag{27a}$$

and

$$\Upsilon_2(\tau,\xi)=\frac{2}{\omega\xi}\int_0^1\{\phi(\xi,\mu)\Xi_1(\mu)[e^{-(\tau-L)/\xi}-e^{-(\tau-L)/\mu}]$$
$$+\phi(-\xi,\mu)\Xi_2(\mu)e^{-(R-\tau)/\mu}[1-e^{-(\tau-L)/\xi}e^{-(\tau-L)/\mu}]\}\mu\,d\mu, \tag{27b}$$

where

$$\Xi_1(\mu)=\Phi(\mu)e^{-L/\mu} \tag{28a}$$

and

$$\Xi_2(\mu)=\Phi(-\mu)e^{R/\mu}. \tag{28b}$$

As discussed in I, we can now solve the $2(N+1)$ linear algebraic equations obtained by considering equations (26) at selected values of ξ, say ξ_β, $\beta=0,1,2,\ldots,N$, to find the desired constants a_α and b_α, $\alpha=0,1,2,\ldots,N$, and thus to establish by way of equations (21) the angular distributions.

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Since we have found the angular distributions, for $\mu>0$,

$$I(L,-\mu)=\Xi_2(\mu)e^{-\Delta/\mu}+\frac{\omega}{2}(1-\mu^2)^{m/2}\sum_{\alpha=0}^{N}a_\alpha P_\alpha(2\mu-1) \tag{29a}$$

and

$$I(R,\mu)=\Xi_1(\mu)e^{-\Delta/\mu}+\frac{\omega}{2}(1-\mu^2)^{m/2}\sum_{\alpha=0}^{N}b_\alpha P_\alpha(2\mu-1), \tag{29b}$$

we can deduce from equations (19) similar results for $I(L,\mu)$ and $I(R,-\mu)$; thus for $\mu>0$

$$I(L,\mu)=\Xi_1(\mu)+\frac{\omega}{2}\left[\rho_1^s(1-\mu^2)^{m/2}\sum_{\alpha=0}^{N}a_\alpha P_\alpha(2\mu-1)+2\delta_{0,m}\rho_1^d\sum_{\alpha=0}^{N}a_\alpha T_{\alpha,0}^0\right] \tag{30a}$$

and

$$I(R,-\mu)=\Xi_2(\mu)+\frac{\omega}{2}\left[\rho_2^s(1-\mu^2)^{m/2}\sum_{\alpha=0}^{N}b_\alpha P_\alpha(2\mu-1)+2\delta_{0,m}\rho_2^d\sum_{\alpha=0}^{N}b_\alpha T_{\alpha,0}^0\right]. \tag{30b}$$

We note that $I(L,\mu)$ and $I(R,\mu)$ can now be considered known. We can thus write, as in I

$$I(\tau,\mu)=\sum_{\beta=0}^{\kappa-1}\left[A(\nu_\beta)\phi(\nu_\beta,\mu)e^{-\tau/\nu_\beta}+A(-\nu_\beta)\phi(-\nu_\beta,\mu)e^{\tau/\nu_\beta}\right]+\int_{-1}^{1}A(\nu)\phi(\nu,\mu)e^{-\tau/\nu}\,d\nu \tag{31}$$

and use the full-range orthogonality theorem of McCormick and Kuščer (1966), at either $\tau=L$ or $\tau=R$, to find the expansion coefficients $A(\nu_\beta)$ and $A(\nu)$. In this way the complete solution for $\tau\in(L,R)$ and $\mu\in(-1,1)$ is available. We prefer here to find $I(\tau,\mu)$ for $\tau\in(L,R)$ in a manner derived from the F_N method. From the analysis given in I, it is clear that equations (20) are valid for values of $\tau\in(L,R)$, in addition to $\tau=L$ and $\tau=R$. Thus we consider equation (20a) with L replaced by τ and equation (20b) with R replaced by τ, i.e.,

$$\int_{-1}^{1}\mu\phi(-\xi,\mu)I(\tau,\mu)\,d\mu=e^{-(R-\tau)/\xi}\left[\int_0^1\mu\phi(-\xi,\mu)I(R,\mu)\,d\mu-\int_0^1\mu\phi(\xi,\mu)I(R,-\mu)\,d\mu\right] \tag{32a}$$

and

$$\int_{-1}^{1}\mu\phi(\xi,\mu)I(\tau,\mu)\,d\mu=e^{-(\tau-L)/\xi}\left[\int_0^1\mu\phi(\xi,\mu)I(L,\mu)\,d\mu-\int_0^1\mu\phi(-\xi,\mu)I(L,-\mu)\,d\mu\right], \tag{32b}$$

with $\xi\in P$. For $\tau\in(L,R)$ we introduce the approximations, for $\mu>0$,

$$I(\tau,-\mu)=\Xi_2(\mu)e^{-(R-\tau)/\mu}+\frac{\omega}{2}(1-\mu^2)^{m/2}\sum_{\alpha=0}^{N}c_\alpha(\tau)P_\alpha(2\mu-1) \tag{33a}$$

and

$$I(\tau,\mu)=\Xi_1(\mu)e^{-(\tau-L)/\mu}+\frac{\omega}{2}(1-\mu^2)^{m/2}\sum_{\alpha=0}^{N}d_\alpha(\tau)P_\alpha(2\mu-1) \tag{33b}$$

into equations (32) to find

$$\sum_{\alpha=0}^{N}\{c_\alpha(\tau)B_\alpha(\xi)-\omega d_\alpha(\tau)A_\alpha(\xi)\}=2W_1(\tau,\xi) \tag{34a}$$

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and

$$\sum_{\alpha=0}^{N} \{d_\alpha(\tau)B_\alpha(\xi) - \omega c_\alpha(\tau)A_\alpha(\xi)\} = 2W_2(\tau,\xi). \tag{34b}$$

Here

$$W_1(\tau,\xi) = \Upsilon_1(\tau,\xi) - \tfrac{1}{2}e^{-(R-\tau)/\xi} \sum_{\alpha=0}^{N} b_\alpha\left[\omega A_\alpha(\xi) - \rho_2^s B_\alpha(\xi) - 2T_{\alpha,0}^0 \delta_{0,m}\rho_2^d B_0(\xi)\right] \tag{35}$$

and

$$W_2(\tau,\xi) = \Upsilon_2(\tau,\xi) - \tfrac{1}{2}e^{-(\tau-L)/\xi} \sum_{\alpha=0}^{N} a_\alpha\left[\omega A_\alpha(\xi) - \rho_1^s B_\alpha(\xi) - 2T_{\alpha,0}^0 \delta_{0,m}\rho_1^d B_0(\xi)\right]. \tag{36}$$

The functions $\Upsilon_1(\tau,\xi)$ and $\Upsilon_2(\tau,\xi)$ clearly can be evaluated by numerical integration, and so for any τ we can solve the system of linear algebraic equations

$$\sum_{\alpha=0}^{N} \left[c_\alpha(\tau)B_\alpha(\xi_\beta) - \omega d_\alpha(\tau)A_\alpha(\xi_\beta)\right] = 2W_1(\tau,\xi_\beta) \tag{37a}$$

and

$$\sum_{\alpha=0}^{N} \left[d_\alpha(\tau)B_\alpha(\xi_\beta) - \omega c_\alpha(\tau)A_\alpha(\xi_\beta)\right] = 2W_2(\tau,\xi_\beta) \tag{37b}$$

where $\beta = 0,1,2,\ldots,N$, to find the coefficients $c_\alpha(\tau)$ and $d_\alpha(\tau)$ required to complete the desired solution. We note that the matrix of coefficients in equations (37) does not depend on τ, and thus only one matrix inversion, for each N, is necessary.

IV. THE COMPLETE SOLUTION

We now use the F_N solutions given by equations (29), (30), and (33) to reconstruct the complete intensity $I(\tau,\mu,\phi)$. Thus on using equation (29) in equation (6) we find, for $\mu > 0$ and $\phi \in [0, 2\pi]$,

$$I(L,-\mu,\phi) = \frac{\omega}{2} \sum_{m=0}^{L} (1-\mu^2)^{m/2} \sum_{\alpha=0}^{N} \left[\cos m(\phi-\phi_0) a_{c,\alpha}^m + \sin m(\phi-\phi_0) a_{s,\alpha}^m\right] P_\alpha(2\mu-1)$$
$$+ \Xi(L,-\mu,\phi) + I_p(L,-\mu) \tag{38a}$$

and

$$I(R,\mu,\phi) = \frac{\omega}{2} \sum_{m=0}^{L} (1-\mu^2)^{m/2} \sum_{\alpha=0}^{N} \left[\cos m(\phi-\phi_0) b_{c,\alpha}^m + \sin m(\phi-\phi_0) b_{s,\alpha}^m\right] P_\alpha(2\mu-1)$$
$$+ \Xi(R,\mu,\phi) + I_p(R,\mu), \tag{38b}$$

where, in general, for $\mu > 0$ and $\phi \in [0, 2\pi]$,

$$\Xi(\tau,-\mu,\phi) = e^{-(R-\tau)/\mu}\left(1-\rho_1^s\rho_2^s e^{-2\Delta/\mu}\right)^{-1}\left[K_2(\mu,\phi) + \rho_2^s e^{-\Delta/\mu}K_1(\mu,\phi) + 2\left(\rho_2^d\Phi_2^0 + \rho_2^s\rho_1^d\Phi_1^0 e^{-\Delta/\mu}\right)\right] \tag{39a}$$

and

$$\Xi(\tau,\mu,\phi) = e^{-(\tau-L)/\mu}\left(1-\rho_1^s\rho_2^s e^{-2\Delta/\mu}\right)^{-1}\left[K_1(\mu,\phi) + \rho_1^s e^{-\Delta/\mu}K_2(\mu,\phi) + 2\left(\rho_1^d\Phi_1^0 + \rho_1^s\rho_2^d\Phi_2^0 e^{-\Delta/\mu}\right)\right]. \tag{39b}$$

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Here $K_1(\mu,\phi)$ and $K_2(\mu,\phi)$ are given by equations (14) while

$$\Phi_1^0=\left[1-2\left(\rho_1^d\rho_2^s+\rho_2^d\rho_1^s\right)J+4\rho_1^d\rho_2^d\left(\rho_1^s\rho_2^sJ^2-K^2\right)\right]^{-1}\left[\left(1-2\rho_1^s\rho_2^dJ\right)R_2+2\rho_2^dKR_1\right] \tag{40a}$$

and

$$\Phi_2^0=\left[1-2\left(\rho_1^d\rho_2^s+\rho_2^d\rho_1^s\right)J+4\rho_1^d\rho_2^d\left(\rho_1^s\rho_2^sJ^2-K^2\right)\right]^{-1}\left[2\rho_1^dKR_2+\left(1-2\rho_2^s\rho_1^dJ\right)R_1\right], \tag{40b}$$

where

$$R_1=\int_0^1 e^{-\Delta/\mu}\left(1-\rho_1^s\rho_2^se^{-2\Delta/\mu}\right)^{-1}\left[K_1^0(\mu)+\rho_1^sK_2^0(\mu)e^{-\Delta/\mu}\right]\mu d\mu, \tag{41a}$$

$$R_2=\int_0^1 e^{-\Delta/\mu}\left(1-\rho_1^s\rho_2^se^{-2\Delta/\mu}\right)^{-1}\left[K_2^0(\mu)+\rho_2^sK_1^0(\mu)e^{-\Delta/\mu}\right]\mu d\mu, \tag{41b}$$

$$K_\alpha^0(\mu)=\frac{1}{2\pi}\int_0^{2\pi}K_\alpha(\mu,\phi)d\phi, \quad \alpha=1 \quad \text{and} \quad 2, \tag{42}$$

$$J=\int_0^1 e^{-2\Delta/\mu}\left(1-\rho_1^s\rho_2^se^{-2\Delta/\mu}\right)^{-1}\mu d\mu, \tag{43a}$$

and

$$K=\int_0^1 e^{-\Delta/\mu}\left(1-\rho_1^s\rho_2^se^{-2\Delta/\mu}\right)^{-1}\mu d\mu. \tag{43b}$$

We note that the constants $\{a_{c,\alpha}^m\}$, $\{a_{s,\alpha}^m\}$, $\{b_{c,\alpha}^m\}$, and $\{b_{s,\alpha}^m\}$ appearing in equations (38) are to be found from appropriate versions of equations (26). Clearly once the constants have been deduced to establish equations (38) we can find $I(L,\mu,\phi)$ and $I(R,-\mu,\phi)$, for $\mu>0$ and $\phi\in[0,2\pi]$, from equations (5). For $\tau\in(L,R)$ we can use equation (33) in equation (6) to find, for all ϕ and $\mu>0$,

$$I(\tau,-\mu,\phi)=\frac{\omega}{2}\sum_{m=0}^{L}\sum_{\alpha=0}^{N}\left(1-\mu^2\right)^{m/2}\left[\cos m(\phi-\phi_0)c_{c,\alpha}^m(\tau)+\sin m(\phi-\phi_0)c_{s,\alpha}^m(\tau)\right]P_\alpha(2\mu-1)$$
$$+\Xi(\tau,-\mu,\phi)+I_p(\tau,-\mu) \tag{44a}$$

and

$$I(\tau,\mu,\phi)=\frac{\omega}{2}\sum_{m=0}^{L}\sum_{\alpha=0}^{N}\left(1-\mu^2\right)^{m/2}\left[\cos m(\phi-\phi_0)d_{c,\alpha}^m(\tau)+\sin m(\phi-\phi_0)d_{s,\alpha}^m(\tau)\right]P_\alpha(2\mu-1)$$
$$+\Xi(\tau,\mu,\phi)+I_p(\tau,\mu), \tag{44b}$$

where the coefficients $\{c_{c,\alpha}^m(\tau), d_{c,\alpha}^m(\tau)\}$, and $\{c_{s,\alpha}^m(\tau), d_{s,\alpha}^m(\tau)\}$ are to be found from equations (37).

V. NUMERICAL RESULTS

In order to demonstrate the computational aspects of the foregoing development, we now consider an atmospheric layer of thickness $\Delta=4$ illuminated by a solar beam from above and by an isotropic incident distribution from below. We take the upper boundary to be $L=0$ and thus write, for $\mu>0$ and $\phi\in[0,2\pi]$,

$$I(0,\mu,\phi)=\pi\delta(\mu-\mu_0)\delta(\phi-\phi_0)+\rho_1^sI(0,-\mu,\phi)+\frac{1}{\pi}\rho_1^d\int_0^{2\pi}\int_0^1 I(0,-\mu',\phi')\mu'd\mu'd\phi' \tag{45}$$

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TABLE 1

THE SCATTERING LAW

l	$(2l+1)f_l$	l	$(2l+1)f_l$
0 ...	1	5 ...	0.04725
1 ...	2.00916	6 ...	0.00671
2 ...	1.56339	7 ...	0.00068
3 ...	0.67407	8 ...	0.00005
4 ...	0.22215		

TABLE 2

EIGENVALUES FOR $\omega=0.9$

m	$\{\nu_\beta\}$
0	3.060603356062
	1.000000000002
1	1.098190371490
2–8 ...	...

and

$$I(4,-\mu,\phi)=0.4+\rho_2^s I(4,\mu,\phi)+\frac{1}{\pi}\rho_2^d\int_0^{2\pi}\int_0^1 I(4,\mu',\phi')\mu' d\mu' d\phi'. \tag{46}$$

We allow an internal source of the form

$$S(\tau)=S_0+S_1\tau+S_2\tau^2+S_3\tau^3, \tag{47}$$

where $S_0=0.01$, $S_1=0.002$, $S_2=0.0004$, and $S_3=-0.0001$. To complete the specification of this application we use the nine-term scattering law shown in Table 1, take $\omega=0.9$, $\mu_0=0.5$, and $\rho_1^s=\rho_2^s=\rho_1^d=\rho_2^d=0.25$, and let $\phi_r=\phi_0$. The scattering law used here is the same as that used in I and is based on the Mie theory for spherical particles with size parameter $\alpha=2$ and index of refraction $m=1.33$. Our choice of a collocation strategy is based on one introduced by Garcia and Siewert (1981), i.e., in equations (26) and (37) we use $\xi_\beta=\nu_\beta$, $\beta=0,1,2,\ldots,\kappa-1$, and

$$\xi_\beta=\frac{1}{2}+\frac{1}{2}\cos\left[\frac{2\beta-2\kappa+1}{2(N+1-\kappa)}\pi\right], \quad \beta=\kappa,\kappa+1,\ldots,N. \tag{48}$$

The points given by equation (48) are the zeros of the Chebyshev polynomial of the first kind $T_{N+1-\kappa}(2x-1)$.

In order to present our numerical results we first list in Table 2 the discrete eigenvalues for this problem. We then write

$$I(\tau,\mu,\phi)=I_*(\tau,\mu,\phi)+\Delta(\tau,\mu,\phi), \tag{49}$$

where, for $\mu>0$,

$$\Delta(\tau,-\mu,\phi)=\pi\left(1-\rho_1^s\rho_2^s e^{-2\Delta/\mu}\right)^{-1}\rho_2^s e^{-\Delta/\mu}e^{-(R-\tau)/\mu}\delta(\mu-\mu_0)\delta(\phi-\phi_0) \tag{50a}$$

and

$$\Delta(\tau,\mu,\phi)=\pi\left(1-\rho_1^s\rho_2^s e^{-2\Delta/\mu}\right)^{-1}e^{-(\tau-L)/\mu}\delta(\mu-\mu_0)\delta(\phi-\phi_0), \tag{50b}$$

and we list in Tables 3, 4, and 5 our final results for selected values of τ, μ, and ϕ. The entries in Tables 3, 4, and 5 were deduced from calculations based on $N=35$, 40, and 50 and are, we believe, correct to ±1 in the last digit reported.

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TABLE 3

THE COMPONENT $I_*(\tau, \mu, \phi)$ OF THE COMPLETE SOLUTION FOR $\phi - \phi_0 = 0$

μ	$\tau=0$	$\tau=1$	$\tau=2$	$\tau=3$	$\tau=4$
−1......	0.39191	0.40746	0.45654	0.52409	0.58833
−0.9 ...	0.44386	0.41415	0.44966	0.51776	0.59433
−0.8 ...	0.48707	0.41804	0.44055	0.50770	0.59577
−0.7 ...	0.53908	0.42469	0.43159	0.49576	0.59620
−0.6 ...	0.60318	0.43531	0.42349	0.48220	0.59640
−0.5 ...	0.68200	0.45092	0.41695	0.46716	0.59690
−0.4 ...	0.77816	0.47253	0.41279	0.45098	0.59811
−0.3 ...	0.89429	0.50121	0.41185	0.43460	0.60022
−0.2 ...	1.0326	0.53817	0.41476	0.42016	0.60333
−0.1 ...	1.1925	0.58508	0.42176	0.41024	0.60763
−0.0 ...	1.3423	0.64441	0.43328	0.40430	0.61460
0.0 ...	0.44240	0.64441	0.43328	0.40430	0.47496
0.1 ...	0.40494	0.71969	0.45028	0.40152	0.44707
0.2 ...	0.36497	0.80384	0.47436	0.40211	0.42988
0.3 ...	0.33039	0.86207	0.50579	0.40670	0.41744
0.4 ...	0.30136	0.88120	0.53889	0.41553	0.40899
0.5 ...	0.27732	0.86659	0.56552	0.42714	0.40417
0.6 ...	0.25761	0.82536	0.57968	0.43848	0.40215
0.7 ...	0.24159	0.76238	0.57776	0.44597	0.40137
0.8 ...	0.22858	0.67970	0.55720	0.44595	0.39964
0.9 ...	0.21778	0.57465	0.51383	0.43391	0.39387
1......	0.20480	0.38405	0.40193	0.38312	0.36985

TABLE 4

THE COMPLETE SOLUTION $I(\tau, \mu, \phi)$ FOR $\phi - \phi_0 = \frac{1}{2}\pi$

μ	$\tau=0$	$\tau=1$	$\tau=2$	$\tau=3$	$\tau=4$
−1......	0.39191	0.40746	0.45654	0.52409	0.58833
−0.9 ...	0.38994	0.39943	0.44482	0.51436	0.58850
−0.8 ...	0.38911	0.39223	0.43281	0.50333	0.58894
−0.7 ...	0.38951	0.38617	0.42074	0.49079	0.58970
−0.6 ...	0.39111	0.38158	0.40894	0.47658	0.59084
−0.5 ...	0.39375	0.37879	0.39792	0.46062	0.59244
−0.4 ...	0.39703	0.37805	0.38830	0.44312	0.59460
−0.3 ...	0.40017	0.37949	0.38075	0.42497	0.59743
−0.2 ...	0.40162	0.38308	0.37559	0.40832	0.60106
−0.1 ...	0.39788	0.38875	0.37271	0.39570	0.60576
−0.0 ...	0.37348	0.39633	0.37193	0.38645	0.61308
0.0 ...	0.20019	0.39633	0.37193	0.38645	0.46886
0.1 ...	0.20629	0.40525	0.37326	0.37951	0.43957
0.2 ...	0.20722	0.41243	0.37663	0.37473	0.42079
0.3 ...	0.20686	0.41353	0.38145	0.37203	0.40627
0.4 ...	0.20608	0.40987	0.38643	0.37121	0.39495
0.5 ...	0.20525	0.40415	0.39064	0.37180	0.38630
0.6 ...	0.20459	0.39812	0.39387	0.37329	0.37990
0.7 ...	0.20420	0.39271	0.39628	0.37532	0.37535
0.8 ...	0.20410	0.38842	0.39822	0.37767	0.37233
0.9 ...	0.20430	0.38549	0.40000	0.38027	0.37056
1......	0.20480	0.38405	0.40193	0.38312	0.36985

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TABLE 5

THE COMPLETE SOLUTION $I(\tau, \mu, \phi)$ FOR $\phi - \phi_0 = \pi$

μ	$\tau=0$	$\tau=1$	$\tau=2$	$\tau=3$	$\tau=4$
−1......	0.39191	0.40746	0.45654	0.52409	0.58833
−0.9 ...	0.36094	0.38929	0.44097	0.51169	0.58437
−0.8 ...	0.34765	0.37676	0.42708	0.49999	0.58447
−0.7 ...	0.33639	0.36524	0.41309	0.48697	0.58549
−0.6 ...	0.32593	0.35460	0.39912	0.47220	0.58712
−0.5 ...	0.31570	0.34491	0.38555	0.45552	0.58928
−0.4 ...	0.30526	0.33626	0.37295	0.43705	0.59197
−0.3 ...	0.29406	0.32858	0.36188	0.41765	0.59524
−0.2 ...	0.28142	0.32163	0.35259	0.39946	0.59924
−0.1 ...	0.26621	0.31512	0.34484	0.38503	0.60423
−0.0 ...	0.24425	0.30873	0.33830	0.37363	0.61184
0.0 ...	0.16788	0.30873	0.33830	0.37363	0.46390
0.1 ...	0.17337	0.30201	0.33277	0.36410	0.43349
0.2 ...	0.17717	0.29408	0.32805	0.35617	0.41350
0.3 ...	0.18033	0.28489	0.32382	0.34966	0.39752
0.4 ...	0.18313	0.27632	0.31993	0.34440	0.38442
0.5 ...	0.18574	0.26987	0.31671	0.34031	0.37367
0.6 ...	0.18830	0.26647	0.31488	0.33753	0.36503
0.7 ...	0.19092	0.26701	0.31549	0.33650	0.35850
0.8 ...	0.19373	0.27329	0.32031	0.33824	0.35442
0.9 ...	0.19705	0.29058	0.33367	0.34531	0.35402
1......	0.20480	0.38405	0.40193	0.38312	0.36985

APPENDIX

THE FUNCTIONS $A_\alpha(\xi)$ AND $B_\alpha(\xi)$

Since we are using the basis functions $P_\alpha(2\mu-1)$, rather than the powers μ^α, in our F_N method here, the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ are different from those reported in I. Here

$$A_\alpha(\xi)=\frac{2}{\omega\xi}\int_0^1 \mu P_\alpha(2\mu-1)\phi(-\xi,\mu)(1-\mu^2)^{m/2}d\mu \tag{A1a}$$

and

$$B_\alpha(\xi)=\frac{2}{\xi}\int_0^1 \mu P_\alpha(2\mu-1)\phi(\xi,\mu)(1-\mu^2)^{m/2}d\mu, \tag{A1b}$$

where the (generalized) functions $\phi(\xi,\mu)$ are those reported by McCormick and Kuščer (1966) and used in I. We can use the explicit expressions for the $\phi(\xi,\mu)$ to deduce a set of recursive relations that is useful for computing the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$. We find, for $\alpha>0$,

$$A_\alpha(\xi)=-\left(\frac{2\alpha-1}{\alpha}\right)(2\xi+1)A_{\alpha-1}(\xi)-\left(\frac{\alpha-1}{\alpha}\right)A_{\alpha-2}(\xi)+(-1)^m\frac{2(2\alpha-1)}{\alpha}\sum_{l=m}^{L}(2l+1)(-1)^l f_l^m g_l^m(\xi)T_{\alpha-1,l}^m \tag{A2a}$$

and

$$B_\alpha(\xi)=\left(\frac{2\alpha-1}{\alpha}\right)(2\xi-1)B_{\alpha-1}(\xi)-\left(\frac{\alpha-1}{\alpha}\right)B_{\alpha-2}(\xi)-\frac{2\omega(2\alpha-1)}{\alpha}\sum_{l=m}^{L}(2l+1)f_l^m g_l^m(\xi)T_{\alpha-1,l}^m, \tag{A2b}$$

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where, for $\alpha \geq 0$ and $l \geq m$,

$$T_{\alpha,l}^{m} = \int_0^1 \mu P_\alpha(2\mu - 1) P_l^m(\mu)(1-\mu^2)^{m/2} d\mu. \tag{A3}$$

The basic properties of the associated Legendre functions $P_l^m(\mu)$ and the Legendre polynomials can now be used to establish a recursive formula that provides a convenient way to compute the numbers $T_{\alpha,l}^m$ required in equations (A2). To avoid a loss of accuracy in this calculation, we find $T_{\alpha,l+1}^m$ from previously obtained values of $T_{\alpha-1,l}^m$, $T_{\alpha,l}^m$, $T_{\alpha+1,l}^m$, and $T_{\alpha,l-1}^m$, viz.,

$$T_{\alpha,l+1}^m = \left[\frac{2l+1}{2(l-m+1)}\right]\left[\left(\frac{\alpha}{2\alpha+1}\right)T_{\alpha-1,l}^m + T_{\alpha,l}^m + \left(\frac{\alpha+1}{2\alpha+1}\right)T_{\alpha+1,l}^m\right] - \left(\frac{l+m}{l-m+1}\right)T_{\alpha,l-1}^m, \tag{A4}$$

where $\alpha \geq 0$ and $l \geq m$. We observe that $T_{\alpha,\beta}^{\beta} = 0$ if $\alpha > 2\beta + 1$, and therefore to initiate our calculation we use

$$T_{0,0}^0 = \tfrac{1}{2}, \tag{A5a}$$

$$T_{1,0}^0 = \tfrac{1}{6}, \tag{A5b}$$

and subsequently

$$T_{\alpha,m+1}^{m+1} = \left[\frac{2(m+1)(2m+1)}{2m+3}\right]T_{\alpha,m}^m - \left(\frac{2}{2m+3}\right)T_{\alpha,m+2}^m. \tag{A6}$$

For the purpose of initiating the use of equations (A2) in a forward manner we note, from I, that either

$$A_0(\xi) = \sum_{l=m}^{L} (2l+1) f_l^m g_l^m(\xi) \int_0^1 \mu(1-\mu^2)^{m/2} P_l^m(\mu) \frac{d\mu}{\mu+\xi} \tag{A7a}$$

or

$$A_0(\xi) = \sum_{l=m}^{L} (2l+1) f_l^m g_l^m(\xi) \Pi_l^m(\xi) - \frac{2}{\omega}\xi\psi(\xi)\log\left(1+\frac{1}{\xi}\right), \tag{A7b}$$

with

$$B_0(\xi) = \omega A_0(\xi) + \left(\frac{2}{2m+1}\right)h_m \tag{A8}$$

can be used. Concerning the use of equations (A2) in a forward or backward manner, we employ the strategy suggested by Garcia and Siewert (1981). Thus for the functions $A_\alpha(\xi)$ we use the forward method for $\xi \in [0, 0.001]$ and the backward method for $\xi > 0.001$; for the functions $B_\alpha(\xi)$ we use the forward method for $\xi \in [0, 1.001]$ and backward for $\xi > 1.001$. In conclusion we note that care must be taken in initiating the use of equations (A2) in a backward manner.

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C. DEVAUX: Laboratoire d'Optique Atmosphérique, Université des Sciences et Techniques de Lille, Villeneuve d'Ascq, France

C. E. SIEWERT: Mathematics and Nuclear Engineering Departments, North Carolina State University, Raleigh, NC 27650

Y. L. YUAN: Nuclear Engineering Department, North Carolina State University, Raleigh, NC 27650

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