On Angular Flux Computations in Neutron-Transport Theory

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ABSTRACT

The use of the F_N method for computing angular fluxes in neutron-transport theory is reviewed, and accurate results for a basic problem are reported.

I. INTRODUCTION

In two recent works[1,2] the F_N method[3-5] was extended in order to compute in an accurate and concise manner particle distribution functions at any position in a finite slab. First of all, in regard to radiative transfer, Devaux, Siewert, and Yuan[1] computed the azimuthally dependent angular intensity $I(\tau,\mu,\phi)$, for all τ, μ, and ϕ, in a finite layer illuminated by a solar beam and an isotropic incident distribution, with internal emission, with specular and diffuse reflection at both boundaries and with eighth-order anisotropic scattering. Subsequently, Garcia, Pomraning, and Siewert[2] used the extended F_N method to compute the distribution function $\psi(\tau,u)$, for all τ and u, relevant to the transport of neutral hydrogen atoms in a hydrogen plasma. In this brief Note we review, in the context of neutron-transport theory, the use of the F_N method for interior angular-flux calculations for the simple problem of a constant source in a finite isotropically scattering slab. To keep this Note brief we assume that Refs. 4 and 5 are available, and we use the notation and basis functions $P_\alpha(2\mu - 1)$ recently introduced for the F_N method.[6] Thus, we let Δ denote the slab thickness and consider, for $c < 1$,

$$\mu \frac{\partial}{\partial x} \psi(x,\mu) + \psi(x,\mu) = \frac{c}{2} \int_{-1}^{1} \psi(x,\mu')d\mu' + S \ , \tag{1}$$

$$\psi(0,\mu) = 0 \ , \quad \mu > 0 \ , \tag{2a}$$

and

$$\psi(\Delta,-\mu) = 0 \ , \quad \mu > 0 \ . \tag{2b}$$

If we substitute the approximations, for $\mu > 0$,

$$\psi(0,-\mu) = \psi(\Delta,\mu) = S[1 - \exp(-\Delta/\mu)] + \sum_{\alpha=0}^{N} a_\alpha P_\alpha(2\mu - 1) \tag{3}$$

into the exact system of singular integral equations and constraints,[4,5,7]

$$\int_{-1}^{1} \mu\phi(\xi,\mu)\psi_*(0,-\mu)d\mu + \exp(-\Delta/\xi) \times \int_{-1}^{1} \mu\phi(-\xi,\mu)\psi_*(\Delta,\mu)d\mu = 0 \ , \tag{4}$$

where $\xi \in \nu_0 \cup [0,1]$, $\phi(\xi,\mu)$ are the elementary solutions reported by Case[8] and $\psi_*(x,\mu) = \psi(x,\mu) - S/(1 - c)$, and evaluate the resulting equations at $N + 1$ selected values of ξ, say $\{\xi_\beta\}$, we find the F_N equations to be solved to establish the surface results:

$$\sum_{\alpha=0}^{N} a_\alpha [B_\alpha(\xi_\beta) + c \exp(-\Delta/\xi_\beta)A_\alpha(\xi_\beta)] = cS\{2[1 - \exp(-\Delta/\xi_\beta)] - R(\Delta,\xi_\beta)\} \ , \tag{5}$$

where, in general,

$$R(x,\xi) = \int_0^1 \mu\{C(x:\mu,\xi) + \exp[-(\Delta - x)/\mu] S(x:\mu,\xi)\}d\mu \ , \tag{6}$$

$$C(x:\mu,\xi) = \frac{\exp(-x/\mu) - \exp(-x/\xi)}{\mu - \xi} \ , \tag{7a}$$

and

$$S(x:\mu,\xi) = \frac{1 - \exp(-x/\mu)\exp(-x/\xi)}{\mu + \xi} \ . \tag{7b}$$

For the collocation points $\{\xi_\beta\}$, we use $\xi_0 = \nu_0$ and for $N > 0$ the zeros of the Chebyshev polynomial of the first kind $T_N(2x - 1)$, i.e.,

$$\xi_\beta = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\beta - 1}{2N}\pi\right) \ , \quad \beta = 1, 2, \ldots, N \ . \tag{8}$$

Therefore, after computing the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ as discussed in Ref. 6, we can readily solve the system of

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[6] R. D. M. GARCIA and C. E. SIEWERT, *Nucl. Sci. Eng.*, **78**, 315 (1981).

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linear algebraic equations given by Eq. (5) to find the constants $\{a_\alpha\}$. We note that Eq. (4) is a special case of the more general equations

$$\int_{-1}^{1} \mu\phi(\xi,\mu)\psi_*(x,-\mu)d\mu + \exp[-(\Delta - x)/\xi] \times \int_{-1}^{1} \mu\phi(-\xi,\mu)\psi_*(\Delta,\mu)d\mu = 0 \quad (9a)$$

and

$$\int_{-1}^{1} \mu\phi(-\xi,\mu)\psi_*(x,-\mu)d\mu + \exp(-x/\xi) \times \int_{-1}^{1} \mu\phi(\xi,\mu)\psi_*(0,\mu)d\mu = 0 \quad (9b)$$

available from Ref. 5. Therefore, we can substitute the approximations, for any $x \in (0,\Delta)$ and $\mu \geq 0$,

$$\psi(x,-\mu) = S\{1 - \exp[-(\Delta - x)/\mu]\} + \sum_{\alpha=0}^{N} c_\alpha(x)P_\alpha(2\mu - 1) \quad (10a)$$

and

$$\psi(x,\mu) = S[1 - \exp(-x/\mu)] + \sum_{\alpha=0}^{N} d_\alpha(x)P_\alpha(2\mu - 1) \quad (10b)$$

into Eqs. (9) to find the system of linear algebraic equations

$$\sum_{\alpha=0}^{N} [c_\alpha(x)B_\alpha(\xi_\beta) - cd_\alpha(x)A_\alpha(\xi_\beta)] = c\left(2S\{1 - \exp[-(\Delta - x)/\xi_\beta]\} - SR(\Delta - x,\xi_\beta) - \exp[-(\Delta - x)/\xi_\beta] \sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi_\beta)\right) \quad (11a)$$

and

$$\sum_{\alpha=0}^{N} [d_\alpha(x)B_\alpha(\xi_\beta) - cc_\alpha(x)A_\alpha(\xi_\beta)] = c\left\{2S[1 - \exp(-x/\xi_\beta)] - SR(x,\xi_\beta) - \exp(-x/\xi_\beta) \sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi_\beta)\right\} . \quad (11b)$$

II. NUMERICAL RESULTS

For a given value of c, we first compute ν_0, the positive zero of

$$\Lambda(z) = 1 + \frac{c}{2} z \int_{-1}^{1} \frac{d\mu}{\mu - z} . \quad (12)$$

We then evaluate the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ at the collocation points defined in Sec. I and solve the system of linear algebraic equations given by Eq. (5) to establish the constants $\{a_\alpha\}$. Having found these constants, we then solve, for selected values of $x \in (0,\Delta)$, the linear system given by Eqs. (11) to find $\{c_\alpha(x)\}$ and $\{d_\alpha(x)\}$.

We note that the functions $B_\alpha(\xi)$ defined for $\xi \in \nu_0 \cup [0,1]$ can be expressed as

$$B_\alpha(\xi) = 2P_\alpha(2\xi - 1) - cU_\alpha(\xi) , \quad (13)$$

where

$$U_\alpha(\xi) = G_\alpha(\xi) + \left[1 + \xi \ln\left(1 + \frac{1}{\xi}\right)\right]P_\alpha(2\xi - 1) \quad (14)$$

and the polynomials $G_\alpha(\xi)$ are those discussed in Ref. 6. In addition, we recall that before restricting ξ to the collocation set $\{\xi_\beta\}$, we had a version of Eq. (5) valid for all $\xi \in \nu_0 \cup [0,1]$, i.e.,

$$\sum_{\alpha=0}^{N} a_\alpha[B_\alpha(\xi) + c\exp(-\Delta/\xi)A_\alpha(\xi)] = cS\{2[1 - \exp(-\Delta/\xi)] - R(\Delta,\xi)\} . \quad (15)$$

It follows that we can use Eq. (13) in Eq. (15) for $\xi = \mu \in [0,1]$ and note Eq. (3) to find

$$\psi(0,-\mu) = \psi(\Delta,\mu) = (1 + c)S[1 - \exp(-\Delta/\mu)] + \frac{c}{2}\left\{\sum_{\alpha=0}^{N} a_\alpha[U_\alpha(\mu) - \exp(-\Delta/\mu)A_\alpha(\mu)] - SR(\Delta,\mu)\right\} . \quad (16)$$

In a similar way we deduce that for $x \in (0,\Delta)$ and $\mu \in [0,1]$

$$\psi(x,-\mu) = (1 + c)S\{1 - \exp[-(\Delta - x)/\mu]\} + \frac{c}{2}\left[\sum_{\alpha=0}^{N} \left(c_\alpha(x)U_\alpha(\mu) + \{d_\alpha(x) - \exp[-(\Delta - x)/\mu]a_\alpha\}A_\alpha(\mu)\right) - SR(\Delta - x,\mu)\right] \quad (17a)$$

and

$$\psi(x,\mu) = (1 + c)S[1 - \exp(-x/\mu)] + \frac{c}{2}\left(\sum_{\alpha=0}^{N} \{d_\alpha(x)U_\alpha(\mu) + [c_\alpha(x) - \exp(-x/\mu)a_\alpha]A_\alpha(\mu)\} - SR(x,\mu)\right) . \quad (17b)$$

In regard to computing the angular flux, we have found Eqs. (16), (17a), and (17b) to be significant improvements, especially as $\mu \to 0$, over the usual expressions given by Eqs. (3), (10a), and (10b). In fact, we have been able to compute angular fluxes from Eqs. (16), (17a), and (17b) correct to, we believe, seven significant figures for all μ. In Tables I through IV we list some selected results, which we believe correct to ±1 in the last digit reported, obtained from Eqs. (16), (17a), and (17b) and N typically between 20 and 30.

In conclusion, we note that the matrix elements in Eqs. (11) do not depend on x, so that one matrix inversion is sufficient for all $x \in (0,\Delta)$.

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TABLE I

The Normalized Angular Flux $\psi(x,\mu)/S$ for $c = 0.1$ and $\Delta = 1$

μ	$x = 0$	$x = \Delta/8$	$x = \Delta/4$	$x = 3\Delta/8$	$x = \Delta/2$
−1.0	0.67303	0.62226	0.56339	0.49603	0.41940
−0.9	0.71419	0.66350	0.60377	0.53441	0.45437
−0.8	0.75958	0.70972	0.64975	0.57880	0.49546
−0.7	0.80937	0.76148	0.70227	0.63051	0.54428
−0.6	0.86328	0.81904	0.76223	0.69110	0.60298
−0.5	0.92003	0.88191	0.83011	0.76218	0.67431
−0.4	0.97627	0.94768	0.90492	0.84463	0.76140
−0.3	1.0248	1.0097	0.98146	0.93609	0.86608
−0.2	1.0537	1.0536	1.0445	1.0231	0.98106
−0.1	1.0566	1.0653	1.0690	1.0691	1.0634
−0.0	1.0452	1.0607	1.0674	1.0709	1.0719
0.0	---	1.0607	1.0674	1.0709	1.0719
0.1	---	0.75344	0.97580	1.0428	1.0634
0.2	---	0.49043	0.75735	0.90244	0.98106
0.3	---	0.35952	0.59980	0.75985	0.86608
0.4	---	0.28313	0.49286	0.64761	0.76140
0.5	---	0.23333	0.41720	0.56147	0.67431
0.6	---	0.19837	0.36127	0.49444	0.60298
0.7	---	0.17249	0.31837	0.44119	0.54428
0.8	---	0.15257	0.28449	0.39803	0.49546
0.9	---	0.13677	0.25707	0.36241	0.45437
1.0	---	0.12393	0.23445	0.33256	0.41940

TABLE II

The Normalized Angular Flux $\psi(x,\mu)/S$ for $c = 0.5$ and $\Delta = 1$

μ	$x = 0$	$x = \Delta/8$	$x = \Delta/4$	$x = 3\Delta/8$	$x = \Delta/2$
−1.0	0.90964	0.84913	0.77111	0.67770	0.56958
−0.9	0.96504	0.90559	0.82670	0.73047	0.61734
−0.8	1.0260	0.96890	0.89007	0.79157	0.67350
−0.7	1.0928	1.0398	0.96257	0.86289	0.74034
−0.6	1.1647	1.1187	1.0455	0.94665	0.82087
−0.5	1.2399	1.2049	1.1397	1.0452	0.91902
−0.4	1.3132	1.2951	1.2440	1.1602	1.0394
−0.3	1.3734	1.3795	1.3513	1.2890	1.1853
−0.2	1.4013	1.4372	1.4407	1.4139	1.3484
−0.1	1.3801	1.4431	1.4742	1.4843	1.4726
−0.0	1.3020	1.4083	1.4581	1.4843	1.4925
0.0	---	1.4083	1.4581	1.4843	1.4925
0.1	---	0.98111	1.3090	1.4278	1.4726
0.2	---	0.63681	1.0093	1.2253	1.3484
0.3	---	0.46635	0.79724	1.0278	1.1853
0.4	---	0.36708	0.65422	0.87411	1.0394
0.5	---	0.30243	0.55333	0.75685	0.91902
0.6	---	0.25706	0.47888	0.66589	0.82087
0.7	---	0.22349	0.42185	0.59379	0.74034
0.8	---	0.19766	0.37684	0.53543	0.67350
0.9	---	0.17718	0.34045	0.48733	0.61734
1.0	---	0.16053	0.31043	0.44705	0.56958

TABLE III

The Normalized Angular Flux $\psi(x,\mu)/S$ for $c = 0.9$ and $\Delta = 1$

μ	$x = 0$	$x = \Delta/8$	$x = \Delta/4$	$x = 3\Delta/8$	$x = \Delta/2$
−1.0	1.4096	1.3305	1.2128	1.0639	0.88773
−0.9	1.4951	1.4193	1.3009	1.1473	0.96262
−0.8	1.5890	1.5189	1.4014	1.2441	1.0508
−0.7	1.6914	1.6305	1.5165	1.3572	1.1560
−0.6	1.8013	1.7548	1.6486	1.4905	1.2830
−0.5	1.9150	1.8906	1.7991	1.6481	1.4383
−0.4	2.0235	2.0323	1.9664	1.8329	1.6299
−0.3	2.1071	2.1641	2.1400	2.0421	1.8642
−0.2	2.1300	2.2498	2.2861	2.2496	2.1315
−0.1	2.0526	2.2398	2.3382	2.3740	2.3484
−0.0	1.8309	2.1324	2.2857	2.3692	2.3960
0.0	---	2.1324	2.2857	2.3692	2.3960
0.1	---	1.4522	2.0080	2.2455	2.3484
0.2	---	0.93948	1.5363	1.9083	2.1315
0.3	---	0.68722	1.2099	1.5934	1.8642
0.4	---	0.54061	0.99129	1.3518	1.6299
0.5	---	0.44524	0.83761	1.1687	1.4383
0.6	---	0.37836	0.72443	1.0271	1.2830
0.7	---	0.32889	0.63786	0.91519	1.1560
0.8	---	0.29085	0.56960	0.82477	1.0508
0.9	---	0.26067	0.51445	0.75033	0.96262
1.0	---	0.23617	0.46898	0.68805	0.88773

TABLE IV

The Normalized Angular Flux $\psi(x,\mu)/S$ for $c = 0.9$ and $\Delta = 10$

μ	$x = 0$	$x = \Delta/8$	$x = \Delta/4$	$x = 3\Delta/8$	$x = \Delta/2$
−1.0	5.7945	7.8168	8.6984	8.9842	8.8180
−0.9	5.6455	7.7552	8.6841	9.0086	8.8877
−0.8	5.4824	7.6859	8.6632	9.0254	8.9477
−0.7	5.3038	7.6087	8.6361	9.0350	8.9979
−0.6	5.1073	7.5232	8.6030	9.0380	9.0386
−0.5	4.8903	7.4287	8.5638	9.0350	9.0706
−0.4	4.6485	7.3243	8.5185	9.0265	9.0950
−0.3	4.3761	7.2089	8.4668	9.0129	9.1131
−0.2	4.0631	7.0809	8.4081	8.9942	9.1256
−0.1	3.6888	6.9385	8.3418	8.9702	9.1329
−0.0	3.1479	6.7789	8.2668	8.9408	9.1353
0.0	---	6.7789	8.2668	8.9408	9.1353
0.1	---	6.5983	8.1818	8.9055	9.1329
0.2	---	6.3850	8.0852	8.8636	9.1256
0.3	---	6.1138	7.9737	8.8142	9.1131
0.4	---	5.7955	7.8420	8.7556	9.0950
0.5	---	5.4615	7.6869	8.6858	9.0706
0.6	---	5.1347	7.5102	8.6028	9.0386
0.7	---	4.8269	7.3172	8.5057	8.9979
0.8	---	4.5427	7.1140	8.3950	8.9477
0.9	---	4.2829	6.9061	8.2725	8.8877
1.0	---	4.0464	6.6977	8.1403	8.8180