Green's Functions for the One-Speed Transport Equation in Spherical Geometry

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(Received 4 May 1967)

Several problems in one-speed neutron transport theory for spherically symmetrical systems are discussed. The singular eigenfunction expansion technique is used to construct a solution for a specific finite-slab Green's function. This slab solution is then used to construct the finite-medium spherical Green's function by extending the point-to-plane transformation concept. For the general case, the expansion coefficients are shown to obey a Fredholm equation, and first-order solutions are obtained; however, in the infinite-medium limit the solution is represented in closed form. In addition, the solution for the angular density in an infinite-medium due to an isotropic point source is developed directly from the set of normal modes of the transport equation. A proof that the result so obtained obeys the proper source condition at the origin is given.

I. INTRODUCTION

The singular eigenfunction expansion technique that was introduced by Case has been used extensively to develop exact solutions for many problems in neutron transport theory.[1–5] In addition, several applications of the method have been made in the field of radiative transfer in stellar atmospheres.[6–8] Although the class of problems for which the Case method has been used is a broad one, the major limitation appears to be the restriction to plane geometries. The purpose of this paper is to present an extension of this method in order to solve for the Green's function in spherical geometry and thus to establish a procedure by which rigorous solutions for such problems can be obtained.

It has been shown that the integral form of the homogeneous Boltzmann equation in spherical geometry can be related to the integral equation for a corresponding problem in slab geometry.[9,10] An extension of this technique to include inhomogeneous source terms is described in this paper. Thus, several problems in spherical geometry can be solved by inspection once the slab solution is known.[9,10,11]

In addition to his work on the integral transport equation, Davison also found a set of solutions to the homogeneous version of the differential transport equation.[9,10] In a more recent work, Mitsis found two classes of eigensolutions of the transport equation in spherical geometry: one regular and the other highly singular at the origin.[12]

In Sec. II we solve the problem of a spherical-shell source in a finite medium by solving the integral equation that defines the density. Several limiting situations are investigated; and we obtain, as a special case, the density due to an isotropic point source in a finite sphere. For the finite-medium Green's function, one must solve a Fredholm equation for the expansion coefficients; hence an approximate solution, in the spirit of the "wide slab," is obtained.[13,14] In the infinite-medium limit the solution is expressed in closed form.

Although the integral equation approach is convenient for obtaining the density in the single-region problems considered here, finding the angular density, the current, or the higher moments requires further work. In order to illustrate a procedure by which the angular density can be obtained directly from the normal modes of the spherical transport equation, Sec. III is devoted to the solution of the infinite-medium point source problem. Thus, by making expansions in terms of the normal modes of the homogeneous equation and by properly determining the expansion coefficients from the boundary conditions, we are able to solve for the angular density in a manner analogous to that used in plane geometry.

[1] K. M. Case, Ann. Phys. (New York) 9, 1 (1960).
[2] P. F. Zweifel, Michigan Memorial Phoenix Project Report, University of Michigan (1964).
[3] N. J. McCormick and I. Kuščer, J. Math. Phys. 6, 1939 (1965).
[4] J. Mika, Nucl. Sci. Eng. 22, 235 (1965).
[5] R. C. Erdmann, Ph.D. thesis, California Institute of Technology (1966).
[6] C. E. Siewert and P. F. Zweifel, Ann. Phys. (New York) 36, 61 (1966).
[7] C. E. Siewert and P. F. Zweifel, J. Math. Phys. 7, 2092 (1966).
[8] C. E. Siewert and S. K. Fraley, Ann. Phys. (New York) 43, 338 (1967).
[9] B. Davison, Canadian Report MT-112, National Research Council of Canada, Division of Atomic Energy (1945).
[10] B. Davison, *Neutron Transport Theory* (Oxford University Press, London, 1957).
[11] A Leonard and T. W. Mullikin, Proc. Natl. Acad. Sci. 52, 683 (1964).
[12] G. J. Mitsis, Argonne National Laboratory Report ANL-6787 (1963).
[13] M. R. Mendelson, Ph.D. thesis, University of Michigan (1964).
[14] K. M. Case, Michigan Memorial Phoenix Project Report, University of Michigan (1961).

The current and higher moments can then be obtained easily by integration. This procedure has additional merit since it may be a more useful way in which to approach the solution for multiregion problems where the integral equation approach becomes unmanageable.[15,16]

II. SOLUTION IN A FINITE SPHERE CONTAINING A SPHERICAL-SHELL SOURCE

A. Integral Equation in Spherical Geometry

We consider a single region of radius R in which a source of neutrons has been placed. The medium scatters neutrons isotropically in the laboratory system and no energy degradation of the neutrons is permitted. The source is taken to be spherically symmetric and has an isotropic emission character. Thus, we write the integral equation for the steady state neutron density as

$$\rho(r_0; r) = \int_0^R \int_0^\pi \int_0^{2\pi} r'^2\, dr' \sin\theta'\, d\theta'\, d\varphi' \times \left[\frac{c}{4\pi}\rho(r_0; r') + \frac{S}{4\pi r_0^2}\delta(r' - r_0)\right] \frac{e^{-|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|^2}, \quad 0 \le r,\ r_0 \le R. \quad (1)$$

The total source strength in Eq. (1) is S and distances are measured in units of mean free paths. In addition, the mean number of secondary neutrons per collision is denoted by c. It is possible to relax the requirement of an isotropic emission character for the source by treating the first collision neutrons as the source in Eq. (1) and then adding in the uncollided source neutrons.[17] The point source at the center of a sphere is a limiting case of Eq. (1).

The technique used to solve Eq. (1) was suggested by Davison and has been used by Mitsis for solving critical problems.[10,12] Its basis lies in the similarity of a modified form of Eq. (1) and the integral transport equation in slab geometry. Since solutions to the latter are readily obtained, one need only determine the relationship between spherical and planar problems in order to obtain an explicit solution to Eq. (1).[11] To cast Eq. (1) into a more appropriate form, we perform the two angular integrations. This yields

$$r\rho(r_0; r) = \frac{c}{2}\int_0^R r'\rho(r_0; r') \times \{E_1(|r - r'|) - E_1(|r + r'|)\}\, dr' + \frac{S}{2r_0}\{E_1(|r - r_0|) - E_1(|r + r_0|)\}, \quad 0 \le r,\ r_0 \le R, \quad (2)$$

where

$$E_1(x) = \int_0^1 \frac{e^{-x/y}}{y}\, dy. \quad (3)$$

Extending the range of r to $-R \le r \le R$ and demanding that $\rho(r_0; -r) = \rho(r_0; r)$ permits us to write

$$r\rho(r_0; r) = \frac{c}{2}\int_{-R}^R t\rho(r_0; t)E_1(|r - t|)\, dt + \frac{S}{2r_0}\{E_1(|r - r_0|) - E_1(|r + r_0|)\}, \quad 0 \le r_0 \le R, \quad -R \le r \le R. \quad (4)$$

In the limit as r_0 approaches zero, Eq. (4) yields, for a centrally located point source,

$$r\rho(0; r) = \frac{c}{2}\int_{-R}^R t\rho(0; t)E_1(|r - t|)\, dt + \frac{Se^{-|r|}}{r}, \quad -R \le r \le R. \quad (5)$$

Thus, by solving Eq. (4), we can obtain the point source solution by taking the limit used to construct Eq. (5).[18,19]

In the finite slab, $-R \le r \le R$, the integral equation that describes the neutron density resulting from a unit plane isotropic source at $r_0 > 0$ is

$$\phi(r_0; r) = \frac{c}{2}\int_{-R}^R \phi(r_0; t)E_1(|r - t|)\, dt + E_1(|r - r_0|). \quad (6)$$

Hence, a linear combination of slab solutions can be used to construct an integral equation identical in form to Eq. (4), i.e.,

$$[\phi(r_0; r) - \phi(-r_0; r)] = \frac{c}{2}\int_{-R}^R [\phi(r_0; t) - \phi(-r_0; t)]E_1(|r - t|)\, dt + E_1(|r - r_0|) - E_1(|r + r_0|). \quad (7)$$

Equating the dependent variables of Eqs. (4) and (7) and taking into account the source normalization, we obtain

$$r\rho(r_0; r) = (S/2r_0)[\phi(r_0; r) - \phi(-r_0; r)]. \quad (8)$$

We note from Eq. (6) that $\phi(-r_0; r) = \phi(r_0; -r)$; this allows us to write Eq. (8) in the alternate form

$$\rho(r_0; r) = (S/2r_0r)[\phi(r_0; r) - \phi(r_0; -r)], \quad 0 \le r,\ r_0 \le R. \quad (9)$$

This result contains the point-to-plane transformation

[15] The problem of a point source in one of two dissimilar half-spaces was solved in Ref. 16; however, the total cross section was taken to be the same in the two regions.

[16] R. C. Erdmann, Trans. Am. Nucl. Soc. 9, 443 (1966).

[17] K. M. Case, F. de Hoffmann, and G. Placzek, *Introduction to the Theory of Neutron Diffusion* (United States Government Printing Office, Washington, 1953), Vol. 1.

[18] The solution to the point source problem in a finite sphere has been obtained independently by Smith (Ref. 19). He used a transform technique similar to that used by Mitsis (Ref. 12).

[19] O. J. Smith (private communication).

for the infinite medium as a special case[20]:

$$\rho(0; r) = -\frac{S}{r}\frac{d}{dr}\phi(0; r), \quad R \to \infty. \tag{10}$$

For finite R, correction terms can be added to Eq. (10) to account for the additional leakage.

B. General Solution for $\rho(r_0; r)$

An expression for $\rho(r_0; r)$ can be obtained once $\phi(r_0; r)$ is determined; this, in turn, is found by using the method suggested by Case.[1] Since this development parallels previous work, the discussion of it is brief and will be presented mainly as an aid in defining the notation.[14]

The finite-slab Green's function is defined by

$$\mu\frac{\partial}{\partial r}\Psi(r_0; r, \mu) + \Psi(r_0; r, \mu) = \frac{c}{2}\int_{-1}^{1}\Psi(r_0; r, \mu')\,d\mu' + \delta(r - r_0), \quad r_0 > 0, \quad -R \le r \le R, \tag{11}$$

with

$$\Psi(r_0; \pm R, \mu) = 0, \quad \mu \lessgtr 0. \tag{12}$$

The neutron density is

$$\phi(r_0; r) = \int_{-1}^{1}\Psi(r_0; r, \mu)\,d\mu. \tag{13}$$

The general solution to Eq. (11) is

$$\Psi(r_0; r, \mu) = (A_+ \pm K_+)\phi_+(\mu)e^{-r/\nu_0} + (A_- \mp K_-)\phi_-(\mu)e^{r/\nu_0} + \int_{-1}^{1}d\nu[A(\nu) \pm K(\nu)]\phi_\nu(\mu)e^{-r/\nu}, \quad r \gtrless r_0. \tag{14}$$

Here,[14]

$$K_\pm = \frac{e^{\pm r_0/\nu_0}}{2N_+}, \tag{15a}$$

$$K(\nu) = \frac{e^{r_0/\nu_0}}{2N(\nu)}, \tag{15b}$$

$$\phi_\pm(\mu) = \frac{c\nu_0}{2}\frac{1}{\nu_0 \mp \mu}, \tag{15c}$$

$$\phi_\nu(\mu) = \frac{c\nu}{2}\frac{P}{\nu - \mu} + \lambda(\nu)\delta(\nu - \mu), \tag{15d}$$

$$\lambda(\nu) = 1 - c\nu\tanh^{-1}\nu, \tag{15e}$$

$$1 - c\nu_0\tanh^{-1}1/\nu_0 = 0, \tag{15f}$$

$$N_+ = \frac{c\nu_0}{2}\left(\frac{c\nu_0^2}{\nu_0^2 - 1} - 1\right), \tag{15g}$$

and

$$N(\nu) = \nu\left[\lambda^2(\nu) + \frac{c^2\pi^2\nu^2}{4}\right]. \tag{15h}$$

[20] K. M. Case and P. F. Zweifel, *An Introduction to Linear Transport Theory* (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1967).

To determine the coefficients $A_\pm$ and $A(\nu)$, we apply the boundary conditions at $\pm R$. Hence

$$0 = (A_+ \pm K_+)\phi_+(\mu)e^{\mp R/\nu_0} + (A_- \mp K_-)e^{\pm R/\nu_0}\phi_-(\mu) + \int_{-1}^{1}[A(\nu) \pm K(\nu)]\phi_\nu(\mu)e^{\mp R/\nu}\,d\nu, \quad \mu \lessgtr 0. \tag{16}$$

The conditions given by Eq. (16) are sufficient to specify uniquely all of the unknown expansion coefficients; however, they are not expressible in closed form. We defer a further discussion on the evaluation of $A_\pm$ and $A(\nu)$ to Sec. IIC.

Since $\phi_\pm(\mu)$ and $\phi_\nu(\mu)$ are normalized to unity, the expression for the density is obtained immediately from Eq. (14):

$$\phi(r_0; r) = (A_+ \pm K_+)e^{-r/\nu_0} + (A_- \mp K_-)e^{r/\nu_0} + \int_{-1}^{1}[A(\nu) \pm K(\nu)]e^{-r/\nu}\,d\nu, \quad r \gtrless r_0. \tag{17}$$

Substituting Eq. (17) into Eq. (9), we obtain the solution for the density in the finite sphere:

$$\rho(r_0; r) = \frac{S}{rr_0}\Bigg[(A_- - A_+)\sinh r/\nu_0 + \frac{1}{2N_+}\left\{\sinh\left(\frac{r + r_0}{\nu_0}\right) - \sinh\frac{|r - r_0|}{\nu_0}\right\} + \int_0^1\left\{[A(-\nu) - A(\nu)]\sinh r/\nu + \frac{1}{2N(\nu)}\left[\sinh\left(\frac{r + r_0}{\nu}\right) - \sinh\frac{|r - r_0|}{\nu}\right]\right\}d\nu\Bigg], \quad 0 \le r, \quad r_0 \le R. \tag{18}$$

C. First-Order Solution for the Expansion Coefficients

The two expressions for $A_\pm$ and $A(\nu)$ given by Eq. (16) can be simplified by using the half-range orthogonality theorem proved by Kuščer, McCormick, and Summerfield, i.e.,[21]

$$\int_0^1(\nu_0 - \mu)\gamma(\mu)\phi_\nu(\mu)\phi_{\nu'}(\mu)\,d\mu = 0, \quad \nu \ne \nu'; \nu, \nu' > 0. \tag{19}$$

Here,

$$(\nu_0 - \mu)\gamma(\mu) = c\mu/2(1 - c)(\nu_0 + \mu)X(-\mu), \tag{20a}$$

$$X(z) = \frac{1}{1 - z}\exp\left[\frac{1}{\pi}\int_0^1 \arg\Lambda^+(\mu)\frac{d\mu}{\mu - z}\right], \tag{20b}$$

and

$$\Lambda^\pm(\mu) = \lambda(\mu) \pm i(\pi c\mu/2). \tag{20c}$$

[21] I. Kuščer, N. J. McCormick, and G. C. Summerfield, Ann. Phys. (New York) **30**, 411 (1964).

To use directly the results for the various normalization integrals and cross products given by Kuščer, McCormick, and Summerfield, we make several changes of variables in Eq. (16) to obtain the two equations (one associated with the upper signs and one with the lower signs)

$$-\int_0^1 d\nu e^{-R/\nu}\phi_{-\nu}(\mu)[A(\pm\nu) \pm K(\pm\nu)]$$
$$= (A_+ \pm K_+)\phi_\mp(\mu)e^{\mp R/\nu_0} + (A_- \mp K_-)\phi_\pm(\mu)e^{\pm R/\nu_0}$$
$$+ \int_0^1 d\nu e^{R/\nu}\phi_\nu(\mu)[A(\mp\nu) \pm K(\mp\nu)], \quad \mu > 0. \quad (21)$$

If we now multiply Eq. (21) by $(\nu_0 - \mu)\gamma(\mu)\phi_+(\mu)$, integrate over μ from 0 to 1, and make use of Eqs. (A2), (A4), and (A5) of Ref. 21, we find

$$A_\mp = K_\mp - (A_\pm + K_\pm)e^{-2(R+\delta)/\nu_0} - \frac{e^{-(R+2\delta)/\nu_0}}{\nu_0 X(-\nu_0)}$$
$$\times \int_0^1 e^{-R/\nu}\nu X(-\nu)[A(\pm\nu) \pm K(\pm\nu)]\, d\nu. \quad (22a)$$

In a similar manner, we take the scalar product of Eq. (21) with $\phi_{\nu'}(\mu)$, $\nu' > 0$, and use Eqs. (A1), (A3), and (A6) of Ref. 21, to obtain

$$A(\mp\nu) = \mp K(\mp\nu) - \frac{2(A_\pm + K_\pm)}{N(\nu)\gamma(\nu)}$$
$$\times e^{-R/\nu_0}e^{-R/\nu}\nu^2 X(-\nu_0)\phi_-(\nu)\phi_+(\nu)$$
$$- \left(\frac{c\nu}{2}\right)^2 \frac{e^{-R/\nu}}{\gamma(\nu)N(\nu)} \int_0^1 d\nu' e^{-R/\nu'}$$
$$\times [A(\pm\nu') + K(\pm\nu')] \frac{\nu'(\nu_0 + \nu')X(-\nu')}{(\nu + \nu')(\nu_0 - \nu')},$$
$$\nu > 0. \quad (22b)$$

Here,

$$\frac{X(-\nu_0)}{X(\nu_0)} = -e^{-2\delta/\nu_0}. \quad (23)$$

The quantity δ is the extrapolation distance $z_0(c)$ as defined in Ref. 17.

It is not possible to solve Eqs. (22) explicitly for the coefficients $A_\pm$ and $A(\nu)$. However, one can reduce these expressions to a Fredholm equation for $A(\nu)$; this suggests that an iterative type solution could be obtained.[22] Since an iterative approach has been used successfully for similar problems, we proceed in this manner.[12,13]

Firstly, we neglect terms of order e^{-R} in Eq. (22b) to obtain $A(\mp\nu) = \mp K(\mp\nu)$. This result is then substituted into Eq. (22a) to give $A_\pm$ accurate to first order in e^{-R}. Finally, when this expression is entered back into Eq. (22b), we find $A(\pm\nu)$ correct to first order. Thus

$$A_\pm = [1 - e^{-4(R+\delta)/\nu_0}]^{-1}\Big\{K_\pm(1 + e^{-4(R+\delta)/\nu_0})$$
$$- 2K_\mp e^{-2(R+\delta)/\nu_0} + \int_0^1 d\nu e^{-R/\nu}\nu X(-\nu)$$
$$\times \frac{2e^{-R/\nu_0}}{\nu_0 X(\nu_0)}\Big[\mp K(\mp\nu) \mp K(\pm\nu)e^{-2(R+\delta)/\nu_0}\Big]\Big\} \quad (24a)$$

and

$$A(\mp\nu) = \mp K(\mp\nu) - [1 - e^{-4(R+\delta)/\nu_0}]^{-1}$$
$$\times [K_\pm - K_\mp e^{-2(R+\delta)/\nu_0}][1 - c]$$
$$\times \left[\frac{2c\nu X(-\nu)}{N(\nu)}\nu_0^2 X(-\nu_0)e^{-R/\nu}e^{-R/\nu_0}\right]. \quad (24b)$$

When Eqs. (24) are substituted into Eq. (18), we find that the first-order solution for the neutron density becomes

$$\rho(r_0; r)$$
$$= \frac{S}{rr_0}\Big[\frac{\sinh((R + \delta - r_0)/\nu_0)}{N_+ \sinh((R + \delta)/\nu_0)} \sinh r/\nu_0$$
$$+ \int_0^1 \frac{d\nu}{N(\nu)} e^{-r_0/\nu} \sinh r/\nu$$
$$+ \frac{e^{\delta/\nu_0}}{\nu_0 X(\nu_0) \sinh((R + \delta)/\nu_0)} \int_0^1 d\nu \frac{\nu X(-\nu)}{N(\nu)} e^{-R/\nu}$$
$$\times \Big\{\sinh\frac{r_0}{\nu}\sinh\frac{r}{\nu_0} + \sinh\frac{r_0}{\nu_0}\sinh\frac{r}{\nu}\Big\}\Big],$$
$$0 \le r < r_0 \le R. \quad (25)$$

The result for $\rho(r_0; r)$, $r > r_0$, is obtained from Eq. (25) by interchanging r and r_0.

An estimate of the accuracy of this "wide-sphere" result can be obtained from comparisons with similar approximation procedures. Mendelson solves several plane geometry problems in the "wide-slab" approximation and shows that for slab widths greater than two mean free paths, the exact and wide-slab solutions are not discernible.[13] Since most spheres are at least four mean free paths in diameter, the "wide-sphere" analysis presented here should yield very accurate descriptions of the density.

D. Special Cases

The results described by Eq. (25) contain essentially all of the physics for solutions of the one-speed Boltzmann equation in spherical geometry.

Interchanging r and r_0 in Eq. (25) and taking the limit as r_0 approaches zero, we find the finite-medium solution (to first order) for a centrally located point

[22] The coefficients $A_\pm$ could then be found in terms of $A(\nu)$.

source[11]; thus

$$\rho(0; r) = \frac{S}{r}\left[\frac{\sinh((R+\delta-r)/\nu_0)}{\nu_0 N_+ \sinh((R+\delta)/\nu_0)} + \int_0^1 \frac{e^{-r/\nu}}{\nu N(\nu)}\, d\nu + \frac{e^{\delta/\nu_0}}{\nu_0 X(\nu_0)\sinh((R+\delta)/\nu_0)}\int_0^1 \frac{\nu X(-\nu)e^{-R/\nu}}{N(\nu)} \times \left\{\frac{1}{\nu_0}\sinh\frac{r}{\nu} + \frac{1}{\nu}\sinh\frac{r}{\nu_0}\right\} d\nu\right]. \quad (26)$$

Taking the limit as R approaches infinity ($c < 1$) in Eq. (25), we find

$$\rho_\infty(r_0; r) = \frac{S}{rr_0}\left(\frac{e^{-r_0/\nu_0}\sinh r/\nu_0}{N_+} + \int_0^1 \frac{e^{-r_0/\nu}\sinh r/\nu}{N(\nu)}\, d\nu\right), \quad r < r_0, \quad (27a)$$

and

$$\rho_\infty(r_0; r) = \frac{S}{rr_0}\left(\frac{e^{-r/\nu_0}\sinh r_0/\nu_0}{N_+} + \int_0^1 \frac{e^{-r/\nu}\sinh r_0/\nu}{N(\nu)}\, d\nu\right), \quad r > r_0. \quad (27b)$$

If one considers only the discrete part of Eq. (25), it is found to satisfy the Helmholtz equation

$$\nabla^2\rho(r_0; r) - \frac{1}{\nu_0^2}\rho(r_0; r) = -\frac{S}{\nu_0 N_+ r_0^2}\delta(r - r_0), \quad (28)$$

and the boundary condition

$$\rho(r_0; R + \delta) = 0. \quad (29)$$

This is just the equation one would solve in the diffusion theory approximation to the Boltzmann equation.[23,24,25]

For a medium with no absorption ($c = 1$), Eq. (25) reduces to

$$\rho(r_0; r) = S\left[\frac{3}{r_0} - \frac{3}{R+\delta} + \frac{1}{rr_0}\int_0^1 \frac{e^{-r_0/\nu}}{N(\nu)}\sinh\frac{r}{\nu}\, d\nu - \frac{1}{R+\delta}\int_0^1 \frac{\nu X(-\nu)e^{-R/\nu}}{N(\nu)} \times \left(\frac{1}{r_0}\sinh\frac{r_0}{\nu} + \frac{1}{r}\sinh\frac{r}{\nu}\right) d\nu\right],$$
$$0 \le r \le r_0 \le R. \quad (30)$$

We note, also, that Eq. (25) can be used to describe the density in a multiplying sphere ($c > 1$) provided, of course, that the sphere is subcritical. Taking cognizance of the fact that ν_0 is imaginary ($\eta_0 \triangleq |\nu_0|$) for $c > 1$, we write Eq. (25) as

$$\rho(r_0; r) = \frac{S}{rr_0}\left[\frac{\sin((R+\delta-r_0)/\eta_0)\sin(r/\eta_0)}{|N_+|\sin((R+\delta)/\eta_0)} + \int_0^1 \frac{e^{-r_0/\nu}}{N(\nu)}\sinh\frac{r}{\nu}\, d\nu + \frac{e^{\delta/\nu_0}}{\nu_0 X(\nu_0)\sin((R+\delta)/\eta_0)}\int_0^1 \frac{\nu X(-\nu)e^{-R/\nu}}{N(\nu)} \times \left\{\sinh\frac{r_0}{\nu}\sin\frac{r}{\eta_0} + \sin\frac{r_0}{\eta_0}\sinh\frac{r}{\nu}\right\} d\nu\right],$$
$$0 \le r \le r_0 \le R. \quad (31)$$

Again, the results for $r > r_0$ are obtained by interchanging r and r_0 in Eqs. (30) and (31).

III. INFINITE-MEDIUM GREEN'S FUNCTION AS OBTAINED FROM THE SPHERICAL NORMAL MODES

A. General Analysis

In the previous section the solution for a spherical-shell source in a finite medium was obtained. The procedure there was to solve the integral equation for the density; the angular density was then available by integration over free-flight paths. This approach obviously has merit since it generates solutions for the classical single-region problems in spherical geometry. However, it would be satisfying to be able to construct solutions directly from the normal modes of the transport equation. Then, by applying the proper boundary conditions, one would attempt to determine the expansion coefficients, perhaps in a manner analogous to the Case technique that has been used in plane geometry. If such a procedure could be established, multidimensional problems or problems with more than one mean free path might become amenable to solution.[15]

With the above philosophy in mind, we prescribe a method by which the infinite-medium Green's function can be obtained from the normal modes of the homogeneous transport equation in spherical geometry. The Green's function considered here is the solution of

$$\mu\frac{\partial}{\partial r}\Psi(r,\mu) + \frac{(1-\mu^2)}{r}\frac{\partial}{\partial\mu}\Psi(r,\mu) + \Psi(r,\mu) = \frac{c}{2}\int_{-1}^{1}\Psi(r,\mu)\, d\mu + \frac{\delta(r)}{4\pi r^2} \quad (32)$$

subject to the constraint that $r^2\Psi(r, \mu)$ must vanish as r increases without bound.[26] In the usual manner, we replace the source term in Eq. (32) by an equivalent

[23] It should be noted, however, that the diffusion parameters are improvements to the usual ones (Refs. 24 and 25).

[24] R. L. Murray, *Nuclear Reactor Physics* (Prentice-Hall. Englewood Cliffs, N.J. 1957).

[25] A. M. Weinberg and E. P. Wigner, *The Physical Theory of Neutron Chain Reactors* (University of Chicago Press, Chicago, 1958).

[26] Obviously, the medium must be nonmultiplying ($c < 1$).

boundary condition, i.e.,[27]

$$\lim_{r\to 0} 4\pi r^2 \Psi(r, \mu) = \delta(1 - \mu). \tag{33}$$

Thus the density and current satisfy, respectively, the following:

$$\lim_{r\to 0} 4\pi r^2 \rho(r) = 1, \tag{34a}$$

$$\lim_{r\to 0} 4\pi r^2 j(r) = 1. \tag{34b}$$

A set of normal modes for the homogeneous spherical transport equation was obtained by Davison.[9] Mitsis, in a more recent study, discussed this same set.[12] There are several interesting aspects to the manner in which Mitsis obtained his results; we simply state the results by expanding $\Psi(r, \mu)$ in terms of this basis set. Thus, we write

$$\Psi(r, \mu) = \sum_{m=0}^{\infty} \left(\frac{2m+1}{2}\right) P_m(\mu) \times \Big\{ c\nu_0 Q_m(\nu_0)[A_+ k_m(r/\nu_0) + (-1)^m B_+ i_m(r/\nu_0)] + \int_0^1 [c\nu Q_m(\nu) + \lambda(\nu) P_m(\nu)] \times [A(\nu) k_m(r/\nu) + (-1)^m B(\nu) i_m(r/\nu)]\, d\nu \Big\}. \tag{35}$$

Here A_+, B_+, $A(\nu)$, and $B(\nu)$ are the arbitrary expansion coefficients; $P_m(\mu)$ and $Q_m(\nu)$ are, respectively, Legendre polynomials and the Legendre functions of the second kind. Also,

$$k_m(x) = (\pi/2x)^{\frac{1}{2}} K_{m+\frac{1}{2}}(x) \tag{36a}$$

and

$$i_m(x) = (\pi/2x)^{\frac{1}{2}} I_{m+\frac{1}{2}}(x). \tag{36b}$$

Following the notation of Watson, we have used $K_{m+\frac{1}{2}}$ and $I_{m+\frac{1}{2}}$ to denote the modified Bessel functions.[28] The fact that $I_{m+\frac{1}{2}}(x)$ diverges at infinity leads us immediately to equate B_+ and $B(\nu)$ to zero. Thus

$$\Psi(r, \mu) = \sum_{m=0}^{\infty} \tfrac{1}{2}(2m+1) P_m(\mu) \Big\{ A_+ c\nu_0 Q_m(\nu_0) k_m(r/\nu_0) + \int_0^1 A(\nu)[c\nu Q_m(\nu) + \lambda(\nu) P_m(\nu)] k_m(r/\nu)\, d\nu \Big\}. \tag{37}$$

The expression for $\Psi(r, \mu)$ given by Eq. (37) has the correct behavior at infinity. The expansion coefficients, A_+ and $A(\nu)$, must be determined, therefore, so that the boundary condition at the origin, Eq. (33), is satisfied. For the sake of brevity, we introduce the notation

$$S_m(r) \triangleq A_+ c\nu_0 Q_m(\nu_0) k_m(r/\nu_0) + \int_0^1 A(\nu)[c\nu Q_m(\nu) + \lambda(\nu) P_m(\nu)] k_m(r/\nu)\, d\nu. \tag{38}$$

Thus,

$$\Psi(r, \mu) = \sum_{m=0}^{\infty} \tfrac{1}{2}(2m+1) P_m(\mu) S_m(r); \tag{39}$$

the source condition can be stated, therefore, as

$$\lim_{r\to 0} 4\pi r^2 S_m(r) = 1. \tag{40}$$

There are several interesting aspects of Eq. (40). We note that the expression must be true for *all m*. Since $k_m(r)$ diverges as r^{-m-1} in the limit of r tending to zero, it is not obvious how $r^2 S_m(r)$ could exist in that limit. There must be, therefore, a very subtle interrelation between A_+ and $A(\nu)$.[29] The procedure we use here is firstly to determine A_+ and $A(\nu)$ such that Eq. (40) is satisfied for the *particular choice* of $m = 1$. This insures that the angular density will satisfy the "weak" or current boundary condition, i.e.,

$$\lim_{r\to 0} 4\pi r^2 j(r) = 1;$$

however, the complete boundary condition is stated by Eq. (33). Thus the $m = 1$ condition is necessary but *not* sufficient. In order to prove that the expansion coefficients, as determined from the "weak" boundary condition, are the correct ones (i.e., that Eq. (40) is valid for *all m*) is a formidable task; we prefer to devote section IIIB to this proof.

For the case $m = 1$, we must satisfy

$$\frac{1}{2\pi^2} = A_+ \nu_0^2 c\nu_0 Q_1(\nu_0) + \int_0^1 A(\nu)\nu^2 [c\nu Q_1(\nu) + \nu\lambda(\nu)]\, d\nu. \tag{41}$$

The form of Eq. (41) is suggestive of an expansion in terms of the eigenfunctions used by Case for problems in plane geometry.[1] We pursue the point further by considering the following full-range expansion:

$$\frac{1}{2\pi^2 \mu} = A_+ \nu_0^2 [\phi_+(\mu) - \phi_-(\mu)] + \int_0^1 A(\nu)\nu^2 [\phi_\nu(\mu) - \phi_{-\nu}(\mu)]\, d\nu, \quad \mu \in [-1, 1]. \tag{42}$$

The validity of this expansion follows from the full-range completeness theorem proved by Case.[1] Since Eq. (42) is a valid expansion, the coefficients, A_+ and

[27] One of the authors (C. E. S.) is indebted to Dr. Z. Akcasu and Dr. G. C. Summerfield for a discussion of this point.

[28] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1945).

[29] That this is true can be seen immediately by considering the cases $m = 0$, 1, and 2.

$A(\nu)$, can be determined immediately by taking full-range scalar products.[2] We find

$$A_+ = \frac{1}{2\pi^2 \nu_0^2 N_+} \quad (43a)$$

and

$$A(\nu) = \frac{1}{2\pi^2 \nu^3}\, g(c, \nu), \quad (43b)$$

where

$$g(c, \nu) = \frac{1}{\lambda^2(\nu) + (\frac{1}{2}c\nu\pi)^2}. \quad (44)$$

If one multiplies Eq. (42) by μ and integrates over μ from -1 to 1, the following results:

$$\frac{1}{2\pi^2} = A_+\nu_0^2 \int_{-1}^{1} \mu\phi_+(\mu)\, d\mu + \int_0^1 A(\nu)\nu^2 \int_{-1}^{1} \mu\phi_\nu(\mu)\, d\mu\, d\nu; \quad (45)$$

this is identical with Eq. (41), and thus Eqs. (43) do, in fact, satisfy the weak boundary condition.

Since the expansion coefficients have been determined (with the *proviso* that the necessary rigor is to be given in Sec. IIIB), the solution is complete; it can be written as

$$\Psi(r, \mu) = \sum_{m=0}^{\infty} \left(\frac{2m+1}{4\pi^2}\right) P_m(\mu) \left\{ \frac{c\nu_0 Q_m(\nu_0)}{\nu_0^2 N_+}\, k_m\left(\frac{r}{\nu_0}\right) + \int_0^1 g(c, \nu)[c\nu Q_m(\nu) + \lambda(\nu)P_m(\nu)] k_m\left(\frac{r}{\nu}\right) \frac{d\nu}{\nu^3} \right\}. \quad (46)$$

The density and current can be obtained from Eq. (46) by inspection; we find

$$\rho(r) = \frac{1}{4\pi r}\left[\frac{1}{\nu_0 N_+}\, e^{-r/\nu_0} + \int_0^1 g(c, \nu) e^{-r/\nu} \frac{d\nu}{\nu^2}\right] \quad (47a)$$

and

$$j(r) = \frac{1-c}{4\pi r}\left[\frac{1}{N_+}\left(1 + \frac{\nu_0}{r}\right) e^{-r/\nu_0} + \int_0^1 g(c, \nu)\left(1 + \frac{\nu}{r}\right) e^{-r/\nu} \frac{d\nu}{\nu}\right]. \quad (47b)$$

In addition, the higher moments of $\Psi(r, \mu)$, if so desired, are obtained trivially from Eq. (46).

It is interesting to note the form of $\Psi(r, \mu)$ in the limit as c tends to zero. It is a simple matter to obtain

$$\lim_{c\to 0} \Psi(r, \mu) \triangleq \Psi_0(r, \mu) = \frac{1}{2\pi^2} \sum_{m=0}^{\infty} \left(\frac{2m+1}{2}\right) P_m(\mu) G_m(r), \quad (48)$$

where

$$G_m(r) \triangleq \int_0^1 k_m\left(\frac{r}{\nu}\right) P_m(\nu) \frac{d\nu}{\nu^3}. \quad (49)$$

The integral above has been evaluated by Harrington, Siewert, and Murray.[30] They found

$$G_m(r) = \frac{\pi}{2r^2}\, e^{-r}. \quad (50)$$

Note that $G_m(r)$ is independent of m. When Eq. (50) is substituted into Eq. (48), we obtain the usual result, i.e.,[9]

$$\Psi_0(r, \mu) = \frac{e^{-r}}{4\pi r^2}\, \delta(1 - \mu). \quad (51)$$

This, of course, represents the uncollided angular density.

B. Boundary Condition at the Origin

The expansion coefficients A_+ and $A(\nu)$ were determined in the previous work by noting the similarity between the current boundary condition and a suitably chosen full-range expansion in terms of the eigenfunctions $\phi_\nu(\mu)$. We would like now to discuss further the analysis of the source condition and to prove that Eqs. (43) are the correct solutions for the expansion coefficients.

We have shown that A_+ and $A(\nu)$ are such that the "weak" boundary condition is satisfied, i.e., these expansion coefficients are solutions to Eq. (41); however, they are *not* the only possible solutions to Eq. (41). This equation obviously has no *unique* solution. We must keep in mind, however, that Eq. (41) is only one in an infinite set of conditions that must be satisfied (i.e., we must consider *all m*).

The necessary and sufficient condition that the complete boundary condition at the origin, viz.,

$$\lim_{r\to 0} 4\pi r^2 \Psi(r, \mu) = \delta(1 - \mu), \quad (52)$$

be satisfied is that

$$\lim_{r\to 0} 4\pi r^2 S_m(r) = 1, \quad (53)$$

where

$$S_m(r) \triangleq A_+ T_m(\nu_0) k_m\left(\frac{r}{\nu_0}\right) + \int_0^1 A(\nu) T_m(\nu) k_m\left(\frac{r}{\nu}\right) d\nu. \quad (54)$$

We have written Eq. (54) in the more tractable form by defining[12]

$$T_m(\nu_0) \triangleq c\nu_0 Q_m(\nu_0) \quad (55)$$

and

$$T_m(\nu) \triangleq c\nu Q_m(\nu) + \lambda(\nu) P_m(\nu). \quad (56)$$

Here $T_m(x)$ are mth degree polynomials of x; they satisfy the same recursion relation as the Legendre

[30] W. J. Harrington, C. E. Siewert, and R. L. Murray (to be published).

polynomials, i.e.,

$$mT_m(x) = (2m - 1)xT_{m-1}(x) - (m - 1)T_{m-2}(x); \tag{57}$$

they begin differently, however:

$$T_0(x) = 1, \quad T_1(x) = x(1 - c). \tag{58}$$

We note that the $T_m(x)$ reduce to the $P_m(x)$ as c vanishes.

The method used to prove Eq. (53) is that of mathematical induction. We verify the validity of this condition for $m = 0$, 1, assume it to be true for $m = k - 1$, $k - 2$, and then deduce that it must be true for $k = m$. Firstly then, we must prove the $m = 0$ condition; this takes the form

$$\lim_{r\to 0} 2\pi^2 r \int_0^1 A(\nu)\nu e^{-r/\nu}\, d\nu = 1, \tag{59}$$

where

$$A(\nu) = g(c, \nu)/2\pi^2\nu^3. \tag{60}$$

For $g(c, \nu)$ we use the power series given by Case, de Hoffmann, and Plazcek, i.e.,[17]

$$g(c, \nu) = 1 + \sum_{\beta=1}^{\infty} \Gamma_\beta \nu^{2\beta}. \tag{61}$$

Thus Eq. (59) can be written as

$$\lim_{r\to 0} r\left[E_0(r) + \sum_{\beta=1}^{\infty} \Gamma_\beta E_{2\beta}(r)\right] = 1, \tag{62}$$

where $E_n(r)$ denotes the exponential integral function.[17] Clearly, Eq. (62) is satisfied. The $m = 1$ condition takes the form

$$\lim_{r\to 0}\left[2\pi^2\nu_0^3(1 - c)A_+ + 2\pi^2(1 - c)\int_0^1 A(\nu)\nu^3\, d\nu\right] = 1. \tag{63}$$

If we substitute the expressions for the expansion coefficients into Eq. (63), we find

$$\int_0^1 g(c, \nu)\, d\nu = \frac{1}{1 - c} - \frac{\nu_0}{N_+}. \tag{64}$$

This result is given explicitly by Case *et al.*; Eq. (63) is thus valid.

The condition to be proved, Eq. (53), has been verified for $m = 0, 1$; we proceed, therefore, with the inductive proof for arbitrary m. If we utilize the recursion and differentiation formulas for the $k_m(x)$, viz.,[31]

$$k_m\left(\frac{r}{x}\right) = (2m - 1)\frac{x}{r} k_{m-1}\left(\frac{r}{x}\right) + k_{m-2}\left(\frac{r}{x}\right) \tag{65a}$$

and

$$\frac{d}{dr}\left[r^{m+1}k_m\left(\frac{r}{x}\right)\right] = -\frac{r^{m+1}}{x} k_{m-1}\left(\frac{r}{x}\right), \tag{65b}$$

as well as Eq. (57), we obtain

$$m\frac{d}{dr}[r^{m+1}S_m(r)] = -(m - 1)\frac{d}{dr}[r^{m+1}S_{m-2}(r)] - (2m - 1)r^{m+1} \times S_{m-1}(r) + (2m - 1)(m - 1)r^m S_{m-2}(r). \tag{66}$$

Integrating Eq. (66) from 0 to r, we find[32]

$$mr^{m+1}S_m(r) - C_m = -(m - 1)r^{m+1}S_{m-2}(r) - \lim_{r\to 0} r^{m+1}S_{m-2}(r) - (2m - 1)\int_0^r t^{m+1}S_{m-1}(t)\, dt + (2m - 1)(m - 1)\int_0^r t^m S_{m-2}(t)\, dt. \tag{67}$$

We now make the inductive assumption that for all k, $0 \le k \le m - 1$, $m \ge 2$,

$$\lim_{r\to 0} 4\pi r^2 S_k(r) = 1. \tag{68}$$

Thus the limit term in Eq. (67) is zero. The quantity C_m is a constant of integration and, as will be proved, must also be zero. We write, therefore, Eq. (67) as

$$mr^2S_m(r) = -(m - 1)r^2S_{m-2}(r) - (2m - 1)\frac{\int_0^r t^{m+1}S_{m-1}(t)\, dt}{r^{m-1}} + (2m - 1)(m - 1)\frac{\int_0^r t^m S_{m-2}(t)\, dt}{r^{m-1}} \tag{69}$$

If, in Eq. (69), we take the limit as r approaches zero and apply L'Hospital's rule to evaluate the two indeterminate forms,

$$\lim_{r\to 0}\frac{\int_0^r t^{m+1}S_{m-1}(t)\, dt}{r^{m-1}} = 0 \tag{70a}$$

and

$$\lim_{r\to 0}\frac{\int_0^r t^m S_{m-2}(t)\, dt}{r^{m-1}} = \frac{1}{4\pi(m - 1)}, \tag{70b}$$

we obtain the desired result, viz.,

$$\lim_{r\to 0} 4\pi r^2 S_m(r) = 1. \tag{71}$$

[31] M. Abramowitz and I. A. Stegun, Eds., *Handbook of Mathematical Functions* (U.S. Department of Commerce, National Bureau of Standards, Washington, 1964), Appl. Math. Ser. 55, Chap. 10.

[32] This approach was suggested by W. J. Harrington.

The proof that the integration constant C_m is zero provides the bridge for the saltus used to develop Eq. (71). This proof not only is necessary, but has the additional merit that it removes the ambiguity associated with the uniqueness of the method used in Sec. IIIA to determine A_+ and $A(\nu)$. We note from Eq. (67) that

$$C_m = m \lim_{r\to 0} r^{m+1} S_m(r). \tag{72}$$

Returning to the definition of $S_m(r)$, Eq. (54), and using the explicit form of the Bessel functions, i.e.,[28]

$$k_m(r) = \frac{\pi e^{-r}}{2r}\sum_{\alpha=0}^{m} W_\alpha^m r^{-\alpha}, \tag{73}$$

we note that Eq. (72) can be written as

$$C_m = mW_m^m \frac{\pi}{2}\left[A_+\nu_0^{m+1}T_m(\nu_0) + \int_0^1 A(\nu)\nu^{m+1}T_m(\nu)\,d\nu\right]. \tag{74}$$

To show that C_m is zero, we must prove that the term in brackets on the right-hand side of Eq. (74) vanishes. Thus, multiplying Eq. (42) by $\mu^{m-1}P_m(\mu)$ and integrating over all μ, we obtain (after a change of variables)

$$\frac{1}{4\pi^2}\int_{-1}^{1} d\mu\mu^{m-2}P_m(\mu) = A_+\frac{c\nu_0^3}{2}\int_{-1}^{1} d\mu\mu^{m-1}\frac{P_m(\mu)}{\nu_0-\mu} + \int_0^1 d\nu A(\nu)\nu^2\int_{-1}^{1} d\mu\mu^{m-1}P_m(\mu)\phi_\nu(\mu). \tag{75}$$

The left-hand side of Eq. (75) is zero obviously ($m > 1$). Noting that

$$\frac{\mu^{m-1}}{\mu-\nu_0} = \mu^{m-2} + \nu_0\mu^{m-3} + \nu_0^2\mu^{m-4} + \cdots \frac{\nu_0^{m-1}}{\mu-\nu_0}, \tag{76}$$

we rewrite Eq. (75) as

$$A_+\frac{c\nu_0^3}{2}\int_{-1}^{1} d\mu\left[-\mu^{m-2} - \nu_0\mu^{m-3} - \cdots + \frac{\nu_0^{m-1}}{\nu_0-\mu}\right]P_m(\mu) + \int_0^1 d\nu A(\nu)\nu^2\int_{-1}^{1} d\mu\mu^{m-1}P_m(\mu)\phi_\nu(\mu) = 0. \tag{77}$$

The continuum term in Eq. (77) also can be decomposed in the manner indicated by Eq. (76). Since to do so only complicates the notation, we do not explicitly write it out. The fact that the Legendre polynomials are orthogonal can be used again to reduce Eq. (77) to

$$A_+\frac{c\nu_0^{m+2}}{2}\int_{-1}^{1} d\mu\frac{P_m(\mu)}{\nu_0-\mu} + \int_0^1 d\nu A(\nu)\nu^{m+1}\int_{-1}^{1} d\mu P_m(\mu)\phi_\nu(\mu) = 0. \tag{78}$$

The final result is thus obtained, viz.,

$$A_+\nu_0^{m+1}T_m(\nu_0) + \int_0^1 A(\nu)\nu^{m+1}T_m(\nu)\,d\nu = 0, \tag{79}$$

or

$$C_m \equiv 0. \tag{80}$$

ACKNOWLEDGMENTS

One of the authors (C. E. S.) would like to acknowledge the helpful discussions regarding Sec. III of this paper that were held with Dr. Z. Akcasu, Dr. G. C. Summerfield, Dr. R. Zelazny, and Dr. P. F. Zweifel at The University of Michigan; also, appreciation for many useful suggestions is extended to O. J. Smith and Dr. W. J. Harrington, and Dr. R. L. Murray of North Carolina State University.

This work was supported in part by the National Science Foundation.