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# Strong evaporation into a half space. II. The three-dimensional BGK model

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## I. Introduction

In a recent paper [1], hereafter referred to as I, the method of elementary solutions [2, 3] was used to solve a strong-evaporation problem based on the one-dimensional BGK model that was formulated and solved by Arthur and Cercignani [4]. Here in order to improve the physical model we use the BGK equation for a three-dimensional gas. Since the aim of this work is to present the analysis required to establish the desired exact solution and to report relevant numerical results, we assume that I and the work of Arthur and Cercignani [4] are available, and thus we abbreviate here the formulation of the problem to be solved. If we consider a liquid evaporating at $x = 0$ into a vacuum which occupies the region $x > 0$ and linearize the distribution function $f(x, \xi)$ and the local Maxwellian distribution $\Phi(x, \xi)$ about the downstream equilibrium condition, we find we must solve

$$(u + c_x) \frac{\partial}{\partial x} h(x, \boldsymbol{c}) + h(x, \boldsymbol{c})$$
$$= \pi^{-3/2} \int_{-\infty}^{\infty} h(x, \boldsymbol{c}') \left[1 + 2\boldsymbol{c} \cdot \boldsymbol{c}' + \frac{2}{3}\left(c'^2 - \frac{3}{2}\right)\left(c^2 - \frac{3}{2}\right)\right] \mathrm{e}^{-c'^2} \mathrm{d}^3 c', \quad (1)$$

where $h(x, \boldsymbol{c})$ is the perturbation from the downstream equilibrium distribution and, in dimensionless units, $x$ is the position and $\boldsymbol{c}$, with components $c_x$, $c_y$ $c_z$ and magnitude $c$, is the velocity. We thus seek a solution of Eq. (1) such that

$$\lim_{x \to \infty} h(x, \boldsymbol{c}) = 0 \quad (2\text{a})$$

and

$$h(0, \boldsymbol{c}) = 2 c_x (u_0 - u) + \Delta\varrho + \left(c^2 - \frac{3}{2}\right) \Delta T, \quad c_x > -u, \quad (2\text{b})$$

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where $u_0$ is the mass speed at the surface,

$$\Delta \varrho = \frac{\varrho_0 - \varrho_\infty}{\varrho_\infty} \tag{3a}$$

and

$$\Delta T = \frac{T_0 - T_\infty}{T_\infty}. \tag{3b}$$

Here $\varrho_0$, $T_0$, $\varrho_\infty$ and $T_\infty$ are respectively the density and temperature of the gas at the surface and downstream. Since we are concerned with temperature-density effects, we can [5] take "moments" of Eq. (1) to obtain equations dependent only on $x$ and $c_x$. We let

$$\psi_1(x, c_x) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-(c_y^2 + c_z^2)}\, h(x, \boldsymbol{c})\, \mathrm{d}c_y\, \mathrm{d}c_z, \tag{4a}$$

and

$$\psi_2(x, c_x) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-(c_y^2 + c_z^2)}\, h(x, \boldsymbol{c})\, (c_y^2 + c_z^2 - 1)\, \mathrm{d}c_y\, \mathrm{d}c_z. \tag{4b}$$

Thus if we multiply Eq. (1) by $\exp(-c_y^2 - c_z^2)$ and integrate over the complete range with respect to $c_y$ and $c_z$, and similarly if we multiply Eq. (1) by $(c_y^2 + c_z^2 - 1)\exp(-c_y^2 - c_z^2)$ and integrate, we obtain two equations which can be written in the form

$$\left[(u + \xi)\frac{\partial}{\partial x} + 1\right] \boldsymbol{\Psi}(x, \xi) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} [\boldsymbol{Q}(\xi)\, \boldsymbol{Q}^T(\xi') + 2\,\xi\xi'\,\boldsymbol{P}] \cdot \boldsymbol{\Psi}(x, \xi')\, \mathrm{e}^{-\xi'^2}\, \mathrm{d}\xi', \tag{5}$$

where

$$\boldsymbol{Q}(\xi) = \begin{vmatrix} \left(\frac{2}{3}\right)^{\frac{1}{2}} \left(\xi^2 - \frac{1}{2}\right) & 1 \\ \left(\frac{2}{3}\right)^{\frac{1}{2}} & 0 \end{vmatrix}, \quad \boldsymbol{P} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \tag{6a, b}$$

and $\boldsymbol{\Psi}(x, \xi)$ represents the column vector whose components are $\psi_1(x, \xi)$ and $\psi_2(x, \xi)$. The boundary conditions given by Eqs. (2a) and (2b) can be expressed in terms of $\boldsymbol{\Psi}(x, \xi)$ by taking appropriate moments; we find

$$\lim_{x \to \infty} \boldsymbol{\Psi}(x, \xi) = \boldsymbol{0}, \tag{7a}$$

and, for $\xi > -u$,

$$\boldsymbol{\Psi}(0, \xi) = \pi^{\frac{1}{2}} \begin{vmatrix} 1 \\ 0 \end{vmatrix} \Delta\varrho + \pi^{\frac{1}{2}} \begin{vmatrix} \xi^2 - \frac{1}{2} \\ 1 \end{vmatrix} \Delta T + \pi^{\frac{1}{2}} \begin{vmatrix} 1 \\ 0 \end{vmatrix} 2\,\xi\,(u_0 - u). \tag{7b}$$

Equations (5) and (7) represent the complete formulation of the problem to be solved.

## II. Elementary solutions

We now seek to establish, for $u > 0$, the elementary solutions of

$$(u+\mu)\frac{\partial}{\partial x}\boldsymbol{\Psi}(x,\mu)+\boldsymbol{\Psi}(x,\mu)=\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}[\boldsymbol{Q}(\mu)\,\boldsymbol{Q}^T(\mu')+2\,\mu\,\mu'\,\boldsymbol{P}]\cdot\boldsymbol{\Psi}(x,\mu')\,\mathrm{e}^{-\mu'^2}\,\mathrm{d}\mu'. \tag{8}$$

Thus if we substitute

$$\boldsymbol{\Psi}(x,\mu:\eta)=\boldsymbol{\Phi}(\eta,\mu)\,\mathrm{e}^{-x/(\eta+u)} \tag{9}$$

into Eq. (8) we find

$$(\eta-\mu)\,\boldsymbol{\Phi}(\eta,\mu)=(\eta+u)\,\boldsymbol{K}(\mu)\,\boldsymbol{M}(\eta) \tag{10}$$

where

$$\boldsymbol{K}(\mu)=\pi^{-\frac{1}{2}}[\boldsymbol{Q}(\mu)-2\,u\,\mu\,\boldsymbol{T}], \tag{11}$$

$$\boldsymbol{T}=\begin{vmatrix}0 & 1\\ 0 & 0\end{vmatrix} \tag{12}$$

and

$$\boldsymbol{M}(\eta)=\int_{-\infty}^{\infty}\boldsymbol{Q}^T(\mu)\,\boldsymbol{\Phi}(\eta,\mu)\,\mathrm{e}^{-\mu^2}\,\mathrm{d}\mu. \tag{13}$$

If we consider first that $\eta=\xi\notin(-\infty,\infty)$, then Eq. (10) yields

$$\boldsymbol{\Phi}(\xi,\mu)=(\xi+u)\,\boldsymbol{K}(\mu)\left(\frac{1}{\xi-\mu}\right)\boldsymbol{M}(\xi) \tag{14}$$

if

$$\boldsymbol{\Lambda}(\xi)\,\boldsymbol{M}(\xi)=\boldsymbol{0}, \tag{15}$$

where

$$\boldsymbol{\Lambda}(z)=\boldsymbol{I}+(z+u)\int_{-\infty}^{\infty}\boldsymbol{\Psi}(\mu)\frac{\mathrm{d}\mu}{\mu-z} \tag{16}$$

and

$$\boldsymbol{\Psi}(\mu)=\pi^{-\frac{1}{2}}\,\boldsymbol{Q}^T(\mu)\,[\boldsymbol{Q}(\mu)-2\,u\,\mu\,\boldsymbol{T}]\,\mathrm{e}^{-\mu^2}. \tag{17}$$

We find that we can express $\boldsymbol{\Lambda}(z)$ as

$$\boldsymbol{\Lambda}(z)=\boldsymbol{Y}(z)+\boldsymbol{\Xi}(z)\,J(z) \tag{18}$$

where

$$J(z)=\pi^{-\frac{1}{2}}(z+u)\int_{-\infty}^{\infty}\mathrm{e}^{-\mu^2}\frac{\mathrm{d}\mu}{\mu-z} \tag{19}$$

and the polynomial matrices are

$$Y(z) = \begin{vmatrix} 1+\frac{2}{3}(z+u)\,z\left(z^2-\frac{1}{2}\right) & \left(\frac{2}{3}\right)^{\frac{1}{2}} z\,(z+u)\,(1-2\,u\,z) \\ \left(\frac{2}{3}\right)^{\frac{1}{2}} z\,(z+u) & 1-2\,u\,(z+u) \end{vmatrix} \tag{20a}$$

and

$$\Xi(z) = \begin{vmatrix} \frac{2}{3}\left[1+\left(z^2-\frac{1}{2}\right)^2\right] & \left(\frac{2}{3}\right)^{\frac{1}{2}}\left(z^2-\frac{1}{2}\right)(1-2\,u\,z) \\ \left(\frac{2}{3}\right)^{\frac{1}{2}}\left(z^2-\frac{1}{2}\right) & 1-2\,u\,z \end{vmatrix} \tag{20b}$$

Computing the determinant of $\Lambda(z)$, we find

$$\Lambda(z) = a(z) + b(z)\,J(z) + c(z)\,J^2(z) \tag{21}$$

where

$$a(z) = 1-2\,u^2 + \left[\frac{u}{3}(2\,u^2-7)\right] z + \left[\frac{1}{3}(2\,u^2-1)\right] z^2, \tag{22a}$$

$$b(z) = \frac{1}{6}(11-10\,u^2) - \frac{10}{3}\,u\,z + \left[\frac{1}{3}(2\,u^2-1)\right] z^2 \tag{22b}$$

and

$$c(z) = \frac{2}{3} - \frac{4}{3}\,u\,z. \tag{22c}$$

For large $z$ we find

$$\Lambda(z) \to -\frac{1}{z^3}\,u\left(u^2-\frac{5}{6}\right) + \ldots \tag{23}$$

so that for $u^2 \neq 5/6$, $\Lambda(z)$ has a triple zero at infinity. The argument principle [6] can be used to show that $\Lambda(z)$ has no zeros in the finite plane. Concerning solutions corresponding to $\eta = \infty$, we conclude that

$$\boldsymbol{\Psi}_1(x,\mu) = \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \quad \boldsymbol{\Psi}_2(x,\mu) = \mu \begin{vmatrix} 1 \\ 0 \end{vmatrix} \quad \text{and} \quad \boldsymbol{\Psi}_3(x,\mu) = \begin{vmatrix} \mu^2-\frac{1}{2} \\ 1 \end{vmatrix} \tag{24a, b and c}$$

satisfy Eq. (8). If $u^2 = 5/6$ then we find that

$$\boldsymbol{\Psi}_4(x,\mu) = (x-u-\mu)\left[\frac{3}{2}\,\boldsymbol{\Psi}_1(x,\mu) - 3\,u\,\boldsymbol{\Psi}_2(x,\mu) + \boldsymbol{\Psi}_3(x,\mu)\right] \tag{25}$$

though clearly not of the form given by Eq. (9) is also a solution. If we now solve Eq. (10) for $\eta \in (-\infty, \infty)$, we find we can express the desired solution to Eq. (8) as

$$\boldsymbol{\Psi}(x, \mu) = \sum_{\alpha=1}^{K} A_\alpha \boldsymbol{\Psi}_\alpha(x, \mu) + \int_{-\infty}^{\infty} \boldsymbol{\Phi}(\eta, \mu) \boldsymbol{A}(\eta) \, \mathrm{e}^{-x/(\eta+u)} \, \mathrm{d}\eta \tag{26}$$

where $K = 3$ for $u^2 \neq 5/6$ and $K = 4$ if $u^2 = 5/6$. In addition $A_\alpha$, $\alpha = 1, 2, \ldots K$, and the two-vector $\boldsymbol{A}(\eta)$ are expansion coefficients to be determined from appropriate boundary conditions, and the $2 \times 2$ matrix solution to Eq. (10) is

$$\boldsymbol{\Phi}(\eta, \mu) = (\eta + u) \boldsymbol{K}(\mu) \, \mathrm{e}^{-\eta^2} P v \left(\frac{1}{\eta - \mu}\right) + \boldsymbol{Q}^{-T}(\eta) \boldsymbol{\lambda}(\eta) \, \delta(\eta - \mu), \tag{27}$$

where

$$\boldsymbol{\lambda}(\eta) = \boldsymbol{I} + (\eta + u) P \int_{-\infty}^{\infty} \boldsymbol{\Psi}(\mu) \frac{\mathrm{d}\mu}{\mu - \eta}. \tag{28}$$

## III. Boundary-value problem

As discussed in the Introduction we now seek a solution to Eq. (8) subject to the boundary conditions

$$\lim_{x \to \infty} \boldsymbol{\Psi}(x, \mu) = \boldsymbol{0} \tag{29a}$$

and, for $\mu > -u$,

$$\boldsymbol{\Psi}(0, \mu) = \boldsymbol{F}(\mu) = \pi^{\frac{1}{2}} \left\{ \begin{vmatrix} 1 \\ 0 \end{vmatrix} \Delta \varrho + \begin{vmatrix} \mu^2 - \frac{1}{2} \\ 1 \end{vmatrix} \Delta T + 2\mu \begin{vmatrix} 1 \\ 0 \end{vmatrix} (u_0 - u) \right\}. \tag{29b}$$

To satisfy Eq. (29a) we write

$$\boldsymbol{\Psi}(x, \mu) = \int_{-u}^{\infty} \boldsymbol{\Phi}(\eta, \mu) \boldsymbol{A}(\eta) \, \mathrm{e}^{-x/(\eta+u)} \, \mathrm{d}\eta \tag{30}$$

where $\boldsymbol{A}(\eta)$ is to be found from Eq. (29b). Setting $x = 0$ in Eq. (30) and using Eqs. (27) and (29b), we now find

$$\boldsymbol{F}(\mu) = \boldsymbol{Q}^{-T}(\mu) \boldsymbol{\lambda}(\mu) \boldsymbol{A}(\mu) + \boldsymbol{K}(\mu) P \int_{-u}^{\infty} (\eta + u) \boldsymbol{A}(\eta) \, \mathrm{e}^{-\eta^2} \frac{\mathrm{d}\eta}{\eta - \mu} \tag{31}$$

or

$$\boldsymbol{Q}^T(\mu) \boldsymbol{F}(\mu) \, \mathrm{e}^{-\mu^2} = \boldsymbol{\lambda}(\mu) \boldsymbol{A}(\mu) \, \mathrm{e}^{-\mu^2} + \boldsymbol{\Psi}(\mu) P \int_{-u}^{\infty} (\eta + u) \boldsymbol{A}(\eta) \, \mathrm{e}^{-\eta^2} \frac{\mathrm{d}\eta}{\eta - \mu}. \tag{32}$$

Now introducing

$$\boldsymbol{N}(z) = \frac{1}{2\pi i} \int_{-u}^{\infty} (\eta + u) \boldsymbol{A}(\eta) \, \mathrm{e}^{-\eta^2} \frac{\mathrm{d}\eta}{\eta - z} \tag{33}$$

and using the Plemelj formulas [7], we find we can write Eq. (32) as

$$(\mu + u)\, \boldsymbol{Q}^T(\mu)\, \boldsymbol{F}(\mu)\, \mathrm{e}^{-\mu^2} = \boldsymbol{\Lambda}^+(\mu)\, \boldsymbol{N}^+(\mu) - \boldsymbol{\Lambda}^-(\mu)\, \boldsymbol{N}^-(\mu) \tag{34}$$

where the superscripts $\pm$ are used to denote limiting values as branch cuts are approached from above (+) and below (−). If we let

$$\boldsymbol{\Gamma}(\mu) = (\mu + u)\, [\boldsymbol{\Phi}^+(\mu)]^T [\boldsymbol{\Lambda}^+(\mu)]^{-1}\, \boldsymbol{Q}^T(\mu)\, \mathrm{e}^{-\mu^2} \tag{35}$$

then we can write the solution [7] to Eq. (34) as

$$\boldsymbol{N}(z) = \boldsymbol{\Phi}^{-T}(z) \left\{ \frac{1}{2\pi i} \int_{-u}^{\infty} \boldsymbol{\Gamma}(\mu)\, \boldsymbol{F}(\mu) \frac{\mathrm{d}\mu}{\mu - z} + \boldsymbol{P}_*(z) \right\} \tag{36}$$

where $\boldsymbol{P}_*(z)$ is a vector of polynomials and $\boldsymbol{\Phi}(z)$ is a canonical solution (with ordered normal form at infinity [7]) of the matrix Riemann-Hilbert problem defined by

$$\boldsymbol{\Phi}^+(\mu) = \boldsymbol{G}(\mu)\, \boldsymbol{\Phi}^-(\mu), \quad \mu \in [-u, \infty), \tag{37a}$$

where

$$\boldsymbol{G}(\mu) = [\boldsymbol{\Lambda}^+(\mu)]^T [\boldsymbol{\Lambda}^-(\mu)]^{-T}. \tag{37b}$$

In order to proceed with the solution of Eq. (31) we now must investigate $\boldsymbol{\Phi}(z)$ so that the form of Eq. (36) as $|z| \to \infty$ can be analyzed and made consistent with the definition of $\boldsymbol{N}(z)$ given by Eq. (33). To this end we first factor the dispersion matrix as

$$\boldsymbol{\Lambda}(z) = \boldsymbol{\Theta}(z)\, \boldsymbol{P}(z)\, \boldsymbol{\Phi}^T(z) \tag{38}$$

where $\boldsymbol{P}(z)$ is a matrix of polynomials and $\boldsymbol{\Theta}(z)$ is a canonical solution (of ordered normal form at infinity) of the Riemann-Hilbert problem defined by

$$\boldsymbol{\Theta}^+(\tau) = \boldsymbol{G}_*(\tau)\, \boldsymbol{\Theta}^-(\tau), \quad \tau \in (-\infty, -u], \tag{39a}$$

where

$$\boldsymbol{G}_*(\tau) = \boldsymbol{\Lambda}^+(\tau)\, [\boldsymbol{\Lambda}^-(\tau)]^{-1}. \tag{39b}$$

If we now investigate $\theta(x) = \arg \Lambda^+(x)$ for $x \in (-\infty, \infty)$ we find we can, for $0 < u^2 \leqq 5/6$, take $\theta(-\infty) = -\pi$ and deduce that $\theta(x)$ varies continuously from $\theta(-\infty) = -\pi$ to $\theta(-u) = 0$ to $\theta(\infty) = 2\pi$, for $u^2 < 5/6$, or to $\theta(\infty) = 3\pi$, for $u^2 = 5/6$. For $u^2 > 5/6$ we find that $\theta(x)$ varies continuously from $\theta(-\infty) = 0$ to $\theta(-u) = 0$ to $\theta(\infty) = 3\pi$. Thus we can write [7]

$$\boldsymbol{\Phi}(z) = \frac{K}{(z+u)^2} \exp\left\{ \frac{1}{\pi} \int_{-u}^{\infty} [\theta(\mu) - 2\pi] \frac{\mathrm{d}\mu}{\mu - z} \right\}, \quad 0 < u^2 < 5/6, \tag{40a}$$

$$\boldsymbol{\Phi}(z) = \frac{K}{(z+u)^3} \exp\left\{ \frac{1}{\pi} \int_{-u}^{\infty} [\theta(\mu) - 3\pi] \frac{\mathrm{d}\mu}{\mu - z} \right\}, \quad u^2 \geqq 5/6, \tag{40b}$$

$$\Theta(z) = \frac{K_*}{(z+u)} \exp\left\{\frac{1}{\pi}\int_{-\infty}^{-u} [\theta(\tau)+\pi]\frac{d\tau}{\tau - z}\right\}, \qquad 0 < u^2 \leqq 5/6, \tag{41 a}$$

and

$$\Theta(z) = K_* \exp\left\{\frac{1}{\pi}\int_{-\infty}^{-u} \theta(\tau)\frac{d\tau}{\tau - z}\right\}, \qquad u^2 > 5/6, \tag{41 b}$$

where $K$ and $K^*$ are constants, $\Phi(z) = \det \boldsymbol{\Phi}(z)$ and $\Theta(z) = \det \boldsymbol{\Theta}(z)$. It is thus apparent that the indices [7] for the two Riemann-Hilbert problems defined by Eqs. (37) and (39) are, respectively, $\varkappa = 2$ for $0 < u^2 < 5/6$ or $\varkappa = 3$ for $u^2 \geqq 5/6$ and $\varkappa^* = 1$ for $0 < u^2 \leqq 5/6$ or $\varkappa^* = 0$ for $u^2 > 5/6$.

Considering the Riemann-Hilbert problem defined by Eqs. (37), we note first that $\boldsymbol{G}(-u) = \boldsymbol{I}$ and that $\boldsymbol{G}(\mu) \to \boldsymbol{I}$ as $\mu \to \infty$. We thus can define $\boldsymbol{G}(\mu) = \boldsymbol{I}$ for $\mu \in (-\infty, -u]$ and view

$$\boldsymbol{\Phi}^+(\mu) = \boldsymbol{G}(\mu)\,\boldsymbol{\Phi}^-(\mu) \tag{42}$$

as being valid for $\mu$ on the entire real axis. The transformation

$$\zeta = \frac{i-z}{i+z} \tag{43}$$

which maps the upper half plane into the interior of the unit circle can now be used to yield a Riemann-Hilbert problem defined on a closed contour. Thus the existence of a canonical solution follows from the work of Mandžavidze and Hvedelidze [8]. The same argument can be used to establish the existence of a solution to the Riemann-Hilbert problem defined by Eqs. (39). We can also follow Muskhelishvili [7] and express the dominant terms of $\boldsymbol{\Phi}(z)$ and $\boldsymbol{\Theta}(z)$, for large $|z|$, as

$$\boldsymbol{\Phi}(z) \to \boldsymbol{K} \begin{vmatrix} z^{-\varkappa_1} & 0 \\ 0 & z^{-\varkappa_2} \end{vmatrix} \tag{44 a}$$

and

$$\boldsymbol{\Theta}(z) \to \boldsymbol{K}_* \begin{vmatrix} z^{-\varkappa_1^*} & 0 \\ 0 & z^{-\varkappa_2^*} \end{vmatrix} \tag{44 b}$$

where $\varkappa_1$, $\varkappa_2$, $\varkappa_1^*$ and $\varkappa_2^*$ are the *partial* indices basic to the two considered Riemann-Hilbert problems.

Noting that

$$\boldsymbol{\Lambda}(z) \to \frac{1}{z}\, u \begin{vmatrix} -1 & \left(\frac{2}{3}\right)^{\frac{1}{2}} \\ 0 & 0 \end{vmatrix} + \frac{1}{z^2} \begin{vmatrix} -\frac{7}{6} & \left(\frac{2}{3}\right)^{\frac{1}{2}}\left(u^2 - \frac{1}{2}\right) \\ -\frac{1}{2}\left(\frac{2}{3}\right)^{\frac{1}{2}} & u^2 - \frac{1}{2} \end{vmatrix} + \ldots \tag{45}$$

as $|z| \to \infty$, we can use Eqs. (38) and (44) to deduce that for large $|z|$

$$P_{\alpha\alpha}(z) \to a_{\alpha\alpha}\, z^{\varkappa_\alpha + \varkappa_\alpha^* - 1} \tag{46}$$

where $P_{11}(z)$ and $P_{22}(z)$ are the diagonal elements of $\boldsymbol{P}(z)$ and $a_{\alpha\alpha}$, $\alpha = 1$ and 2, are constants. If we could show that $P_{11}(z)\, P_{22}(z) \not\equiv 0$ then Eq. (46) would yield $\varkappa_1 + \varkappa_1^* \geqq 1$ and $\varkappa_2 + \varkappa_2^* \geqq 1$ or, since $\varkappa_1 + \varkappa_2 = \varkappa$ and $\varkappa_1^* + \varkappa_2^* = \varkappa^*$,

$$\varkappa_1 + \varkappa_1^* \geqq 1 \tag{47 a}$$

and

$$\varkappa_1 + \varkappa_1^* \leqq \varkappa + \varkappa^* - 1. \tag{47 b}$$

We note that the partial indices are ordered, i.e. $\varkappa_1 \leqq \varkappa_2$ and $\varkappa_1^* \leqq \varkappa_2^*$, so that Eqs. (40), (41) and (47) can be used to deduce that $\varkappa_1 = 1$, $\varkappa_2 = 1$, $\varkappa_1^* = 0$ and $\varkappa_2^* = 1$ for $u^2 < 5/6$, that $\varkappa_1 = 1$, $\varkappa_2 = 2$, $\varkappa_1^* = 0$ and $\varkappa_2^* = 1$ for $u^2 = 5/6$ and that $\varkappa_1 = 1$, $\varkappa_2 = 2$, $\varkappa_1^* = 0$ and $\varkappa_2^* = 0$ for $u^2 > 5/6$. Though we have been unable to prove that $P_{11}(z)\, P_{22}(z) \not\equiv 0$, we proceed with our analysis with the *assumption* that such is the case. We recall that partial indices are crucial to the arguments used by Muskhelishvili [7] for determining if a system of singular integral equations, such as Eq. (31), is solvable and, if so, whether uniquely or not. Therefore our assumption here that $P_{11}(z)\, P_{22}(z) \not\equiv 0$, which allowed us to deduce the partial indices, is a significant flaw [9] in this analysis. However, in Appendix C we discuss some evidence based on numerical computations that we believe strongly supports our assumption.

It now follows from Eqs. (36) and (44 a) that $z\,\boldsymbol{N}(z)$ will be bounded at $\infty$ only if $\boldsymbol{P}_*(z) \equiv \boldsymbol{0}$,

$$\int_{-u}^{\infty} \boldsymbol{\Gamma}(\mu)\, \boldsymbol{F}(\mu)\, \mathrm{d}\mu = \boldsymbol{0}, \quad \text{all } u, \tag{48}$$

and

$$\begin{vmatrix} 0 \\ 1 \end{vmatrix}^T \int_{-u}^{\infty} \boldsymbol{\Gamma}(\mu)\, \boldsymbol{F}(\mu)\, \mu\, \mathrm{d}\mu = 0 \tag{49}$$

if $u^2 \geqq 5/6$. From Eq. (29 b) we see that $\boldsymbol{F}(\mu)$ contains two constants $\Delta\varrho$ and $\Delta T$ that can be determined uniquely from the system of two linear algebraic equations resulting from substituting Eq. (29 b) into Eq. (48), provided of course that the matrix of coefficients is not singular. In a similar vein it appears that there is no solution for the considered problem if $u^2 \geqq 5/6$ since Eqs. (48) and (49) would yield, upon substitution of Eq. (29 b), three linear algebraic equations for the two constants $\Delta\varrho$ and $\Delta T$. To be conclusive we clearly must demonstrate that these three equations are linearly independent.

## IV. *H* Matrices

In regard to the Riemann-Hilbert problems defined by Eqs. (37) and (39) we observe [10] that if $\boldsymbol{\Phi}(z)$ and $\boldsymbol{\Theta}(z)$ are canonical solutions with ordered normal form at infinity, then so are

$$\boldsymbol{\Phi}(z)\begin{vmatrix} a & 0 \\ P(z) & b \end{vmatrix} \quad \text{and} \quad \boldsymbol{\Theta}(z)\begin{vmatrix} c & 0 \\ P_*(z) & d \end{vmatrix}$$

where $a$, $b$, $c$ and $d$ are constants and the polynomials $P(z)$ and $P_*(z)$ are, respectively, of degrees $\varkappa_2 - \varkappa_1$, and $\varkappa_2^* - \varkappa_1^*$. For this reason we find it sufficiently general to consider the matrix $\boldsymbol{P}(z)$ in Eq. (38) a constant. Thus since $\boldsymbol{\Lambda}(-u) = \boldsymbol{I}$, we write Eq. (38) as

$$\boldsymbol{\Lambda}(z) = \boldsymbol{\Theta}(z)\,\boldsymbol{\Theta}^{-1}(-u)\,\boldsymbol{\Phi}^{-T}(-u)\,\boldsymbol{\Phi}^{T}(z). \tag{50}$$

Introducing the definitions

$$\boldsymbol{H}_2^{-T}(z) = \boldsymbol{\Phi}^{-T}(-u)\,\boldsymbol{\Phi}^{T}(z) \tag{51a}$$

and

$$\boldsymbol{H}_1^{-1}(z) = \boldsymbol{\Theta}(z)\,\boldsymbol{\Theta}^{-1}(-u) \tag{51b}$$

we deduce the desired factorization of $\boldsymbol{\Lambda}(z)$, i.e.

$$\boldsymbol{\Lambda}(z) = \boldsymbol{H}_1^{-1}(z)\,\boldsymbol{H}_2^{-T}(z). \tag{52}$$

Now since

$$\boldsymbol{H}_2^{-T}(z) = \boldsymbol{H}_1(z)\,\boldsymbol{\Lambda}(z) \tag{53}$$

vanishes at infinity we can write

$$\boldsymbol{H}_2^{-T}(z) = \frac{1}{2\pi i}\int_{-u}^{\infty} \boldsymbol{H}_1(\mu)\,[\boldsymbol{\Lambda}^+(\mu) - \boldsymbol{\Lambda}^-(\mu)]\frac{d\mu}{\mu - z} \tag{54}$$

or

$$\boldsymbol{H}_2^{-T}(z) = \boldsymbol{I} + (z+u)\int_{-u}^{\infty} \boldsymbol{H}_1(\mu)\,\boldsymbol{\Psi}(\mu)\frac{d\mu}{\mu - z}. \tag{55}$$

In a similar manner, we write

$$\boldsymbol{H}_1^{-1}(z) = \boldsymbol{H}_1^{-1}(\infty) + \frac{1}{2\pi i}\int_{-\infty}^{-u} [\boldsymbol{\Lambda}^+(\tau) - \boldsymbol{\Lambda}^-(\tau)]\,\boldsymbol{H}_2^{T}(\tau)\frac{d\tau}{\tau - z} \tag{56}$$

or

$$\boldsymbol{H}_1^{-1}(z) = \boldsymbol{I} - (z+u)\int_{u}^{\infty} \boldsymbol{\Psi}(-\tau)\,\boldsymbol{H}_2^{T}(-\tau)\frac{d\tau}{\tau + z}. \tag{57}$$

Thus $\boldsymbol{H}_1^{-1}(z)$ and $\boldsymbol{H}_2^{-T}(z)$ can be readily computed from Eqs. (55) and (57) for all appropriate $z$ once the coupled set

$$\boldsymbol{H}_1^{-1}(\mu) = \boldsymbol{I} - (\mu+u)\int_{u}^{\infty} \boldsymbol{\Psi}(-\tau)\,\boldsymbol{H}_2^{T}(-\tau)\frac{d\tau}{\tau + \mu} \tag{58a}$$

and

$$\boldsymbol{H}_2^{-T}(-\tau) = \boldsymbol{I} - (\tau-u)\int_{-u}^{\infty} \boldsymbol{H}_1(\mu)\,\boldsymbol{\Psi}(\mu)\frac{d\mu}{\mu + \tau} \tag{58b}$$

has been solved for $\mu \in [-u, \infty)$ and $\tau \in [u, \infty)$. In Appendix A we discuss a convenient way to develop the desired solutions to Eqs. (58).

If we now consider Eqs. (55) and (57) for large $|z|$ and write

$$\boldsymbol{H}_1^{-1}(z) = \boldsymbol{X}_0 + \frac{1}{z}\boldsymbol{X}_1 - \frac{1}{z^2}\boldsymbol{X}_2 + \dots \tag{59 a}$$

and

$$\boldsymbol{H}_2^{-T}(z) = \boldsymbol{Y}_0 - \frac{1}{z}\boldsymbol{Y}_1 - \frac{1}{z^2}\boldsymbol{Y}_2 + \dots \tag{59 b}$$

where

$$\boldsymbol{X}_0 = \boldsymbol{I} - \int_u^\infty \boldsymbol{\Psi}(-\tau)\,\boldsymbol{H}_2^T(-\tau)\,\mathrm{d}\tau, \tag{60 a}$$

$$\boldsymbol{Y}_0 = \boldsymbol{I} - \int_{-u}^\infty \boldsymbol{H}_1(\mu)\,\boldsymbol{\Psi}(\mu)\,\mathrm{d}\mu, \tag{60 b}$$

and, for $\alpha \geqq 1$,

$$\boldsymbol{X}_\alpha = \int_u^\infty (\tau - u)\,\boldsymbol{\Psi}(-\tau)\,\boldsymbol{H}_2^T(-\tau)\,\tau^{\alpha-1}\,\mathrm{d}\tau \tag{61 a}$$

and

$$\boldsymbol{Y}_\alpha = \int_{-u}^\infty (\mu + u)\,\boldsymbol{H}_1(\mu)\,\boldsymbol{\Psi}(\mu)\,\mu^{\alpha-1}\,\mathrm{d}\mu, \tag{61 b}$$

then Eqs. (44), (45) and (51) can be used to establish the following identities that provide useful checks for computed values of $\boldsymbol{H}_1(\mu)$ and $\boldsymbol{H}_2^T(-\tau)$:

$$\boldsymbol{Y}_0 = \boldsymbol{0}, \quad \text{all } u, \tag{62 a}$$

$$\boldsymbol{X}_0^T \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \boldsymbol{0}, \quad u^2 \leqq 5/6, \tag{62 b}$$

$$\boldsymbol{Y}_1 \begin{vmatrix} \sqrt{\dfrac{2}{3}} \\ 1 \end{vmatrix} = \boldsymbol{0}, \quad u^2 \geqq 5/6, \tag{62 c}$$

$$\boldsymbol{X}_0\,\boldsymbol{Y}_1 = u \begin{vmatrix} 1 & -\left(\dfrac{2}{3}\right)^{\frac{1}{2}} \\ 0 & 0 \end{vmatrix}, \quad \text{all } u, \tag{62 d}$$

and

$$\boldsymbol{X}_0\,\boldsymbol{Y}_2 \begin{vmatrix} \left(\dfrac{2}{3}\right)^{\frac{1}{2}} \\ 1 \end{vmatrix} = - \begin{vmatrix} \left(\dfrac{2}{3}\right)^{\frac{1}{2}}\left(u^2 - \dfrac{5}{3}\right) \\ u^2 - \dfrac{5}{6} \end{vmatrix}, \quad u^2 \geqq 5/6. \tag{62 e}$$

## V. Final results

We first note that Eq. (48) is equivalent to

$$\int_{-u}^{\infty} (\mu + u)\, \boldsymbol{H}_1(\mu)\, \boldsymbol{Q}^T(\mu)\, \boldsymbol{F}(\mu)\, \mathrm{e}^{-\mu^2}\, \mathrm{d}\mu = \boldsymbol{0} \tag{63}$$

and thus on using Eq. (29 b) in Eq. (63) we find

$$\int_{-u}^{\infty} (\mu + u)\, \boldsymbol{H}_1(\mu)\, \boldsymbol{Q}^T(\mu) \left\{ \begin{vmatrix} 1 \\ 0 \end{vmatrix} \Delta\varrho + \begin{vmatrix} \mu^2 - \dfrac{1}{2} \\ 1 \end{vmatrix} \Delta T \right\} \mathrm{e}^{-\mu^2}\, \mathrm{d}\mu = \boldsymbol{W}(u) \tag{64}$$

where

$$\boldsymbol{W}(u) = 2\,(u - u_0) \int_{-u}^{\infty} (\mu + u)\, \boldsymbol{H}_1(\mu)\, \boldsymbol{Q}^T(\mu)\, \mu\, \mathrm{e}^{-\mu^2}\, \mathrm{d}\mu \begin{vmatrix} 1 \\ 0 \end{vmatrix}. \tag{65}$$

Thus for $u^2 \in (0, 5/6)$ we can solve the two linear algebraic equations given by Eq. (64) to find $\Delta\varrho$ and $\Delta T$. Our numerical results for $u_0 = 0$ and various values of $u$ are shown, along with the results of I and those of Ytrehus [12], in Table I. It is apparent that the BGK model considered here is a significant improvement over the one-dimensional gas considered in I. To conclude we note that Eq. (36) can be written as

$$2\pi i\, \boldsymbol{N}(z) = \boldsymbol{\Lambda}^{-1}(z)\, \boldsymbol{H}_1^{-1}(z) \int_{-u}^{\infty} (\mu + u)\, \boldsymbol{H}_1(\mu)\, \boldsymbol{Q}^T(\mu)\, \boldsymbol{F}(\mu)\, \mathrm{e}^{-\mu^2} \frac{\mathrm{d}\mu}{\mu - z}, \tag{66}$$

and thus since

$$\boldsymbol{N}^+(\eta) - \boldsymbol{N}^-(\eta) = (\eta + u)\, \boldsymbol{A}(\eta)\, \mathrm{e}^{-\eta^2} \tag{67}$$

we can readily compute $\boldsymbol{A}(\eta)$ to establish the complete solution $\boldsymbol{\Psi}(x, \mu)$. For $u^2 = 5/6$ and selected values of $u^2 > 5/6$ we have found that the three equations found after substituting Eq. (29 b), with $u_0 = 0$, into Eqs. (48) and (49) cannot be solved.

Table I
Density and temperature perturbations.

| $u$ | $\varrho_\infty/\varrho_0$ | | | $T_\infty/T_0$ | | |
|---|---|---|---|---|---|---|
| | Previous work [1] | Present work | Ytrehus [12] | Previous work [1] | Present work | Ytrehus [12] |
| 0.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| 0.1 | 0.8812 | 0.8614 | 0.8494 | 0.9195 | 0.9552 | 0.9567 |
| 0.2 | 0.7961 | 0.7652 | 0.7283 | 0.8421 | 0.9101 | 0.9152 |
| 0.3 | 0.7322 | 0.6949 | 0.6303 | 0.7695 | 0.8648 | 0.8756 |
| 0.4 | 0.6824 | 0.6418 | 0.5501 | 0.7024 | 0.8196 | 0.8378 |
| 0.5 | 0.6427 | 0.6004 | 0.4841 | 0.6407 | 0.7748 | 0.8016 |
| 0.6 | 0.6104 | 0.5676 | 0.4292 | 0.5845 | 0.7308 | 0.7671 |
| 0.7 | 0.5838 | 0.5412 | 0.3834 | 0.5334 | 0.6877 | 0.7342 |
| 0.8 | 0.5616 | 0.5196 | 0.3447 | 0.4870 | 0.6458 | 0.7028 |
| 0.9 | 0.5429 | 0.5020 | 0.3120 | 0.4449 | 0.6052 | 0.6729 |

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## Appendix A – The $L$ equations

As for the scalar case discussed in I, we have found that an iterative solution of Eqs. (58) converges slowly, and thus we discuss here an analogous transformation that improves the rate of convergence. We introduce, for $u^2 < 5/6$,

$$\boldsymbol{L}_1^{-1}(z) = \boldsymbol{D}(z)\,\boldsymbol{H}_1^{-1}(z) \tag{A-1 a}$$

and

$$\boldsymbol{L}_2^{-T}(z) = (1 - z - u)\,\boldsymbol{H}_2^{-T}(z), \tag{A-1 b}$$

where

$$\boldsymbol{D}(z) = \begin{vmatrix} 1 & 0 \\ 0 & z+1+u \end{vmatrix}. \tag{A-2}$$

Noting that $\boldsymbol{L}_1^{-1}(z)$ and $\boldsymbol{L}_2^{-T}(z)$ are bounded at infinity, we can now use the Cauchy integral formula [6] to establish the following coupled non-linear integral equations:

$$\boldsymbol{L}_1^{-1}(\mu) = \boldsymbol{I} - (\mu + u) \int_u^\infty (\tau + 1 - u)\,\boldsymbol{D}(-\tau)\,\boldsymbol{\Psi}(-\tau)\,\boldsymbol{L}_2^T(-\tau)\frac{\mathrm{d}\tau}{\tau+\mu} \tag{A-3 a}$$

and

$$\boldsymbol{L}_2^{-T}(-\tau) = \boldsymbol{I} - (\tau - u) \int_{-u}^\infty (1 - \mu - u)\,\boldsymbol{L}_1(\mu)\,\boldsymbol{D}(\mu)\,\boldsymbol{\Psi}(\mu)\frac{\mathrm{d}\mu}{\mu+\tau} \tag{A-3 b}$$

where $\mu \in [-u, \infty)$ and $\tau \in [u, \infty)$. We have found, for $u^2 < 5/6$, that a numerical solution of Eqs. (A-3), by an iterative procedure, converges rapidly, and thus we have used, for $u^2 < 5/6$, this method to find $\boldsymbol{H}_1(\mu)$ and $\boldsymbol{H}_2(-\tau)$. For $u^2 = 5/6$ we define our $\boldsymbol{L}$ functions differently. Noting an earlier work on $\boldsymbol{L}$ equations [11], Eq. (45) and the fact that for large $|z|$

$$\Lambda(z) \to \frac{1}{z^4}, \quad u^2 = 5/6, \tag{A-4}$$

we let

$$\boldsymbol{L}_1^{-1}(z) = \boldsymbol{D}(z)\,\boldsymbol{H}_1^{-1}(z) \tag{A-5a}$$

and

$$\boldsymbol{L}_2^{-T}(z) = (1 - z - u)\,\boldsymbol{A}^T \boldsymbol{H}_2^{-T}(z)\,\boldsymbol{A}\,\boldsymbol{E}(z) \tag{A-5b}$$

where

$$\boldsymbol{E}(z) = \begin{vmatrix} 1 - z - u & 0 \\ 0 & 1 \end{vmatrix} \tag{A-6}$$

and

$$\boldsymbol{A} = \sqrt{\frac{3}{5}} \begin{vmatrix} \sqrt{\frac{2}{3}} & -1 \\ 1 & \sqrt{\frac{2}{3}} \end{vmatrix}. \tag{A-7}$$

Again $\boldsymbol{L}_1^{-1}(z)$ and $\boldsymbol{L}_2^{-T}(z)$ are bounded at infinity, and we find, for $u^2 = 5/6$, that

$$\boldsymbol{L}_1^{-1}(\mu) = \boldsymbol{I} - (\mu + u) \int_u^\infty (\tau + 1 - u)\,\boldsymbol{D}(-\tau)\,\boldsymbol{\Psi}(-\tau)\,\boldsymbol{A}\,\boldsymbol{E}(-\tau)\,\boldsymbol{L}_2^T(-\tau)\,\boldsymbol{A}^T \cdot \frac{\mathrm{d}\tau}{\tau + \mu} \tag{A-8a}$$

and

$$\boldsymbol{L}_2^{-T}(-\tau) = \boldsymbol{I} - (\tau - u) \int_{-u}^\infty (1 - \mu - u)\,\boldsymbol{A}^T \boldsymbol{L}_1(\mu)\,\boldsymbol{D}(\mu)\,\boldsymbol{\Psi}(\mu)\,\boldsymbol{A}\,\boldsymbol{E}(\mu) \frac{\mathrm{d}\mu}{\mu + \tau} \tag{A-8b}$$

can be solved iteratively in a rapidly converging manner. Finally for $u^2 > 5/6$ we let

$$\boldsymbol{L}_1^{-1}(z) = \boldsymbol{H}_1^{-1}(z) \tag{A-9a}$$

and

$$\boldsymbol{L}_2^{-T}(z) = (1 - z - u)\,\boldsymbol{A}^T \boldsymbol{H}_2^{-T}(z)\,\boldsymbol{A}\,\boldsymbol{E}(z) \tag{A-9b}$$

and find

$$\boldsymbol{L}_1^{-1}(\mu) = \boldsymbol{I} - (\mu + u) \int_u^\infty (\tau + 1 - u)\,\boldsymbol{\Psi}(-\tau)\,\boldsymbol{A}\,\boldsymbol{E}(-\tau)\,\boldsymbol{L}_2^T(-\tau)\,\boldsymbol{A}^T \frac{\mathrm{d}\tau}{\tau + \mu} \tag{A-10a}$$

and

$$\boldsymbol{L}_2^{-T}(-\tau) = \boldsymbol{I} - (\tau - u) \int_{-u}^\infty (1 - \mu - u)\,\boldsymbol{A}^T \boldsymbol{L}_1(\mu)\,\boldsymbol{\Psi}(\mu)\,\boldsymbol{A}\,\boldsymbol{E}(\mu) \frac{\mathrm{d}\mu}{\mu + \tau}. \tag{A-10b}$$

## Appendix B: An approximate solution

We now use the $F_N$ method [13, 14] to construct an approximate, but accurate, solution for $\boldsymbol{\Psi}(0,-\mu)$, $\mu \in [u, \infty)$, and the desired $\Delta\varrho$ and $\Delta T$. Setting $x=0$ in Eq. (30), we write

$$\boldsymbol{\Psi}(0,\mu) = \int_{-u}^{\infty} \boldsymbol{\Phi}(\eta,\mu)\, \boldsymbol{A}(\eta)\, \mathrm{d}\eta. \tag{B-1}$$

Thus following our previous work on the one-dimensional gas [14], we assume for the moment that $\boldsymbol{\Psi}(0,\mu)$ is known for all $\mu \in (-\infty, \infty)$ and use the theory of Muskhelishvili [7] to show that the system of singular integral equations given by Eq. (B-1) has a unique solution provided

$$\int_{-\infty}^{\infty} (\mu+u)\, \boldsymbol{Q}^T(\mu)\, \boldsymbol{\Psi}(0,\mu)\, \mathrm{e}^{-\mu^2}\, \mathrm{d}\mu = \boldsymbol{0}, \tag{B-2a}$$

$$\begin{vmatrix} 0 \\ 1 \end{vmatrix}^T \int_{-\infty}^{\infty} \mu\,(\mu+u)\, \boldsymbol{Q}^T(\mu)\, \boldsymbol{\Psi}(0,\mu)\, \mathrm{e}^{-\mu^2}\, \mathrm{d}\mu = 0 \tag{B-2b}$$

and

$$\int_{-\infty}^{\infty} (\mu+u)\, \boldsymbol{\Phi}^\dagger(-\eta,\mu)\, \boldsymbol{Q}^T(\mu)\, \boldsymbol{\Psi}(0,\mu)\, \mathrm{e}^{-\mu^2}\, \mathrm{d}\mu = 0, \quad \eta \in [u,\infty), \tag{B-3}$$

where

$$\boldsymbol{\Phi}^\dagger(\eta,\mu) = (\eta+u)\, P v\left(\frac{1}{\eta-\mu}\right) \boldsymbol{I} + \boldsymbol{\lambda}(\eta)\, \boldsymbol{\Psi}^{-1}(\eta)\, \delta(\eta-\mu). \tag{B-4}$$

If we now substitute the approximation

$$\boldsymbol{\Psi}(0,-\mu) = \boldsymbol{K}(-\mu) \sum_{\alpha=0}^{N} \left(\frac{1}{\nu_\alpha+\mu}\right) \boldsymbol{A}_\alpha, \quad \mu \in [u,\infty), \quad \nu_\alpha \in [-u,\infty), \tag{B-5}$$

where $\boldsymbol{K}(\mu)$ is given by Eq. (11), and Eq. (29b) into Eqs. (B-2) and (B-3) we find

$$-\sum_{\alpha=0}^{N} \boldsymbol{I}_0(\nu_\alpha)\, \boldsymbol{A}_\alpha + \begin{vmatrix} M_2 - \frac{1}{2} M_0 \\ M_0 \end{vmatrix} \Delta\varrho + \begin{vmatrix} M_4 - M_2 + \frac{5}{4} M_0 \\ M_2 - \frac{1}{2} M_0 \end{vmatrix} \Delta T = 2u \begin{vmatrix} M_3 - \frac{1}{2} M_1 \\ M_1 \end{vmatrix}, \tag{B-6a}$$

$$\sum_{\alpha=0}^{N} \begin{vmatrix} 0 \\ 1 \end{vmatrix}^T \boldsymbol{I}_1(\nu_\alpha)\, \boldsymbol{A}_\alpha + M_1\, \Delta\varrho + (M_3 - \tfrac{1}{2} M_1)\, \Delta T = 2u\, M_2 \tag{B-6b}$$

and, for $\eta \in [u, \infty)$,

$$\sum_{\alpha=0}^{N} \boldsymbol{\Gamma}_\alpha(\eta)\, \boldsymbol{A}_\alpha + \begin{vmatrix} R_2(\eta) - \frac{1}{2} R_0(\eta) \\ R_0(\eta) \end{vmatrix} \Delta\varrho + \begin{vmatrix} R_4(\eta) - R_2(\eta) + \frac{5}{4} R_0(\eta) \\ R_2(\eta) - \frac{1}{2} R_0(\eta) \end{vmatrix} \Delta T$$
$$= 2u \begin{vmatrix} R_3(\eta) - \frac{1}{2} R_1(\eta) \\ R_1(\eta) \end{vmatrix}. \tag{B-7}$$

Here

$$M_j = \frac{1}{\sqrt{\pi}} \int_{-u}^{\infty} c^j (c+u)\, \mathrm{e}^{-c^2}\, \mathrm{d}c, \tag{B-8}$$

$$\boldsymbol{I}_\beta(z) = \operatorname{diag}\left\{\sqrt{\frac{3}{2}}, 1\right\} \frac{1}{\pi} \int_{u}^{\infty} \mu^\beta (\mu - u)\, \boldsymbol{\Psi}(-\mu) \frac{\mathrm{d}\mu}{\mu + z}, \tag{B-9}$$

$$R_\alpha(\eta) = \int_{-u}^{\infty} \mu^\alpha (\mu + u)\, \mathrm{e}^{-\mu^2} \frac{\mathrm{d}\mu}{\mu + \eta} \tag{B-10}$$

and

$$\boldsymbol{\Gamma}_\alpha(\eta) = -\frac{1}{\sqrt{\pi}} \operatorname{diag}\left\{\sqrt{\frac{3}{2}}, 1\right\} \left(\frac{1}{\nu_\alpha + \eta}\right)$$

$$\cdot \left[\int_{u}^{\infty} (\mu - u)\, \boldsymbol{\Psi}(-\mu) \frac{\mathrm{d}\mu}{\mu + \nu_\alpha} + \int_{-u}^{\infty} (\mu + u)\, \boldsymbol{\Psi}(\mu) \frac{\mathrm{d}\mu}{\mu + \eta}\right]. \tag{B-11}$$

If we now consider Eq. (B-7) at $N$ selected values of $\eta_j, j = 1, 2, \ldots, N$, along with Eqs. (B-6) then clearly we find $2N + 3$ linear algebraic equations for the $2N + 4$ unknowns $\Delta\varrho$, $\Delta T$, $\{A_{1,\alpha}\}$ and $\{A_{2,\alpha}\}$, where $A_{1,\alpha}$ and $A_{2,\alpha}$ denote the elements of $\boldsymbol{A}_\alpha$. In order to generate an additional equation, we use a combination of the two components of Eq. (B-7) evaluated at $\eta = u$. Thus if, for example, we let $\nu_\alpha = \tau_\alpha - u$ and $\eta_j = \tau_j + u$, where $\{\tau_j\}$ are the positive zeros of the Hermite polynomials $H_{2N}(\xi)$, then we can solve the linear algebraic equations

$$-\sum_{\alpha=0}^{N} \boldsymbol{I}_0(\nu_\alpha) \boldsymbol{A}_\alpha + \begin{vmatrix} M_2 - \frac{1}{2} M_0 \\ M_0 \end{vmatrix} \Delta\varrho + \begin{vmatrix} M_4 - M_2 + \frac{5}{4} M_0 \\ M_2 - \frac{1}{2} M_0 \end{vmatrix} \Delta T$$

$$= 2u \begin{vmatrix} M_3 - \frac{1}{2} M_1 \\ M_1 \end{vmatrix}, \tag{B-12a}$$

$$\sum_{\alpha=0}^{N} \begin{vmatrix} 0 \\ 1 \end{vmatrix}^T \boldsymbol{I}_1(\nu_\alpha) \boldsymbol{A}_\alpha + M_1 \Delta\varrho + (M_3 - \tfrac{1}{2} M_1) \Delta T = 2u M_2, \tag{B-12b}$$

$$\boldsymbol{W}^T \left\{\sum_{\alpha=0}^{N} \boldsymbol{\Gamma}_\alpha(u) \boldsymbol{A}_\alpha + \begin{vmatrix} R_2(u) - \frac{1}{2} R_0(u) \\ R_0(u) \end{vmatrix} \Delta\varrho + \begin{vmatrix} R_4(u) - R_2(u) + \frac{5}{4} R_0(u) \\ R_2(u) - \frac{1}{2} R_0(u) \end{vmatrix} \Delta T\right\}$$

$$= 2u\, \boldsymbol{W}^T \begin{vmatrix} R_3(u) - \frac{1}{2} R_1(u) \\ R_1(u) \end{vmatrix} \tag{B-12c}$$

and, for $j = 1, 2, \ldots, N$,

$$\sum_{\alpha=0}^{N} \boldsymbol{\Gamma}_\alpha(\eta_j)\, \boldsymbol{A}_\alpha + \begin{vmatrix} R_2(\eta_j) - \frac{1}{2} R_0(\eta_j) \\ R_0(\eta_j) \end{vmatrix} \Delta\varrho + \begin{vmatrix} R_4(\eta_j) - R_2(\eta_j) + \frac{5}{4} R_0(\eta_j) \\ R_2(\eta_j) - \frac{1}{2} R_0(\eta_j) \end{vmatrix} \Delta T$$
$$= 2u \begin{vmatrix} R_3(\eta_j) - \frac{1}{2} R_1(\eta_j) \\ R_1(\eta_j) \end{vmatrix} \tag{B-12d}$$

to find the desired constants $\Delta\varrho$, $\Delta T$ and $\{\boldsymbol{A}_\alpha\}$. We have used, for $l \in [0, 1]$,

$$\boldsymbol{W} = \begin{vmatrix} 1-l \\ l \end{vmatrix} \tag{B-13}$$

and the $F_N$ method discussed here to reproduce the exact results given in Table I with $N \leqq 4$.

## C. Indices

If we write Eq. (58 a), for $\nu \in [-u, \infty)$, as

$$\boldsymbol{H}_1(\nu) = \boldsymbol{I} + (\nu + u) \int_u^\infty \boldsymbol{\Psi}(-\tau)\, \boldsymbol{H}_2^T(-\tau) \frac{\mathrm{d}\tau}{\tau + \nu} \boldsymbol{H}_1(\nu) \tag{C-1}$$

we can multiply Eq. (C-1) by $\boldsymbol{\Psi}(\nu)\, Pv\left(\frac{1}{\nu - \mu}\right)$ and integrate to find the linear singular integral equation, for $\mu \in [-u, \infty)$,

$$\boldsymbol{H}_1(\mu)\, \boldsymbol{\lambda}(\mu) = \boldsymbol{I} + (\mu + u)\, P \int_{-u}^\infty \boldsymbol{H}_1(\nu)\, \boldsymbol{\Psi}(\nu) \frac{\mathrm{d}\nu}{\nu - \mu}. \tag{C-2a}$$

In a similar manner we find, for $\tau \in [u, \infty)$,

$$\boldsymbol{\lambda}(-\tau)\, \boldsymbol{H}_2^T(-\tau) = \boldsymbol{I} + (\tau - u)\, P \int_u^\infty \boldsymbol{\Psi}(-\nu)\, \boldsymbol{H}_2^T(-\nu) \frac{\mathrm{d}\nu}{\nu - \tau}. \tag{C-2b}$$

Clearly if Eqs. (58) have solutions then so do Eqs. (C-2). Following Muskhelishvili [7] we can now show that Eqs. (C-2) have solutions only if $\varkappa_1^* \geqq 0$ and $\varkappa_1 \geqq 0$. Thus the fact that we have computed solutions of Eqs. (58) provides good evidence that the partial indices are non-negative. If $\varkappa_1^* \geqq 0$ then clearly the partial indices relevant to $\boldsymbol{\Theta}(z)$ are determined, i.e. $\varkappa_1^* = 0$ for all $u$, $\varkappa_2^* = 1$ for $u^2 \leqq 5/6$ and $\varkappa_2^* = 0$ for $u^2 > 5/6$. For $u^2 > 5/6$ we can show, if $\varkappa_1 \geqq 0$, that $\varkappa_1 = 0$ is not possible; thus for this case $\varkappa_1 = 1$ and $\varkappa_2 = 2$. Given that we have solutions of Eqs. (58) that, by way of Eqs. (55) and (57), provide a factorization of $\Lambda(z)$ means that

$$\boldsymbol{H}_2^{-1}(z) = \boldsymbol{\Phi}(z)\,\boldsymbol{P}(z) \tag{C-3}$$

where $\boldsymbol{P}(z)$ is a matrix of polynomials. Since we have verified numerically for $u^2 \leqq 5/6$, that $\boldsymbol{H}_2^{-1}(\infty) = \boldsymbol{0}$ we conclude from Eq. (C-3) that $\varkappa_1 > 0$ and thus that $\varkappa_1 = \varkappa_2 = 1$ for $u^2 < 5/6$ and $\varkappa_1 = 1$ and $\varkappa_2 = 2$ for $u^2 = 5/6$.

## References

[1] C. E. Siewert and J. R. Thomas, Jr., Z. angew. Math. Phys. *32*, 421 (1981).
[2] K. M. Case, Ann. Phys., *9*, 1 (1960).
[3] C. Cercignani, Ann. Phys., *20*, 219 (1962).
[4] M. D. Arthur and C. Cercignani, Z. angew. Math. Phys. *31*, 634 (1980).
[5] J. T. Kriese, T. S. Chang, and C. E. Siewert, Int. J. Engng. Sci., *12*, 441 (1974).
[6] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, New York (1953).
[7] N. I. Muskhelishvili, *Singular Integral Equations*, Noordhoff, Groningen, Holland (1953).
[8] G. F. Mandžavidze and B. V. Hvedelidze, Dokl. Akad. Nauk. SSSR *123*, 791 (1958).
[9] E. E. Burniston, C. E. Siewert, P. Silvennoinen, and P. F. Zweifel, Nucl. Sci. Eng. *45*, 331 (1971).
[10] C. E. Siewert and E. E. Burniston, Ap. J., *174*, 629 (1972).
[11] J. T. Kriese and C. E. Siewert, Ap. J., *164*, 389 (1971).
[12] T. Ytrehus, in: *Rarefied Gas Dynamics*, J. L. Potter, ed., AIAA, New York (1977).
[13] C. E. Siewert, R. D. M. Garcia and P. Grandjean, J. Math. Phys., *21*, 2760 (1980).
[14] S. K. Loyalka, C. E. Siewert and J. R. Thomas, Jr., Z. angew. Math. Phys., *32*, 745 (1981).

## Abstract

Evaporation of a liquid into a vacuum occupying a half space is investigated on the basis of the BGK equation for a three-dimensional gas linearized about a drifting Maxwellian distribution. The theory of singular integral equations is used and numerical results are given.

## Zusammenfassung

Die Verdampfung einer Flüssigkeit im Vakuum, die den Halbraum füllt, wird untersucht mit Benützung der BGK-Gleichung für ein dreidimensionales Gas, linearisiert in bezug auf eine mitbewegte Maxwell-Verteilung. Es wird die Theorie der singulären Integralgleichungen benützt, und es werden numerische Resultate gegeben.

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