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On partial indices for a matrix Riemann-Hilbert problem

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I. Introduction

In a recent paper [1] Siewert and Kelley used a solution, similar to one obtained by Cercignani [2], of the Riemann-Hilbert problem defined by

$$\boldsymbol{\Phi}^+(\mu) = \boldsymbol{G}(\mu)\, \boldsymbol{\Phi}^-(\mu), \qquad \mu \in [0, \infty)\,, \tag{1}$$

where

$$\boldsymbol{G}(\mu) = \boldsymbol{\Lambda}^+(\mu)[\boldsymbol{\Lambda}^-(\mu)]^{-1}\,, \tag{2}$$

$$\boldsymbol{\Lambda}(z) = \boldsymbol{I} + z \int_{-\infty}^{\infty} \boldsymbol{\Psi}(\mu) \frac{\mathrm{d}\mu}{\mu - z}\,, \tag{3}$$

$$\boldsymbol{\Psi}(\mu) = (\pi)^{-1/2}\, \boldsymbol{Q}^T(\mu)\, \boldsymbol{Q}(\mu)\, \mathrm{e}^{-\mu^2} \tag{4}$$

and

$$\boldsymbol{Q}(\mu) = \begin{vmatrix} \left(\dfrac{2}{3}\right)^{1/2} \left(\mu^2 - \dfrac{1}{2}\right) & 1 \\ \left(\dfrac{2}{3}\right)^{1/2} & 0 \end{vmatrix}, \tag{5}$$

to construct a *canonical* solution of the problem. In the process of developing their canonical solution, Siewert and Kelley [1] used the fact, proved previously by Kriese, Chang and Siewert [3], that the partial indices relevant to the considered Riemann-Hilbert problem are $\varkappa_1 = \varkappa_2 = 1$. Here we wish to show that the partial indices $\varkappa_1$ and $\varkappa_2$ can be determined directly from the solutions given by Cercignani [2] and Siewert and Kelley [1]. In order to condense our presentation, we assume that the paper by Siewert and Kelley, to which we hereafter refer as I, is available, and we use, with the inclusion of I in the equation numbers, the results developed there.

II. Analysis

The solution, before it was converted to a *canonical* solution, developed in I is written as

$$\boldsymbol{\Phi}(z) = \boldsymbol{S}^{-1}(z) \begin{vmatrix} U_1(z) & 0 \\ 0 & U_2(z) \end{vmatrix} \boldsymbol{S}(z)\,, \tag{6}$$

where $\boldsymbol{S}(z)$ is given by Eq. (I-12) and $U_\alpha(z)$, $\alpha = 1$ and 2, are given by Eqs. (I-39) and (I-40). If we let $\boldsymbol{\Phi}_0(z)$ denote a canonical solution then, from I,

$$\boldsymbol{\Phi}_0(z) = [z(z-x_1)]^{-2}\,\boldsymbol{\Phi}(z)\,\boldsymbol{P}(z)\,, \tag{7}$$

where $\boldsymbol{P}(z)$ is a matrix of polynomials with $\det \boldsymbol{P}(z) \propto z^2(z-x_1)^2$. From Eq. (7) and I it is apparent that $\boldsymbol{P}(z)$ must be determined so that

$$\boldsymbol{\Phi}(\xi)\,\boldsymbol{P}(\xi) = \boldsymbol{0}, \quad \xi = 0 \quad \text{and} \quad \xi = x_1 \tag{8a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\xi}[\boldsymbol{\Phi}(\xi)\,\boldsymbol{P}(\xi)] = \boldsymbol{0}, \quad \xi = 0 \quad \text{and} \quad \xi = x_1\,. \tag{8b}$$

We note from Muskhelishvili [4] that as $|z| \to \infty$

$$\boldsymbol{\Phi}_0(z) \to \boldsymbol{K}\,\mathrm{diag}\{z^{-\varkappa_1}, z^{-\varkappa_2}\}, \qquad \det \boldsymbol{K} \neq 0\,, \tag{9}$$

where $\varkappa_1$ and $\varkappa_2 \geqq \varkappa_1$ are the partial indices. We note also that $\varkappa_1 + \varkappa_2 = \varkappa$, where $\varkappa$ can be computed from $\det \boldsymbol{G}(\mu)$; it is also apparent from Eq. (7) that $\varkappa = 2$, which agrees with the calculation of Kriese, Chang and Siewert [3].

After using Eqs. (6) and (9) along with the explicit expressions for $\boldsymbol{S}(z)$ and $U_\alpha(z)$ given in I, we can readily deduce from Eq. (7) that as $|z| \to \infty$

$$\boldsymbol{P}(z) \to z^3 \begin{vmatrix} a_{11}\,z^{-\varkappa_1} & a_{12}\,z^{-\varkappa_2} \\ a_{21}\,z^{-\varkappa_1} & a_{22}\,z^{-\varkappa_2} \end{vmatrix}. \tag{10}$$

It is clear from Eq. (10) that $\varkappa_1 > -2$ for otherwise the second column of $\boldsymbol{P}(z)$ would be identically zero, which is not allowed. From I, we observe that $U_2(z)$ has a double zero at $z = 0$, and thus we conclude from Eqs. (8), after using Eq. (6), that

$$\begin{vmatrix} \left(\frac{3}{2}\right)^{1/2} & 1 \\ 0 & 0 \end{vmatrix} \bar{\boldsymbol{P}}(0) = \boldsymbol{0} \tag{11a}$$

and

$$\begin{vmatrix} \left(\frac{3}{2}\right)^{1/2} & 1 \\ 0 & 0 \end{vmatrix} \frac{\mathrm{d}}{\mathrm{d}\xi}\boldsymbol{P}(\xi) = \boldsymbol{0}, \qquad \xi = 0\,. \tag{11b}$$

Now since $U_1(x_1) = 0$, we can conclude from Eq. (8a) that

$$\begin{vmatrix} 0 & 0 \\ 1 & \alpha \end{vmatrix} \boldsymbol{P}(x_1) = \boldsymbol{0}\,, \tag{12}$$

where, from I,

$$\alpha = -\left(\frac{1}{6}\right)^{1/2}\left[R(x_1) + x_1^2 + \frac{1}{2}\right]. \tag{13}$$

If we assume for the moment that the elements of the second column of $\boldsymbol{P}(z)$ are linear in z, then Eqs. (11) fix the ratio $P_{12}(z)/P_{22}(z)$ to be $-(2/3)^{1/2}$. From Eq. (12) we thus find a contradiction since $\alpha \neq (2/3)^{1/2}$. It follows that the second column of $\boldsymbol{P}(z)$ must be at least quadratic in z and therefore that $\varkappa_1 = \varkappa_2 = 1$, which agrees with the conclusion of Kriese, Chang and Siewert [3].

III. Concluding remarks

It is worthwhile to recall from the work of Muskhelishvili [4] that the partial indices play a crucial role in the theory of systems of singular-integral equations. Muskhelishvili [4] in fact uses these indices to determine if a given system can be solved, and if so the number and type of constraints that may be required to yield a unique solution. Though there are problems in regard to coupled systems in particle transport theory, see for example [3, 5, 6], for which the partial indices have been determined, there are also important cases, even for two-vector problems, where these indices remain elusive [7, 8]. In our continuing work we intend to see if the idea developed here can be used to determine the partial indices for Riemann-Hilbert problems more difficult than the one considered at present.

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Abstract

An analytical solution of the matrix Riemann-Hilbert problem relevant to the BGK model in the field of rarefied-gas dynamics is used to deduce the partial indices basic to a canonical solution of the considered Riemann-Hilbert problem.

Sommario

Si fa uso di una soluzione analitica del problema di Riemann-Hilbert che origina dal modello BGK nel campo della dinamica dei gas rarefatti per dedurre gli indici necessari per ottenere una soluzione canonica del problema di Riemann-Hilbert.

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