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ASTRONOMY AND ASTROPHYSICS

On the Phase Matrix Basic to the Scattering of Polarized Light

C. E. Siewert

Department of Physics and Astronomy, Free University, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands

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Summary. A special representation of the scattering matrix, basic to studies of polarized light, that allows the components in a Fourier decomposition to be expressed analytically is reviewed. In order to provide a convenient method of using the representation for practical calculations, a basic set of orthogonality and recursive relations is reported and used to provide working formulas for the basic constants and basic matrices required in the established formalism. The final results are also used to provide an independent verification of known symmetry relations satisfied by the phase matrix.

Key words: polarization – radiative transfer – phase matrix – multiple scattering – dust – planetary atmospheres

I. Introduction

We consider the equation of transfer (Chandrasekhar, 1950)

$$\mu\frac{\partial}{\partial\tau}\boldsymbol{I}(\tau,\mu,\phi)+\boldsymbol{I}(\tau,\mu,\phi)=\frac{\omega}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\boldsymbol{P}(\mu,\mu',\phi-\phi')\boldsymbol{I}(\tau,\mu',\phi')d\mu'd\phi' \tag{1}$$

where the four-vector $\boldsymbol{I}(\tau,\mu,\phi)$ has the Stokes parameters $I(\tau,\mu,\phi)$, $Q(\tau,\mu,\phi)$, $U(\tau,\mu,\phi)$, and $V(\tau,\mu,\phi)$ as components, $\boldsymbol{P}(\mu.\mu',\phi-\phi')$ is the *phase matrix* and ω is the albedo for single scattering. As discussed by Chandrasekhar (1950) and Hovenier (1971) the phase matrix is related to the *scattering matrix* $\boldsymbol{F}(\cos\Theta)$ by

$$\boldsymbol{P}(\mu,\mu',\phi-\phi')=\boldsymbol{L}(\pi-i_2)\boldsymbol{F}(\cos\Theta)\boldsymbol{L}(-i_1). \tag{2}$$

Here Θ is the angle between light rays before and after scattering, and the matrices $\boldsymbol{L}(\pi-i_2)$ and $\boldsymbol{L}(-i_1)$ are those required to rotate meridian planes before and after scattering onto a local scattering plane. In a recent paper (Siewert, 1981) that commences with the "circularly polarized" representation of the radiation field and a particular expansion of the corresponding phase matrix introduced by Kuscer and Ribaric (1959), the equation of transfer for a Stokes representation was derived in such a way that the components in a Fourier decomposition of the phase matrix $\boldsymbol{P}(\mu,\mu',\phi-\phi')$ could be expressed analytically. Here we restate the final result of the previous paper (Siewert, 1981), and we develop explicit expressions for the basic coefficients in our analytical representation of the phase matrix corresponding to a scattering matrix of the form considered by Hovenier (1971), i.e.

$$\boldsymbol{F}(\xi)=\begin{vmatrix} a_1(\xi) & b_1(\xi) & 0 & 0\\ b_1(\xi) & a_2(\xi) & 0 & 0\\ 0 & 0 & a_3(\xi) & b_2(\xi)\\ 0 & 0 & -b_2(\xi) & a_4(\xi)\end{vmatrix} \tag{3}$$

where $\xi=\cos\Theta$. The six functions appearing in Eq. (3) are real valued for $\xi\in[-1,1]$, and $\boldsymbol{F}(\xi)$ is normalized such that

$$\int_{-1}^{1} a_1(\xi)d\xi=2. \tag{4}$$

In addition we note that van de Hulst (1957) has shown for the scattering model we consider that $b_1(\pm1)=b_2(\pm1)=0$ and $a_2(\pm1)=\pm a_3(\pm1)$.

In our previous work (Siewert, 1981) and more explicitly in a following paper (Siewert and Pinheiro, 1981) it was shown that corresponding to a representation of the scattering matrix $\boldsymbol{F}(\xi)$ defined by

$$a_1(\xi)=\sum_{l=0}^{\infty}\beta_l P_l(\xi),\ \beta_0=1, \tag{5a}$$

$$a_2(\xi)=\sum_{l=2}^{\infty}\left[\frac{(l-2)!}{(l+2)!}\right]^{1/2}\{\alpha_l R_l^2(\xi)+\zeta_l T_l^2(\xi)\} \tag{5b}$$

$$a_3(\xi)=\sum_{l=2}^{\infty}\left[\frac{(l-2)!}{(l+2)!}\right]^{1/2}\{\zeta_l R_l^2(\xi)+\alpha_l T_l^2(\xi)\} \tag{5c}$$

$$a_4(\xi)=\sum_{l=0}^{\infty}\delta_l P_l(\xi) \tag{5d}$$

$$b_1(\xi)=\sum_{l=2}^{\infty}\left[\frac{(l-2)!}{(l+2)!}\right]^{1/2}\gamma_l P_l^2(\xi) \tag{5e}$$

and

$$b_2(\xi)=-\sum_{l=2}^{\infty}\left[\frac{(l-2)!}{(l+2)!}\right]^{1/2}\varepsilon_l P_l^2(\xi) \tag{5f}$$

the phase matrix could be expressed as

$$\boldsymbol{P}(\mu,\mu',\phi-\phi')=\frac{1}{2}\boldsymbol{C}^0(\mu,\mu')+\sum_{m=1}^{\infty}[\boldsymbol{C}^m(\mu,\mu')\cos m(\phi-\phi')+\boldsymbol{S}^m(\mu,\mu')\sin m(\phi-\phi')]. \tag{6}$$

Here

$$\boldsymbol{C}^m(\mu,\mu')=\boldsymbol{A}^m(\mu,\mu')+\boldsymbol{D}\boldsymbol{A}^m(\mu,\mu')\boldsymbol{D}, \tag{7a}$$

$$\boldsymbol{S}^m(\mu,\mu')=\boldsymbol{A}^m(\mu,\mu')\boldsymbol{D}-\boldsymbol{D}\boldsymbol{A}^m(\mu,\mu'), \tag{7b}$$

$$\boldsymbol{A}^m(\mu,\mu')=\sum_{l=m}^{\infty}\boldsymbol{\Pi}_l^m(\mu)\boldsymbol{B}_l^m\boldsymbol{\Pi}_l^m(\mu'), \tag{8}$$

$$\boldsymbol{D}=\mathrm{diag}\{1,1,-1,-1\}, \tag{9}$$

$$\boldsymbol{\Pi}_l^m(\mu)=\begin{vmatrix} P_l^m(\mu) & 0 & 0 & 0\\ 0 & R_l^m(\mu) & -T_l^m(\mu) & 0\\ 0 & -T_l^m(\mu) & R_l^m(\mu) & 0\\ 0 & 0 & 0 & P_l^m(\mu)\end{vmatrix} \tag{10}$$

and

$$\boldsymbol{B}_l^m=\frac{(l-m)!}{(l+m)!}\boldsymbol{B}_l \tag{11}$$

with

$$\boldsymbol{B}_l=\begin{vmatrix} \beta_l & \gamma_l & 0 & 0\\ \gamma_l & \alpha_l & 0 & 0\\ 0 & 0 & \zeta_l & -\varepsilon_l\\ 0 & 0 & \varepsilon_l & \delta_l\end{vmatrix}. \tag{12}$$

We use $P_l(\mu)$ to denote the Legendre polynomial of order l and

$$P_l^m(\mu)=(1-\mu^2)^{m/2}\frac{d^m}{d\mu^m}P_l(\mu) \tag{13}$$

to denote the associated Legendre function. In addition

$$R_l^m(\mu)=-\frac{1}{2}(i)^m\left[\frac{(l+m)!}{(l-m)!}\right]^{1/2}\{P_{m,2}^l(\mu)+P_{m,-2}^l(\mu)\} \tag{14a}$$

and

$$T_l^m(\mu)=-\frac{1}{2}(i)^m\left[\frac{(l+m)!}{(l-m)!}\right]^{1/2}\{P_{m,2}^l(\mu)-P_{m,-2}^l(\mu)\} \tag{14b}$$

where, for $l\geqq\sup(|m|,|n|)$,

$$P_{m,n}^l(\mu)=A_{m,n}^l(1-\mu)^{-(n-m)/2}(1+\mu)^{-(n+m)/2}\frac{d^{l-n}}{d\mu^{l-n}}[(1-\mu)^{l-m}(1+\mu)^{l+m}], \tag{15}$$

with

$$A_{m,n}^l=\frac{(-1)^{l-m}(i)^{n-m}}{2^l(l-m)!}\left[\frac{(l-m)!(l+n)!}{(l+m)!(l-n)!}\right]^{1/2}, \tag{16}$$

are the generalized spherical functions discussed by Gel'fand and Sapiro (1956). We note that Deuze (1974) has reported, for Mie scattering, a related development relevant to the case of unpolarized incident light and a special ground condition.

It is clear that if we were interested only in the azimuthally symmetric component in a Fourier representation of the four-vector $\boldsymbol{I}(\tau, \mu, \phi)$ we would require only the leading term in Eq. (6). For this special case of $m=0$ (Herman and Lenoble, 1968; van de Hulst, 1980) we observe that the formalism requires only the Legendre polynomials and the associated Legendre functions $P_l^2(\mu)$. The general case, however, does require the associated Legendre functions $P_l^m(\mu)$ and the combinations of generalized spherical functions $R_l^m(\mu)$ and $T_l^m(\mu)$.

Having reviewed the theory given previously (Siewert, 1981) we proceed in Sect. II to discuss the matter of computing the basic coefficients $\{\alpha_l, \beta_l, \gamma_l, \delta_l, \varepsilon_l, \zeta_l\}$ for a given phase matrix. Section III is devoted to the development of recursive relations that are especially useful for computations of the functions $R_l^m(\mu)$ and $T_l^m(\mu)$ and subsequently the matrices $\boldsymbol{\Pi}_l^m(\mu)$ required in Eq. (8). Finally Sect. IV is devoted to a discussion of symmetry relations satisfied by the phase matrix.

II. The Basic Constants

It is clear that the foregoing analytical representation of the phase matrix $\boldsymbol{P}(\mu, \mu', \phi-\phi')$ yields a convenient starting point for workers interested in developing analytical, or at least semi-analytical, solutions for the four-vector $\boldsymbol{I}(\tau, \mu, \phi)$. On the other hand, the analytical representation for $\boldsymbol{P}(\mu, \mu', \phi-\phi')$ can also prove useful to workers who wish to develop strictly numerical methods for finding $\boldsymbol{I}(\tau, \mu, \phi)$. The analytical development clearly is an attractive alternative to multiplying Eq. (2), for a given scattering matrix $\boldsymbol{F}(\xi)$, by $\sin m(\phi-\phi')$ and $\cos m(\phi-\phi')$ and integrating, for fixed values of μ and μ', over $\phi-\phi'$ to find the Fourier components in an expansion of the phase matrix. With the analytical method we have the possibility of obtaining the Fourier components with a high degree of accuracy at modest computational expense. To use the analytical development, however, we must find the basic constants $\{\alpha_l, \beta_l, \gamma_l, \delta_l, \varepsilon_l, \zeta_l\}$ from the given scattering matrix $\boldsymbol{F}(\xi)$. In regard to the special case of Mie scattering (van de Hulst, 1957) we note that Herman (1965) and Domke (1975) have reported explicit representations for these constants in terms of the complex coefficients basic to the Mie series. In addition Herman (1980) and co-workers have used orthogonality relations to deduce the basic constants $\{\alpha_l, \beta_l, \gamma_l, \delta_l, \varepsilon_l, \zeta_l\}$ in the context of Mie scattering. We follow this latter approach here for a more general scattering model.

Since the Legendre polynomials and the associated Legendre functions satisfy the orthogonality relations

$$\int_{-1}^{1} P_l(\xi)P_{l'}(\xi)d\xi = \left(\frac{2}{2l+1}\right)\delta_{l,l'} \tag{17}$$

and

$$\int_{-1}^{1} P_l^2(\xi)P_{l'}^2(\xi)d\xi = \left(\frac{2}{2l+1}\right)\left[\frac{(l+2)!}{(l-2)!}\right]\delta_{l,l'} \tag{18}$$

we can readily deduce from Eqs. (5a, d, e, and f) the following integral expressions for the constants β_l, δ_l, γ_l, and ε_l:

$$\beta_l = \left(\frac{2l+1}{2}\right)\int_{-1}^{1} a_1(\xi)P_l(\xi)d\xi, \tag{19a}$$

$$\delta_l = \left(\frac{2l+1}{2}\right)\int_{-1}^{1} a_4(\xi)P_l(\xi)d\xi, \tag{19b}$$

$$\gamma_l = \left(\frac{2l+1}{2}\right)\left[\frac{(l-2)!}{(l+2)!}\right]^{1/2}\int_{-1}^{1} b_1(\xi)P_l^2(\xi)d\xi. \tag{19c}$$

and

$$\varepsilon_l = -\left(\frac{2l+1}{2}\right)\left[\frac{(l-2)!}{(l+2)!}\right]^{1/2}\int_{-1}^{1} b_2(\xi)P_l^2(\xi)d\xi. \tag{19d}$$

Of course we can use the recursion formulas

$$(l+1)P_{l+1}(\xi) = (2l+1)\xi P_l(\xi) - lP_{l-1}(\xi), \quad l \geqq 0, \tag{20}$$

and

$$(l-1)P_{l+1}^2(\xi) = (2l+1)\xi P_l^2(\xi) - (l+2)(1-\delta_{2,l})P_{l-1}^2(\xi), \quad l \geqq 2, \tag{21}$$

with $P_0(\xi)=1$ and $P_2^2(\xi)=3(1-\xi^2)$ to compute the Legendre polynomials and associated Legendre functions required in Eqs. (5) and (19). Also we can use the orthogonality relation, for $l, l' \geqq \sup(|m|, |n|)$,

$$(-1)^{m-n}\int_{-1}^{1} P_{m,n}^l(\xi)P_{m,n}^{l'}(\xi)d\xi = \left(\frac{2}{2l+1}\right)\delta_{l,l'} \tag{22}$$

given by Gel'fand and Sapiro (1956) to find from Eqs. (5b and c)

$$\zeta_l = \left(\frac{2l+1}{2}\right)\left[\frac{(l-2)!}{(l+2)!}\right]^{1/2}\int_{-1}^{1} \{a_3(\xi)R_l^2(\xi) + a_2(\xi)T_l^2(\xi)\}d\xi \tag{23a}$$

and

$$\alpha_l=\left(\frac{2l+1}{2}\right)\left[\frac{(l-2)!}{(l+2)!}\right]^{1/2}\int_{-1}^{1}\{a_2(\xi)R_l^2(\xi)+a_3(\xi)T_l^2(\xi)\}d\xi. \tag{23b}$$

Gel'fand and Sapiro (1956) give, for $l \geqq \sup(|m|,|n|)$, the recursion formula

$$e_{m,n}^l P_{m,n}^{l+1}(\xi)=(2l+1)\xi P_{m,n}^l(\xi)-f_{m,n}^l P_{m,n}^{l-1}(\xi)-\left[\frac{mn(2l+1)}{l(l+1)}\right]P_{m,n}^l(\xi), \tag{24}$$

with

$$e_{m,n}^l=\left(\frac{1}{l+1}\right)[(l+m+1)(l-m+1)(l+n+1)(l-n+1)]^{1/2} \tag{25a}$$

and

$$f_{m,n}^l=\frac{1}{l}[(l+m)(l-m)(l+n)(l-n)]^{1/2}, \tag{25b}$$

which we find we can use to establish, for $l \geqq 2$,

$$\left(\frac{l-1}{l+1}\right)[(l+3)(l-1)]^{1/2}R_{l+1}^2(\xi)=(2l+1)\xi R_l^2(\xi)-\left(\frac{l+2}{l}\right)[(l+2)(l-2)]^{1/2}R_{l-1}^2(\xi)-\left[\frac{4(2l+1)}{l(l+1)}\right]T_l^2(\xi) \tag{26a}$$

and

$$\left(\frac{l-1}{l+1}\right)[(l+3)(l-1)]^{1/2}T_{l+1}^2(\xi)=(2l+1)\xi T_l^2(\xi)-\left(\frac{l+2}{l}\right)[(l+2)(l-2)]^{1/2}T_{l-1}^2(\xi)-\left[\frac{4(2l+1)}{l(l+1)}\right]R_l^2(\xi). \tag{26b}$$

Equations (26) with

$$R_2^2(\xi)=\frac{\sqrt{6}}{2}(1+\xi^2) \quad \text{and} \quad T_2^2(\xi)=\sqrt{6}\xi \tag{27a and b}$$

can be used to compute the polynomials required in Eqs. (5) and (23).

III. The Basic Matrices

While it is apparent that the basic matrices $\boldsymbol{\Pi}_l^m(\mu)$ defined by Eq. (10) are available from the definitions given by Eqs. (13)–(16), these definitions are not at all ideal for actual numerical computations. For this reason we wish to develop here some recursion formulas for the matrices $\boldsymbol{\Pi}_l^m(\mu)$. With these recursive relations we can compute the basic matrices $\boldsymbol{\Pi}_l^m(\mu)$ in an accurate and economical manner. Thus Eqs. (6)–(9) can subsequently be used to evaluate the Fourier components of the phase matrix efficiently. First of all for $m=0$ we let $\boldsymbol{\Pi}_l^0(\mu)=\boldsymbol{\Pi}_l(\mu)$ and note that

$$\boldsymbol{\Pi}_0(\mu)=\mathrm{diag}\{1,0,0,1\}, \quad \boldsymbol{\Pi}_1(\mu)=\mathrm{diag}\{\mu,0,0,\mu\} \tag{28a, b}$$

and

$$\boldsymbol{\Pi}_2(\mu)=\mathrm{diag}\{P_2(\mu),R_2(\mu),R_2(\mu),P_2(\mu)\} \tag{28c}$$

where

$$R_2(\mu)=R_2^0(\mu)=\frac{\sqrt{6}}{4}(1-\mu^2). \tag{29}$$

For $l \geqq 2$ we find that

$$\boldsymbol{\Pi}_{l+1}(\mu)=\boldsymbol{X}_l^{-1}[(2l+1)\mu\boldsymbol{\Pi}_l(\mu)-\boldsymbol{Y}_l\boldsymbol{\Pi}_{l-1}(\mu)] \tag{30}$$

where

$$\boldsymbol{X}_l=\mathrm{diag}\{l+1,[(l+3)(l-1)]^{1/2},[(l+3)(l-1)]^{1/2},l+1\} \tag{31a}$$

and

$$Y_l=\operatorname{diag}\{l,(l^2-4)^{1/2},(l^2-4)^{1/2},l\}. \tag{31b}$$

For $m=1$ we find

$$\boldsymbol{\Pi}_1^1(\mu)=\operatorname{diag}\{(1-\mu^2)^{1/2},0,0,(1-\mu^2)^{1/2}\} \tag{32a}$$

and

$$\boldsymbol{\Pi}_2^1(\mu)=\begin{vmatrix} 3\mu(1-\mu^2)^{1/2} & 0 & 0 & 0 \\ 0 & R_2^1(\mu) & -T_2^1(\mu) & 0 \\ 0 & -T_2^1(\mu) & R_2^1(\mu) & 0 \\ 0 & 0 & 0 & 3\mu(1-\mu^2)^{1/2} \end{vmatrix} \tag{32b}$$

with

$$R_2^1(\mu)=-\tfrac{1}{2}\mu\sqrt{6}(1-\mu^2)^{1/2} \quad \text{and} \quad T_2^1(\mu)=-\tfrac{1}{2}\sqrt{6}(1-\mu^2)^{1/2}, \tag{33a and b}$$

and for $l\geqq 2$ we find

$$\boldsymbol{\Pi}_{l+1}^1(\mu)=[\boldsymbol{X}_l^1]^{-1}[(2l+1)\mu\boldsymbol{\Pi}_l^1(\mu)-\boldsymbol{Y}_l^1\boldsymbol{\Pi}_{l-1}^1(\mu)+\boldsymbol{Z}_l^1\boldsymbol{\Pi}_l^1(\mu)] \tag{34}$$

where, in general,

$$\boldsymbol{X}_l^m=\operatorname{diag}\left\{l+1-m,\left(\frac{l-m+1}{l+1}\right)[(l+3)(l-1)]^{1/2},\left(\frac{l-m+1}{l+1}\right)[(l+3)(l-1)]^{1/2},l+1-m\right\} \tag{35a}$$

$$\boldsymbol{Y}_l^m=(1-\delta_{m,l})\operatorname{diag}\left\{l+m,\left(\frac{l+m}{l}\right)(l^2-4)^{1/2},\left(\frac{l+m}{l}\right)(l^2-4)^{1/2},l+m\right\} \tag{35b}$$

and

$$\boldsymbol{Z}_l^m=\frac{2m(2l+1)}{l(l+1)}\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}. \tag{35c}$$

Finally for the general case $m\geqq 2$ we start with

$$\boldsymbol{\Pi}_m^m(\mu)=(1-\mu^2)^{m/2}\begin{vmatrix} (2m-1)!! & 0 & 0 & 0 \\ 0 & K_m\left\{\dfrac{1+\mu^2}{1-\mu^2}\right\} & -K_m\left\{\dfrac{2\mu}{1-\mu^2}\right\} & 0 \\ 0 & -K_m\left\{\dfrac{2\mu}{1-\mu^2}\right\} & K_m\left\{\dfrac{1+\mu^2}{1-\mu^2}\right\} & 0 \\ 0 & 0 & 0 & (2m-1)!! \end{vmatrix} \tag{36}$$

with $(2m-1)!!=1\cdot 3\cdot 5 \ldots (2m-1)$,

$$K_m=\frac{(2m)!}{2^m}[(m-2)!(m+2)!]^{-1/2}, \tag{37}$$

and for $l\geqq m$

$$\boldsymbol{\Pi}_{l+1}^m(\mu)=[\boldsymbol{X}_l^m]^{-1}[(2l+1)\mu\boldsymbol{\Pi}_l^m(\mu)-\boldsymbol{Y}_l^m\boldsymbol{\Pi}_{l-1}^m(\mu)+\boldsymbol{Z}_l^m\boldsymbol{\Pi}_l^m(\mu)]. \tag{38}$$

As we have developed a procedure for deducing our basic constants $\{\alpha_l, \beta_l, \gamma_l, \delta_l, \varepsilon_l, \zeta_l\}$ or $\boldsymbol{B}_l$ for a general scattering matrix $\boldsymbol{F}(\xi)$ and an efficient way to compute the basic matrices $\boldsymbol{\Pi}_l^m(\mu)$, we now have a complete and computationally viable theory for using Eqs. (7) in Eq. (6) to yield the desired analytical Fourier decomposition of the phase matrix. To complete this section we note that Vestrucci (1981) by comparing his results to those obtained with direct integration by de Haan (1981) has shown for two Mie-scattering test cases [based on truncating Eqs. (5) after $l=L=11$ and $l=L=47$] that the reported analytical formulation is a sound method for computing the components in a Fourier decomposition of the phase matrix $\boldsymbol{P}(\mu, \mu', \phi-\phi')$.

IV. Symmetry Relations

To conclude this work we note that with the relations, for $l \geqq m$,

$$P_l^m(-\mu) = (-1)^{l-m} P_l^m(\mu) \tag{39a}$$

$$R_l^m(-\mu) = (-1)^{l-m} R_l^m(\mu) \tag{39b}$$

and

$$T_l^m(-\mu) = -(-1)^{l-m} T_l^m(\mu) \tag{39c}$$

we can readily observe that the phase matrix as given by Eq. (6) satisfies the symmetry relations (A–G) derived by Hovenier (1969). We write a basic set of these symmetry relations here as

$$\boldsymbol{P}(\mu, \mu', \phi - \phi') = \boldsymbol{D}\boldsymbol{P}(\mu, \mu', \phi' - \phi)\boldsymbol{D} \tag{40a}$$

$$\boldsymbol{P}(\mu, \mu', \phi - \phi') = \boldsymbol{E}\boldsymbol{P}^T(\mu', \mu, \phi - \phi')\boldsymbol{E} \tag{40b}$$

and

$$\boldsymbol{P}(\mu, \mu', \phi - \phi') = \boldsymbol{D}\boldsymbol{P}(-\mu, -\mu', \phi - \phi')\boldsymbol{D} \tag{40c}$$

where

$$\boldsymbol{E} = \mathrm{diag}\{1, 1, -1, 1\}. \tag{41}$$

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