On inverse problems for plane-parallel media with nonuniform surface illumination

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Elementary considerations are used to solve the inverse problem in linear transport theory for the case of variable illumination over the surface of a plane-parallel layer. The developed formalism yields as a special case the inverse solution for the classical searchlight problem.

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I. INTRODUCTION

The inverse problem in radiation transport theory is concerned with the determination of scattering and absorbing properties of a medium from a set of measurable radiation quantities. In the past few years considerable work regarding exact solutions of such inverse problems has been reported.[1–8] However, all of these papers[1–8] have dealt with the case of an infinite plane-parallel layer illuminated uniformly over each of the two free surfaces. From a practical and/or experimental point of view, such problems cannot be easily realized, and so here we report a solution for a class of inverse problems that allows the incident radiation to vary over the surfaces.

We employ a notational scheme similar to that used by Rybicki[9] in a study of the searchlight problem, and thus we write the radiation transport equation as

$$\mu \frac{\partial}{\partial z} I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}) + \boldsymbol{\omega}\cdot \frac{\partial}{\partial \boldsymbol{\rho}} I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}) + I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}) = \frac{c}{4\pi} \int\int I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}') p(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}') d\,\boldsymbol{\Omega}', \tag{1}$$

where z and $\boldsymbol{\rho}$, which lies in the x–y plane, locate in optical units the position in the homogeneous medium and $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\mu, \phi)$, with $\mu = \cos(\theta)$, is a unit vector that defines the direction of propagation (see Fig. 1). In addition, $\boldsymbol{\omega}$ is the projection of $\boldsymbol{\Omega}$ in the x–y plane and $c < 1$ is the albedo for single scattering. We consider that $I(z, \boldsymbol{\rho}, \boldsymbol{\Omega})$ satisfies Eq. (1) subject to the boundary conditions

$$I(0, \boldsymbol{\rho}, \boldsymbol{\Omega}) = I_1(\boldsymbol{\rho}, \boldsymbol{\Omega}), \quad \mu > 0, \phi \in [0, 2\pi], \tag{2a}$$

and

$$I(a, \boldsymbol{\rho}, \boldsymbol{\Omega}) = I_2(\boldsymbol{\rho}, \boldsymbol{\Omega}), \quad \mu < 0, \phi \in [0, 2\pi], \tag{2b}$$

where $I_1(\boldsymbol{\rho}, \boldsymbol{\Omega})$ and $I_2(\boldsymbol{\rho}, \boldsymbol{\Omega})$ are assumed to be given and to have two-dimensional Fourier transforms. Expanding the scattering law in terms of Legendre polynomials, we write

$$p(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}') = \sum_{l=0}^{\infty} \beta_l P_l(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}'), \quad \beta_0 = 1, \tag{3}$$

or, if we use the addition theorem,

$$p(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}') = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \beta_l^m P_l^m(\mu) P_l^m(\mu') \cos[m(\phi - \phi')]. \tag{4}$$

Here we use $P_l^m(\mu)$ to denote the associated Legendre functions,

$$P_l^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} P_l(\mu) \tag{5}$$

and

$$\beta_l^m = (2 - \delta_{0,m}) \frac{(l-m)!}{(l+m)!} \beta_l. \tag{6}$$

We assume that, in general, the quantities $I(0, \boldsymbol{\rho}, \boldsymbol{\Omega})$, for $\mu < 0$ and $\phi \in [0, 2\pi]$, and $I(a, \boldsymbol{\rho}, \boldsymbol{\Omega})$, for $\mu > 0$ and $\phi \in [0, 2\pi]$, can be determined experimentally, and we seek to express the single-scattering albedo c and the coefficients β_l in the Legendre expansion of the scattering law in terms of these quantities.

II. ANALYSIS

We can multiply Eqs. (1) and (2) by $\exp(i\mathbf{k}\cdot\boldsymbol{\rho})$ and integrate, for fixed z, over the x–y plane to find

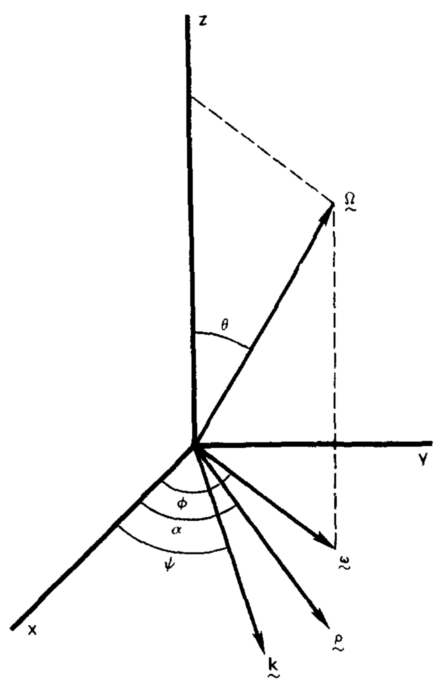

FIG. 1. The geometry for $\boldsymbol{\Omega}$, $\boldsymbol{\omega}$, $\boldsymbol{\rho}$, and $\mathbf{k}$.

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$$\mu \frac{\partial}{\partial z} \Psi(z,\mu,\phi) + [1 - if(\mu,\phi)]\Psi(z,\mu,\phi) = \frac{c}{4\pi}\int_0^{2\pi}\int_{-1}^{1} \Psi(z,\mu',\phi')p(\mathbf{\Omega}\cdot\mathbf{\Omega}')d\mu' d\phi' \quad (7)$$

and, for $\mu > 0$ and $\phi \in [0, 2\pi]$,

$$\Psi(0,\mu,\phi) = \Psi_1(\mu,\phi) \quad (8a)$$

and

$$\Psi(a, -\mu,\phi) = \Psi_2(\mu,\phi), \quad (8b)$$

where we suppress the dependence on the vector **k**, which is in the x–y plane as shown in Fig. 1, and write

$$\Psi(z,\mu,\phi) = \iint I(z, \boldsymbol{\rho}, \mathbf{\Omega}) e^{i\mathbf{k}\cdot\boldsymbol{\rho}} d\boldsymbol{\rho}, \quad (9)$$

$$\Psi_1(\mu,\phi) = \iint I_1[\boldsymbol{\rho}, \mathbf{\Omega}(\mu,\phi)] e^{i\mathbf{k}\cdot\boldsymbol{\rho}} d\boldsymbol{\rho}, \quad (10a)$$

and

$$\Psi_2(\mu,\phi) = \iint I_2[\boldsymbol{\rho}, \mathbf{\Omega}(-\mu,\phi)] e^{i\mathbf{k}\cdot\boldsymbol{\rho}} d\boldsymbol{\rho}. \quad (10b)$$

In addition,

$$f(\mu,\phi) = \mathbf{k}\cdot\boldsymbol{\omega} = k(1-\mu^2)^{1/2}\cos(\phi - \psi), \quad (11)$$

with $k = |\mathbf{k}|$. We now follow an earlier work[4] and let

$$F(z,\mu,\phi) = \mu \frac{\partial}{\partial z}\Psi(z,\mu,\phi) \quad (12)$$

so that we can change μ to $-\mu$ in Eq. (7) and write

$$F(z,-\mu,\phi) + [1 - if(\mu,\phi)]\Psi(z,-\mu,\phi) = \frac{c}{4\pi}\sum_{l=0}^{\infty}\sum_{m=0}^{l}(-1)^{l-m}\beta_l^m P_l^m(\mu) \times \int_0^{2\pi}\Psi_l^m(z,\phi')\cos[m(\phi-\phi')]d\phi', \quad (13)$$

where

$$\Psi_l^m(z,\phi) = \int_{-1}^{1} P_l^m(\mu)\Psi(z,\mu,\phi)d\mu. \quad (14)$$

We can multiply Eq. (13) by $\Psi(z,\mu,\phi)$ and integrate over all μ and ϕ to find

$$T_0(z) + \int_0^{2\pi}\int_{-1}^{1}[1 - if(\mu,\phi)]\Psi(z,\mu,\phi)\Psi(z,-\mu,\phi)d\mu\, d\phi = \frac{c}{4\pi}\sum_{l=0}^{\infty}\sum_{m=0}^{l}(-1)^{l-m}\beta_l^m[C_l^m(z) + S_l^m(z)], \quad (15)$$

where

$$C_l^m(z) = \left(\int_0^{2\pi}\Psi_l^m(z,\phi)\cos(m\phi)d\phi\right)^2, \quad (16a)$$

$$S_l^m(z) = \left(\int_0^{2\pi}\Psi_l^m(z,\phi)\sin(m\phi)d\phi\right)^2, \quad (16b)$$

and

$$T_0(z) = \int_0^{2\pi}\int_{-1}^{1}\Psi(z,\mu,\phi)F(z,-\mu,\phi)d\mu\, d\phi. \quad (17)$$

If we now differentiate Eqs. (15) and (17) and use Eq. (13) we can deduce that $T_0(z)$ is a constant, and on considering Eq. (15) at $z = 0$ and $z = a$ and subtracting the two resulting equations, we find

$$S_0 = \frac{c}{8\pi}\sum_{l=0}^{\infty}\sum_{m=0}^{l}(-1)^{l-m}\beta_l^m \times [C_l^m(0) - C_l^m(a) + S_l^m(0) - S_l^m(a)], \quad (18)$$

where

$$S_0 = \int_0^{2\pi}\int_0^1 [1 - if(\mu,\phi)][\Psi(0,\mu,\phi)\Psi(0,-\mu,\phi) - \Psi(a,\mu,\phi)\Psi(a,-\mu,\phi)]d\mu\, d\phi. \quad (19)$$

As we consider $I(z, \boldsymbol{\rho}, \mathbf{\Omega})$ to be known on the boundaries, $z = 0$ and $z = a$, the unknowns in Eq. (18) are c and the coefficients $\{\beta_l\}$. Thus we define

$$K_l^m = (2 - \delta_{0,m})(-1)^{l-m}\frac{(l-m)!}{(l+m)!} \times [C_l^m(0) - C_l^m(a) + S_l^m(0) - S_l^m(a)] \quad (20)$$

and write Eq. (18) as

$$S_0 = \frac{c}{8\pi}\left\{[\Psi_0(0)]^2 - [\Psi_0(a)]^2 + \sum_{l=1}^{\infty}\beta_l\sum_{m=0}^{l}K_l^m\right\}, \quad (21)$$

where

$$\Psi_0(z) = \int_0^{2\pi}\int_{-1}^{1}\Psi(z,\mu,\phi)d\mu\, d\phi. \quad (22)$$

Clearly for the case of isotropic scattering $\beta_l = 0$, $l \geqslant 1$, and Eq. (21) yields the concise result

$$c = 8\pi\{[\Psi_0(0)]^2 - [\Psi_0(a)]^2\}^{-1}S_0. \quad (23)$$

On the other hand, if we assume that $\beta_l = 0$ only for $l > L$, then Eq. (21) is a single equation for the $L + 1$ unknowns c and $\{\beta_l\}$. As the equation is linear in c and $\{c\beta_l\}$ we clearly can consider utilizing $L + 1$ different experiments, $L + 1$ different values of **k**, or a combination of the two to generate $L + 1$ linear algebraic equations which, in principle, yield without approximation the desired solution of the inverse problem.

To complete this section we note for $k = 0$ that Eqs. (7) and (8) reduce to forms previously considered[1–8] and thus that in principle several solution techniques may apply, for example, in the event that a sufficient number of independent experiments is considered, that the boundary conditions lead to a radiation field that has sensitive dependence on the azimuthal angle, or that the scattering law is limited to three terms. By developing a solution here for the $k \neq 0$ case we clearly introduce the possibility of determining the required scattering coefficients from a single experiment, for a general class of boundary conditions and for a scattering law more general than the three-term model.

III. THE SEARCHLIGHT PROBLEM

As a special case of the foregoing we now consider the classical searchlight problem. We thus write

$$I_1(\boldsymbol{\rho}, \mathbf{\Omega}) = \frac{1}{2\pi\rho}\delta(\rho)\delta(\mu - \mu_0)\delta(\phi - \phi_0) \quad (24a)$$

and

$$I_2(\boldsymbol{\rho}, \mathbf{\Omega}) = 0, \quad (24b)$$

where we use the polar coordinates $\rho = |\boldsymbol{\rho}|$ and α to locate a field point in the x–y plane. Using Eqs. (24) in Eqs. (10), we obtain

$$\Psi_1(\mu, \phi) = \delta(\mu - \mu_0)\delta(\phi - \phi_0) \tag{25a}$$

and

$$\Psi_2(\mu, \phi) = 0, \tag{25b}$$

so that Eq. (19) becomes

$$S_0 = [1 - if(\mu_0, \phi_0)]\Psi(0, -\mu_0, \phi_0). \tag{26}$$

Equations (16) now yield

$$C_l^m(0) = \Big(P_l^m(\mu_0)\cos(m\phi_0) + (-1)^{l-m} \times \int_0^{2\pi}\int_0^1 \Psi(0, -\mu, \phi)P_l^m(\mu)\cos(m\phi)d\mu\, d\phi\Big)^2, \tag{27a}$$

$$S_l^m(0) = \Big(P_l^m(\mu_0)\sin(m\phi_0) + (-1)^{l-m} \times \int_0^{2\pi}\int_0^1 \Psi(0, -\mu, \phi)P_l^m(\mu)\sin(m\phi)d\mu\, d\phi\Big)^2, \tag{27b}$$

$$C_l^m(a) = \Big(\int_0^{2\pi}\int_0^1 \Psi(a, \mu, \phi)P_l^m(\mu)\cos(m\phi)d\mu\, d\phi\Big)^2, \tag{28a}$$

and

$$S_l^m(a) = \Big(\int_0^{2\pi}\int_0^1 \Psi(a, \mu, \phi)P_l^m(\mu)\sin(m\phi)d\mu\, d\phi\Big)^2, \tag{28b}$$

and Eq. (21) can be written as

$$8\pi[1 - if(\mu_0, \phi_0)]\,\Psi(0, -\mu_0, \phi_0) = c\Big[C_0^0(0) - C_0^0(a) + \sum_{l=1}^{\infty}\beta_l \sum_{m=0}^{l} K_l^m\Big]. \tag{29}$$

IV. CONCLUDING REMARKS

For the searchlight problem we note that Eq. (29) is our basic result for finding c and the coefficients $\{\beta_l\}$ in terms of the intensities on the two surfaces $z = 0$ and $z = a$. To use the equation we must in general be able to measure $I[0, \boldsymbol{\rho}, \boldsymbol{\Omega}(-\mu, \phi)]$ and $I[a, \boldsymbol{\rho}, \boldsymbol{\Omega}(\mu, \phi)]$, for $\mu\in[0, 1]$ and $\phi\in[0, 2\pi]$, experimentally and compute the quantities $C_l^m(0)$, $C_l^m(a)$, $S_l^m(0)$, and $S_l^m(a)$ with reasonable accuracy. As a first test of the solution we have considered the case of isotropic scattering and used the Monte Carlo method to solve, for given values of μ_0, ϕ_0, and c, the direct problem. For numerous cases studied we found that the value of c computed from

$$c = 8\pi[1 - if(\mu_0, \phi_0)]\,\Psi(0, -\mu_0, \phi_0)[C_0^0(0) - C_0^0(a)]^{-1} \tag{30}$$

agreed with the given value, for various choices of **k**, with an accuracy consistent with the accuracy of the Monte Carlo results for the exiting intensities. More complete testing of the general formulation is clearly required in order to evaluate the extent to which basic results for practical experiments can be extracted from this exact solution.

It is clear that the inverse solution developed here for the infinite plane-parallel case requires that the incident radiation be specified over the entire boundary and that the exiting radiation be measured experimentally over the entire surface. However, in the event that there is absorption in the layer and the incident radiation is sufficiently localized (as, for example, in the searchlight problem) the case of a plane-parallel body finite in the transverse directions can be well approximated by the infinite plane-parallel case, and the developed inverse solution can be used with confidence.

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