Journal of Applied Mathematics and Physics (ZAMP)
Vol. 33, November 1982
0044-2275/82/006807-12 $ 3.90/0

On the scattering of polarized light

By C. E. Siewert and F. J. V. Pinheiro, Depts. of Mathematics and Nuclear Engineering, North Carolina State University, Raleigh, USA

I. Introduction

In a recent paper, hereafter referred to as [1], the Kuščer-Ribarič formulation [2] of the equation of transfer relevant to the scattering of polarized light was used to reduce the computation of the desired density vector $\boldsymbol{I}(\tau, \mu, \varphi)$ to a set of φ-independent problems, each based on real quantities. Here we wish to develop the basic analysis we use to establish the azimuthally symmetric component of the complete solution.

To summarize the results of [1], we consider the equation of transfer

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}(\tau, \mu, \varphi) + \boldsymbol{I}(\tau, \mu, \varphi) = \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \boldsymbol{P}(\mu, \mu', \varphi - \varphi') \boldsymbol{I}(\tau, \mu', \varphi') \, \mathrm{d}\mu' \mathrm{d}\varphi', \quad (1)$$

where the density vector $\boldsymbol{I}(\tau, \mu, \varphi)$ has the four Stokes parameters [3, 4] I, Q, U and V as components, and $\boldsymbol{P}(\mu, \mu', \varphi - \varphi')$ is the phase matrix. We use here the expansion of the phase matrix introduced by Kuščer and Ribarič [2] and shown in [1] to yield the convenient representation

$$\boldsymbol{P}(\mu, \mu', \varphi - \varphi') \quad (2)$$
$$= \sum_{m=0}^{L} \tfrac{1}{2} (2 - \delta_{0,m}) [\boldsymbol{C}^m(\mu, \mu') \cos m(\varphi - \varphi') + \boldsymbol{S}^m(\mu, \mu') \sin m(\varphi - \varphi')]$$

where

$$\boldsymbol{C}^m(\mu, \mu') = \boldsymbol{A}^m(\mu, \mu') + \boldsymbol{D}\boldsymbol{A}^m(\mu, \mu') \boldsymbol{D}, \quad (3a)$$

$$\boldsymbol{S}^m(\mu, \mu') = \boldsymbol{A}^m(\mu, \mu') \boldsymbol{D} - \boldsymbol{D}\boldsymbol{A}^m(\mu, \mu'), \quad (3b)$$

$$\boldsymbol{D} = \operatorname{diag}\{1, 1, -1, -1\} \quad (4)$$

and

$$\boldsymbol{A}^m(\mu, \mu') = \sum_{l=m}^{L} \boldsymbol{\Pi}_l^m(\mu) \boldsymbol{B}_l^m \boldsymbol{\Pi}_l^m(\mu'). \quad (5)$$

Here the given constants used to define the phase matrix are those introduced by Herman and Lenoble [5], i.e.

$$\boldsymbol{B}_l^m = \frac{(l-m)!}{(l+m)!}\boldsymbol{B}_l \tag{6}$$

with

$$\boldsymbol{B}_l = \begin{vmatrix} \beta_l & \gamma_l & 0 & 0 \\ \gamma_l & \alpha_l & 0 & 0 \\ 0 & 0 & \zeta_l & -\varepsilon_l \\ 0 & 0 & \varepsilon_l & \delta_l \end{vmatrix}, \tag{7}$$

$\beta_0 = 1$ and $\gamma_l = \alpha_l = \zeta_l = \varepsilon_l = 0$ for $l = 0$ and 1. In addition

$$\boldsymbol{\Pi}_l^m(\mu) = \begin{vmatrix} P_l^m(\mu) & 0 & 0 & 0 \\ 0 & R_l^m(\mu) & -T_l^m(\mu) & 0 \\ 0 & -T_l^m(\mu) & R_l^m(\mu) & 0 \\ 0 & 0 & 0 & P_l^m(\mu) \end{vmatrix}, \tag{8}$$

where $P_l^m(\mu)$ is the associated Legendre function, i.e. if $P_l(\mu)$ is the Legendre polynomial of order l, then for $l \geqq m$

$$P_l^m(\mu) = (1-\mu^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}\mu^m}P_l(\mu)\,. \tag{9}$$

Further $R_l^m(\mu)$ and $T_l^m(\mu)$ are defined in terms of the generalized spherical functions discussed by Gel'fand and Šapiro [6], i.e. $R_l^m(\mu) = T_l^m(\mu) = 0$ for $l = 0$ and 1, and for $l \geqq m$

$$R_l^m(\mu) = -\frac{1}{2}(i)^m\sqrt{\frac{(l+m)!}{(l-m)!}}\,[P_{m,2}^l(\mu) + P_{m,-2}^l(\mu)] \tag{10a}$$

and

$$T_l^m(\mu) = -\frac{1}{2}(i)^m\sqrt{\frac{(l+m)!}{(l-m)!}}\,[P_{m,2}^l(\mu) - P_{m,-2}^l(\mu)] \tag{10b}$$

where

$$P_{m,\pm 2}^l(\mu) = A^{\pm}\left(\frac{1-\mu}{1+\mu}\right)^{m/2}(1-\mu^2)^{\mp 1}\frac{\mathrm{d}^{l\mp 2}}{\mathrm{d}\mu^{l\mp 2}}[(1-\mu)^{l-m}(1+\mu)^{l+m}] \tag{11}$$

and

$$A^{\pm} = \frac{-(-1)^{l-m}(i)^{-m}}{2^l(l-m)!}\sqrt{\frac{(l-m)!\,(l\pm 2)!}{(l+m)!\,(l\mp 2)!}}\,. \tag{12}$$

We note that the phase matrix given by Eq. (2) obeys the seven symmetry relations (A, B, …, G) derived by Hovenier [7]. We seek, in general, a solution to Eq. (1) for $\tau \in [L, R]$ subject to the boundary conditions, for $\mu > 0$ and $\varphi \in [0, 2\pi]$,

$$\boldsymbol{I}(L, \mu, \varphi) = \pi\,\delta(\mu - \mu_0)\,\delta(\varphi - \varphi_0)\,\boldsymbol{F} + \boldsymbol{F}_1(\mu, \varphi) \tag{13a}$$

and

$$I(R, -\mu, \varphi) = F_2(\mu, \varphi) + \frac{\lambda_0}{\pi} L \int_0^{2\pi} \int_0^1 I(R, \mu', \varphi') \, \mu' \, d\mu' \, d\varphi' . \tag{13b}$$

Here the flux constant has components F_I, F_Q, F_U and F_V, λ_0 is the Lambert reflection constant, $L = \text{diag}\{1, 0, 0, 0\}$ and $F_1(\mu, \varphi)$ and $F_2(\mu, \varphi)$ are considered given. As we intend to develop our solution in detail, we consider first the azimuthally symmetric component

$$I(\tau, \mu) = \frac{1}{2\pi} \int_0^{2\pi} I(\tau, \mu, \varphi) \, d\varphi \tag{14}$$

of the complete solution.

II. The elementary solutions

If we integrate Eqs. (1) and (13) over φ we find that $I(\tau, \mu)$ is defined by

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\omega}{2} \sum_{l=0}^{L} \Pi_l(\mu) \, B_l \int_{-1}^{1} \Pi_l(\mu') \, I(\tau, \mu') \, d\mu' , \tag{15}$$

and the boundary conditions, for $\mu > 0$,

$$I(L, \mu) = \tfrac{1}{2} \, \delta(\mu - \mu_0) \, F + F_1(\mu) \tag{16a}$$

and

$$I(R, -\mu) = F_2(\mu) + 2\lambda_0 \, L \int_0^1 I(R, \mu') \, \mu' \, d\mu' \tag{16b}$$

where

$$F_\alpha(\mu) = \frac{1}{2\pi} \int_0^{2\pi} F_\alpha(\mu, \varphi) \, d\varphi . \tag{17}$$

Here

$$\Pi_l(\mu) = \text{diag}\,\{P_l(\mu), R_l(\mu), R_l(\mu), P_l(\mu)\} , \tag{18}$$

where $R_l(\mu) = R_l^0(\mu) = 0$ for $l = 0$ and 1, and for $l \geqq 2$

$$R_l(\mu) = \sqrt{\frac{(l-2)!}{(l+2)!}} \, P_l^2(\mu) . \tag{19}$$

It is apparent that for the special case $\mu_0 = 1$ and $F_1(\mu, \varphi)$ and $F_2(\mu, \varphi)$ independent of φ that $I(\tau, \mu, \varphi) = I(\tau, \mu)$; otherwise, of course, $I(\tau, \mu)$ is only one component of the complete solution. We note that Domke [8, 9, 10, 11] has reported elementary solutions of Eq. (15) and extensive analysis basic to the Kuščer-Ribarič formulation [2] of the general polarization problem.

Although $I(\tau, \mu)$ is four-vector, it is clear from Eqs. (15) and (16) that the coupling is not complete. Thus we study here the two-vector problem

$$\mu \frac{\partial}{\partial \tau} \Psi(\tau, \mu) + \Psi(\tau, \mu) = \frac{\omega}{2} \sum_{l=0}^{L} P_l(\mu) \, C_l \int_{-1}^{1} P_l(\mu') \, \Psi(\tau, \mu') \, d\mu' , \tag{20}$$

where

$$\boldsymbol{P}_l(\mu) = \operatorname{diag}\{P_l(\mu), R_l(\mu)\}, \tag{21}$$

and the matrices $\boldsymbol{C}_l$, with elements C_l^{ij}, are constants. Clearly, if we use

$$\boldsymbol{C}_l = \begin{vmatrix} \beta_l & \gamma_l \\ \gamma_l & \alpha_l \end{vmatrix} \tag{22}$$

the two components of $\boldsymbol{\Psi}(\tau, \mu)$ are the Stokes parameters $I(\tau, \mu)$ and $Q(\tau, \mu)$. On the other hand, if we use

$$\boldsymbol{C}_l = \begin{vmatrix} \delta_l & \varepsilon_l \\ -\varepsilon_l & \zeta_l \end{vmatrix}, \tag{23}$$

then the two components of $\boldsymbol{\Psi}(\tau, \mu)$ represent the Stokes parameters $V(\tau, \mu)$ and $U(\tau, \mu)$. In the manner of Case [12], we now substitute

$$\boldsymbol{\Psi}_\xi(\tau, \mu) = \boldsymbol{\Phi}(\xi, \mu)\, \mathrm{e}^{-\tau/\xi} \tag{24}$$

into Eq. (20) to find

$$(\xi - \mu)\, \boldsymbol{\Phi}(\xi, \mu) = \frac{\omega \xi}{2} \sum_{l=0}^{L} \boldsymbol{P}_l(\mu)\, \boldsymbol{C}_l\, \boldsymbol{G}_l(\xi)\, \boldsymbol{M}(\xi), \tag{25}$$

where the 2×2 matrices $\boldsymbol{G}_l(\xi)$ are defined such that

$$\int_{-1}^{1} \boldsymbol{P}_l(\mu)\, \boldsymbol{\Phi}(\xi, \mu)\, \mathrm{d}\mu = \boldsymbol{G}_l(\xi)\, \boldsymbol{M}(\xi). \tag{26}$$

To normalize the vectors $\boldsymbol{\Phi}(\xi, \mu)$ we use the particularly convenient condition

$$\int_{-1}^{1} \boldsymbol{K}(\mu)\, \boldsymbol{\Phi}(\xi, \mu)\, \mathrm{d}\mu = \boldsymbol{M}(\xi), \tag{27}$$

where

$$\boldsymbol{K}(\mu) = \operatorname{diag}\{1, R_2(\mu)\}, \tag{28}$$

which allows us to deduce $\boldsymbol{G}_2(\xi)$ in a simple manner. If we note Eq. (19) and use, for $l \geqq m$, the orthogonality relation

$$\int_{-1}^{1} P_l^m(\mu)\, P_{l'}^m(\mu)\, \mathrm{d}\mu = \frac{(l+m)!}{(l-m)!} \left(\frac{2}{2l+1}\right) \delta_{l,l'} \tag{29}$$

we can deduce a similar expression for the matrices $\boldsymbol{P}_l(\mu)$, i.e.

$$\int_{-1}^{1} \boldsymbol{P}_l(\mu)\, \boldsymbol{P}_{l'}(\mu)\, \mathrm{d}\mu = \left(\frac{2}{2l+1}\right) \operatorname{diag}\{1, (1-\delta_{0,l})(1-\delta_{1,l})\}\, \delta_{l,l'}. \tag{30}$$

We can also use, for $l \geqq m$, the recursive relation

$$(2l+1)\,\mu\,P_l^m(\mu) = (l+1-m)\,P_{l+1}^m(\mu) + (l+m)(1-\delta_{m,l})\,P_{l-1}^m(\mu) \tag{31}$$

to deduce, for $l \geqq 1$, that

$$(2l+1)\,\mu\,\boldsymbol{P}_l(\mu) = \boldsymbol{K}_l\,\boldsymbol{P}_{l+1}(\mu) + \boldsymbol{J}_l\,\boldsymbol{P}_{l-1}(\mu)\,, \tag{32}$$

where

$$\boldsymbol{K}_l = \mathrm{diag}\,\{l+1, \sqrt{(l-1)(l+3)}\} \tag{33}$$

and

$$\boldsymbol{J}_l = \mathrm{diag}\,\{l, \sqrt{l^2-4}\}\,. \tag{34}$$

Since $\boldsymbol{K}_1$ is singular, it is apparent that

$$\boldsymbol{P}_0(\mu) = \boldsymbol{\Delta} = \mathrm{diag}\,\{1, 0\}\,, \tag{35a}$$

$$\boldsymbol{P}_1(\mu) = \boldsymbol{\Delta}\,\mu \tag{35b}$$

and

$$\boldsymbol{P}_2(\mu) = \mathrm{diag}\,\{P_2(\mu), R_2(\mu)\} \tag{35c}$$

are required before Eq. (32) can be used to generate the remaining $\boldsymbol{P}$ matrices.

We now multiply Eq. (25) by $\boldsymbol{P}_l(\mu)$, integrate over μ from -1 to 1 and use Eqs. (30) and (32) to find, for $l \geqq 1$, that

$$[\xi\,\boldsymbol{h}_l\,\boldsymbol{G}_l(\xi) - \boldsymbol{K}_l\,\boldsymbol{G}_{l+1}(\xi) - \boldsymbol{J}_l\,\boldsymbol{G}_{l-1}(\xi)]\,\boldsymbol{M}(\xi) = \boldsymbol{0} \tag{36}$$

where

$$\boldsymbol{h}_l = (2l+1)\left[\boldsymbol{I} - \omega\left(\frac{1}{2l+1}\right)\boldsymbol{C}_l\right]. \tag{37}$$

To avoid constraining ξ at this point, we take the matrices $\boldsymbol{G}_l(\xi)$ to satisfy, for $l \geqq 1$,

$$\xi\,\boldsymbol{h}_l\,\boldsymbol{G}_l(\xi) = \boldsymbol{K}_l\,\boldsymbol{G}_{l+1}(\xi) + \boldsymbol{J}_l\,\boldsymbol{G}_{l-1}(\xi)\,. \tag{38}$$

Again, since $\boldsymbol{K}_1$ is singular it is clear that $\boldsymbol{G}_0(\xi)$, $\boldsymbol{G}_1(\xi)$ and $\boldsymbol{G}_2(\xi)$ are required before we can use Eq. (38) to generate the remaining $\boldsymbol{G}_l(\xi)$. On multiplying Eq. (27) by $\boldsymbol{\Delta}$ and using Eqs. (25) and (26), we see that we can take

$$\boldsymbol{G}_0(\xi) = \boldsymbol{\Delta} \tag{39a}$$

and

$$\boldsymbol{G}_1(\xi) = k_0\,\xi\,\boldsymbol{\Delta}\,. \tag{39b}$$

We can also deduce from Eqs. (25), (26) and (27) that

$$G_2(\xi) = \operatorname{diag}\left\{\tfrac{1}{2}\,(\xi^2 k_0 k_1 - 1), 1\right\}, \tag{39c}$$

where, in general,

$$k_l = (2l+1)\left[1 - \omega\left(\frac{1}{2l+1}\right) C_l^{11}\right]. \tag{40}$$

Returning now to Eq. (25), we first consider that $\xi \notin [-1, 1]$ and write

$$\boldsymbol{\Phi}(\xi, \mu) = \frac{\omega\,\xi}{2}\left(\frac{1}{\xi - \mu}\right) \sum_{l=0}^{L} \boldsymbol{P}_l(\mu)\, \boldsymbol{C}_l\, \boldsymbol{G}_l(\xi)\, \boldsymbol{M}(\xi)\,. \tag{41}$$

Thus on multiplying Eq. (41) by $\boldsymbol{K}(\mu)$ and integrating we find that ξ must be a zero of

$$\Lambda(z) = \det \boldsymbol{\Lambda}(z) \tag{42}$$

where

$$\boldsymbol{\Lambda}(z) = \boldsymbol{I} + \frac{\omega}{2} z \int_{-1}^{1} \boldsymbol{K}(\mu) \sum_{l=0}^{L} \boldsymbol{P}_l(\mu)\, \boldsymbol{C}_l\, \boldsymbol{G}_l(z) \frac{\mathrm{d}\mu}{\mu - z}\,. \tag{43}$$

If we write, for $l \geqq 1$,

$$\frac{1}{z-\mu}\left[\boldsymbol{G}_l(z) - \boldsymbol{G}_l(\mu)\right] = a_l(\mu, z)\, \boldsymbol{A}_l^l + a_{l-2}(\mu, z)\, \boldsymbol{A}_{l-2}^l + \ldots\,, \tag{44}$$

where

$$a_l(\mu, z) = \mu^{l-1} + \mu^{l-2} z + \ldots + z^{l-1} \tag{45}$$

then Eq. (43) yields

$$\boldsymbol{\Lambda}(z) = \boldsymbol{I} + z \int_{-1}^{1} \boldsymbol{\Psi}(\mu) \frac{\mathrm{d}\mu}{\mu - z} - \frac{\omega z^2}{2} (\boldsymbol{I} - \boldsymbol{\Delta}) \sum_{l=2}^{L} \boldsymbol{W}_l\, \boldsymbol{C}_l\, \boldsymbol{A}_l^l\,, \tag{46}$$

where

$$\boldsymbol{\Psi}(\mu) = \frac{\omega}{2} \boldsymbol{K}(\mu) \sum_{l=0}^{L} \boldsymbol{P}_l(\mu)\, \boldsymbol{C}_l\, \boldsymbol{G}_l(\mu) \tag{47}$$

and, for $l \geqq 2$,

$$\boldsymbol{W}_l = \int_{-1}^{1} R_2(\mu)\, \boldsymbol{P}_l(\mu)\, \mu^{l-2}\, \mathrm{d}\mu\,. \tag{48}$$

We find we can write Eq. (46) as

$$\boldsymbol{\Lambda}(z) = \boldsymbol{T}(z) + z \int_{-1}^{1} \boldsymbol{\Psi}(\mu) \frac{\mathrm{d}\mu}{\mu - z}\,, \tag{49}$$

where

$$T(z) = \begin{vmatrix} 1 & 0 \\ \alpha z^2 & 1 \end{vmatrix} . \tag{50}$$

Here α can be expressed as

$$\alpha = -\frac{\omega}{10} k_0 k_1 \sum_{l=2}^{L} \begin{vmatrix} C_l^{21} \\ C_l^{22} \end{vmatrix}^T D_l , \tag{51}$$

where

$$D_l = \left[\frac{1}{l(2l+1)}\right] \operatorname{diag}\{\sqrt{l^2-4}, l\} \, h_{l-1} \, D_{l-1} \tag{52}$$

with

$$D_2 = \begin{vmatrix} 1 \\ 0 \end{vmatrix} . \tag{53}$$

If we now consider Eq. (25) for $\xi = v \in (-1, 1)$, we can readily generalize our previous work [13] to find the following 2×2 solution

$$\Phi(v, \mu) = \frac{\omega v}{2} \left(\frac{P}{v-\mu}\right) G(\mu, v) + \delta(v-\mu) \, K^{-1}(v) \, \lambda(v) , \tag{54}$$

where

$$G(\mu, v) = \sum_{l=0}^{L} P_l(\mu) \, C_l \, G_l(v) \tag{55}$$

and

$$\lambda(v) = T(v) + vP \int_{-1}^{1} \Psi(\mu) \frac{d\mu}{\mu - v} . \tag{56}$$

Thus to express the solution to Eq. (20) in terms of the elementary solutions we have found we write

$$\Psi(\tau, \mu) = \sum_{\beta=0}^{\varkappa-1} \{A(v_\beta) \, \Phi(v_\beta, \mu) \, e^{-\tau/v_\beta} + A(-v_\beta) \, \Phi(-v_\beta, \mu) \, e^{\tau/v_\beta}\} + \int_{-1}^{1} \Phi(v, \mu) \, A(v) \, e^{-\tau/v} \, dv \tag{57}$$

where $\{v_\beta\}$ denote the $\varkappa$ "positive" zeros of $\Lambda(z)$, $A(v_\beta)$, $A(-v_\beta)$ and the two-vector $A(v)$ are expansion coefficients to be determined by the boundary conditions and

$$\Phi(v_\beta, \mu) = \frac{\omega v_\beta}{2} \left(\frac{1}{v_\beta - \mu}\right) G(\mu, v_\beta) \, M(v_\beta) , \tag{58}$$

where $M(v_\beta)$ is a null-vector of $\Lambda(v_\beta)$.

III. Discrete eigenvalues

Now that we have established Eq. (57) it is clear that a computation of the discrete eigenvalues is crucial to our study. Thus we first seek to establish $\varkappa$, the number of pairs of zeros $\pm \nu_\beta$ of $\Lambda(z)$. Note that $\Lambda(z)=\Lambda(-z)$. If we let

$$\boldsymbol{X}_l(\mu, z) = \frac{1}{\mu - z} [\boldsymbol{K}(\mu) \boldsymbol{P}_l(\mu) - \boldsymbol{K}(z) \boldsymbol{P}_l(z)] \tag{59}$$

then clearly Eq. (43) can be written as

$$\Lambda(z) = \boldsymbol{I} + z \boldsymbol{\Psi}(z) \int_{-1}^{1} \frac{\mathrm{d}\mu}{\mu - z} + \omega z \sum_{l=0}^{L} \boldsymbol{\Gamma}_l(z) \boldsymbol{C}_l \boldsymbol{G}_l(z) , \tag{60}$$

where

$$\boldsymbol{\Gamma}_l(z) = \tfrac{1}{2} \int_{-1}^{1} \boldsymbol{X}_l(\mu, z) \, \mathrm{d}\mu . \tag{61}$$

It is not difficult to show that $\boldsymbol{\Gamma}_0(z) = \boldsymbol{0}$,

$$\boldsymbol{\Gamma}_1(z) = \boldsymbol{\Delta} , \tag{62}$$

$$\boldsymbol{\Gamma}_2(z) = \operatorname{diag}\left\{\tfrac{3}{2} z, \tfrac{1}{8} z(3z^2 - 5)\right\} \tag{63}$$

and, for $l \geqq 1$, that

$$(2l+1) z \boldsymbol{\Gamma}_l(z) = \boldsymbol{K}_l \boldsymbol{\Gamma}_{l+1}(z) + \boldsymbol{J}_l \boldsymbol{\Gamma}_{l-1}(z) + \delta_{2,l} \operatorname{diag}\{0, -1\} . \tag{64}$$

If we let

$$\boldsymbol{Y}(z) = \boldsymbol{I} + \omega z \sum_{l=0}^{L} \boldsymbol{\Gamma}_l(z) \boldsymbol{C}_l \boldsymbol{G}_l(z) \tag{65}$$

then we can write

$$\Lambda(z) = \boldsymbol{Y}(z) + z \boldsymbol{\Psi}(z) \int_{-1}^{1} \frac{\mathrm{d}\mu}{\mu - z} \tag{66}$$

and

$$\Lambda(z) = a(z) + b(z) z \int_{-1}^{1} \frac{\mathrm{d}\mu}{\mu - z} + c(z) \left(z \int_{-1}^{1} \frac{\mathrm{d}\mu}{\mu - z} \right)^2 , \tag{67}$$

where the polynomials $a(z)$, $b(z)$ and $c(z)$ are given by

$$a(z) = \det \boldsymbol{Y}(z) , \tag{68a}$$

$$b(z) = Y_{11}(z) \Psi_{22}(z) + Y_{22}(z) \Psi_{11}(z) - Y_{12}(z) \Psi_{21}(z) - Y_{21}(z) \Psi_{12}(z) \tag{68b}$$

and

$$c(z) = \det \boldsymbol{\Psi}(z) . \tag{68c}$$

For the case $\Lambda(\infty) \neq 0$, we readily compute $\varkappa$ by using the argument principle [14]. Thus, since $\Lambda(z) = \Lambda(-z)$ and $\overline{\Lambda(\bar{z})} = \Lambda(z)$,

$$\varkappa = \frac{1}{\pi} \Delta \arg \Lambda^+(\tau) \Big|_0^1 , \tag{69}$$

i.e. $\pi\varkappa$ is the change in the argument, as τ varies from 0 to 1, of

$$\Lambda^+(\tau) = R(\tau) + i\,I(\tau) , \tag{70}$$

where

$$R(\tau) = a(\tau) + b(\tau)\,\tau \ln\left(\frac{1-\tau}{1+\tau}\right) + c(\tau)\,\tau^2 \left\{\left[\ln\left(\frac{1-\tau}{1+\tau}\right)\right]^2 - \pi^2\right\} \tag{71}$$

and

$$I(\tau) = \pi\tau\left[b(\tau) + 2c(\tau)\,\tau \ln\left(\frac{1-\tau}{1+\tau}\right)\right] . \tag{72}$$

If we let

$$\Theta(\tau) = \arg \Lambda^+(\tau) , \tag{73}$$

with $\Theta(0) = 0$ since $\Lambda(0) = \boldsymbol{I}$, we can factor $\Lambda(z)$ in the manner [15]

$$\Lambda(z) = \Lambda(\infty)\, X(z)\, X(-z) \prod_{\alpha=1}^{\varkappa} (\nu_{\alpha-1}^2 - z^2) , \tag{74}$$

where

$$X(z) = \frac{1}{(1-z)^{\varkappa}} \exp\left(\frac{1}{\pi}\int_0^1 \Theta(\tau)\frac{d\tau}{\tau - z}\right) . \tag{75}$$

Now, as discussed previously [15], we can consider Eq. (74) at $\varkappa$ values of z to establish $\varkappa$ algebraic equations that can be solved to yield the discrete eigenvalues $\{\nu_\alpha\}$. It is clear that the determination of $\varkappa$ is vital to this analysis. Thus once the constants ω and $\boldsymbol{C}_l$ are prescribed a careful analytical or numerical study of Eqs. (71) and (72) is necessary in order to deduce $\varkappa$ from Eq. (69).

IV. Exit distributions

We now consider the basic problem defined by Eq. (20) and the boundary conditions

$$\boldsymbol{\Psi}(L, \mu) = \boldsymbol{\Psi}_1(\mu), \quad \mu > 0 , \tag{76a}$$

and

$$\boldsymbol{\Psi}(R, -\mu) = \boldsymbol{\Psi}_2(\mu), \quad \mu > 0, \tag{76b}$$

where, at least for the moment, $\boldsymbol{\Psi}_1(\mu)$ and $\boldsymbol{\Psi}_2(\mu)$ are considered given. Thus if we write

$$\boldsymbol{\Psi}(\tau, \mu) = \sum_{\beta=0}^{\varkappa-1} \{A(\nu_\beta)\, \boldsymbol{\Phi}(\nu_\beta, \mu)\, \mathrm{e}^{-\tau/\nu_\beta} + A(-\nu_\beta)\, \boldsymbol{\Phi}(-\nu_\beta, \mu)\, \mathrm{e}^{\tau/\nu_\beta}\} + \int_{-1}^{1} \boldsymbol{\Phi}(\nu, \mu)\, \boldsymbol{A}(\nu)\, \mathrm{e}^{-\tau/\nu}\, \mathrm{d}\nu, \tag{77}$$

then we must determine the expansion coefficients $A(\pm \nu_\beta)$ and $\boldsymbol{A}(\nu)$ so that Eqs. (76) are satisfied. If this could be done then Eq. (77) would yield the desired result for all $\tau \in [L, R]$ and all $\mu \in [-1, 1]$. On the other hand, if we are primarily interested in the surface quantities $\boldsymbol{\Psi}(L, -\mu)$ and $\boldsymbol{\Psi}(R, \mu)$, $\mu > 0$, we can use an alternative procedure. We note that a full-range orthogonality relation concerning the $\boldsymbol{\Phi}(\xi, \mu)$ can be readily established from Eq. (25). Thus if we multiply Eq. (25) by $\boldsymbol{\Phi}^T(\xi', \mu)\, \boldsymbol{E}/\xi$, where $\boldsymbol{E}$ is a diagonal constant and T is used to denote the transpose operation, and integrate over μ we find

$$\int_{-1}^{1} \boldsymbol{\Phi}^T(\xi', \mu)\, \boldsymbol{E}\, \boldsymbol{\Phi}(\xi, \mu) \left(1 - \frac{\mu}{\xi}\right) \mathrm{d}\mu = \frac{\omega}{2} \boldsymbol{M}^T(\xi') \sum_{l=0}^{L} \boldsymbol{G}_l^T(\xi')\, \boldsymbol{E}\boldsymbol{C}_l\, \boldsymbol{G}_l(\xi)\, \boldsymbol{M}(\xi). \tag{78}$$

Now on interchanging ξ and ξ' in the transpose of Eq. (78) and subtracting the resulting equation from Eq. (78) we find

$$\left(\frac{1}{\xi'} - \frac{1}{\xi}\right) \int_{-1}^{1} \boldsymbol{\Phi}^T(\xi', \mu)\, \boldsymbol{E}\boldsymbol{\Phi}(\xi, \mu)\, \mu\, \mathrm{d}\mu = \frac{\omega}{2} \boldsymbol{M}^T(\xi') \sum_{l=0}^{L} \boldsymbol{G}_l^T(\xi')\, [\boldsymbol{E}\boldsymbol{C}_l - \boldsymbol{C}_l^T \boldsymbol{E}]\, \boldsymbol{G}_l(\xi)\, \boldsymbol{M}(\xi), \tag{79}$$

and we obtain the desired result, i.e. for $\xi, \xi' \in \{\pm \nu_\beta\} \cup (-1, 1)$

$$\int_{-1}^{1} \boldsymbol{\Phi}^T(\xi', \mu)\, \boldsymbol{E}\, \boldsymbol{\Phi}(\xi, \mu)\, \mu\, \mathrm{d}\mu = \boldsymbol{0}, \qquad \xi \neq \xi', \tag{80}$$

if

$$\boldsymbol{E}\boldsymbol{C}_l = \boldsymbol{C}_l^T \boldsymbol{E}. \tag{81}$$

Clearly for $\boldsymbol{C}_l$ as given by Eq. (22) we can take $\boldsymbol{E} = \boldsymbol{I}$; whereas for $\boldsymbol{C}_l$ as given by Eq. (23) we can take $\boldsymbol{E} = \operatorname{diag}\{1, -1\}$.

If we multiply Eq. (77) by $\mu\, \boldsymbol{\Phi}^T(\xi, \mu)\, \boldsymbol{E}$ and integrate over μ we see that

$$\int_{-1}^{1} \mu\, \boldsymbol{\Phi}^T(-\xi, \mu)\, \boldsymbol{E}\, [\boldsymbol{\Psi}(L, \mu) - \mathrm{e}^{-\Delta/\xi}\, \boldsymbol{\Psi}(R, \mu)]\, \mathrm{d}\mu = \boldsymbol{0} \tag{82a}$$

and

$$\int_{-1}^{1} \mu\, \boldsymbol{\Phi}^T(\xi, \mu)\, \boldsymbol{E}\, [\boldsymbol{\Psi}(R, \mu) - e^{-\Delta/\xi}\, \boldsymbol{\Psi}(L, \mu)]\, d\mu = \mathbf{0} \tag{82b}$$

for all $\xi \in \{\nu_\beta\} \cup (0, 1)$. Here $\Delta = R - L$. If we use Eqs. (76) we can write Eqs. (82) as

$$\int_0^1 \mu\, \boldsymbol{\Phi}^T(\xi, \mu)\, \boldsymbol{E}\, \boldsymbol{\Psi}(L, -\mu)\, d\mu + e^{-\Delta/\xi} \int_0^1 \mu\, \boldsymbol{\Phi}^T(-\xi, \mu)\, \boldsymbol{E}\, \boldsymbol{\Psi}(R, \mu)\, d\mu = \boldsymbol{L}_1(\xi) \tag{83a}$$

and

$$\int_0^1 \mu\, \boldsymbol{\Phi}^T(\xi, \mu)\, \boldsymbol{E}\, \boldsymbol{\Psi}(R, \mu)\, d\mu + e^{-\Delta/\xi} \int_0^1 \mu\, \boldsymbol{\Phi}^T(-\xi, \mu)\, \boldsymbol{E}\, \boldsymbol{\Psi}(L, -\mu)\, d\mu = \boldsymbol{L}_2(\xi) \tag{83b}$$

where the known terms are

$$\boldsymbol{L}_1(\xi) = \int_0^1 \mu\, \boldsymbol{\Phi}^T(-\xi, \mu)\, \boldsymbol{E}\, \boldsymbol{\Psi}_1(\mu)\, d\mu + e^{-\Delta/\xi} \int_0^1 \mu\, \boldsymbol{\Phi}^T(\xi, \mu)\, \boldsymbol{E}\, \boldsymbol{\Psi}_2(\mu)\, d\mu \tag{84a}$$

and

$$\boldsymbol{L}_2(\xi) = \int_0^1 \mu\, \boldsymbol{\Phi}^T(-\xi, \mu)\, \boldsymbol{E}\, \boldsymbol{\Psi}_2(\mu)\, d\mu + e^{-\Delta/\xi} \int_0^1 \mu\, \boldsymbol{\Phi}^T(\xi, \mu)\, \boldsymbol{E}\, \boldsymbol{\Psi}_1(\mu)\, d\mu\,. \tag{84b}$$

Equations (83) are a system of singular integral equations and constraints for the desired surface results. In a following work we intend to introduce a concise approximate solution of Eqs. (83) and to report numerical results for basic problems.

Acknowledgement

This work was supported by CNEN and CTA, both of Brazil, and the U.S. National Science Foundation.

References

[1] C. E. Siewert, The Astrophysical J. *245*, 1080 (1981).
[2] I. Kuščer and M. Ribarič, Optica Acta *6*, 42 (1959).
[3] S. Chandrasekhar, *Radiative Transfer*, Oxford University Press, London 1950.
[4] H. C. van de Hulst, *Multiple Light Scattering*, Vols. 1 and 2, Academic Press, New York 1980.
[5] M. Herman and J. Lenoble, JQSRT *8*, 355 (1968).
[6] I. M. Gel'fand and Z. Ya. Šapiro, Amer. Math. Soc. Translations *2*, 207 (1956).
[7] J. W. Hovenier, J. Atmospheric Sci. *26*, 488 (1969).
[8] H. Domke, JQSRT *15*, 669 (1975).
[9] H. Domke, JQSRT *15*, 681 (1975).
[10] H. Domke, JQSRT *16*, 973 (1976).
[11] H. Domke, Astrophys. *10*, 125 (1975).

[12] K. M. Case, Ann. Phys. *9*, 1 (1960).
[13] G. R. Bond and C. E. Siewert, The Astrophysical J. *164*, 97 (1971).
[14] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, New York 1953.
[15] C. E. Siewert, J. Math. Phys. *21*, 2468 (1980).

Abstract

The elementary solutions basic to the azimuthally symmetric component of the radiation density vector are developed for a general model concerning the scattering of polarized light.

Zusammenfassung

Die Elementarlösungen, die die Basis der azimuthal symmetrischen Komponente des Vektors der Strahlungsdichte bilden, werden hergeleitet für ein allgemeines Modell für die Streuung von polarisiertem Licht.

(Received: September 8, 1981.)