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CONCISE AND ACCURATE SOLUTIONS FOR CHANDRASEKHAR'S X AND Y FUNCTIONS

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ABSTRACT

The F_N method is used in a new way to develop, for the case of isotropic scattering, an efficient and accurate procedure for computing the X and Y functions introduced by Chandrasekhar to solve the standard problem in radiative transfer.

Subject headings: numerical methods — radiative transfer

I. INTRODUCTION

In his basic book on radiative transfer Chandrasekhar (1950) was able to express, for the case of isotropic scattering, the solution to his standard problem in terms of two basic functions—the X and Y functions. Thus the radiation field reflected by and transmitted through a finite plane-parallel layer for which the intensity $I(\tau, \mu)$ satisfies

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\omega}{2} \int_{-1}^{1} I(\tau, \mu')\, d\mu' \tag{1}$$

subject to the boundary conditions

$$I(0, \mu) = \delta(\mu - \mu_0), \qquad \mu > 0, \tag{2a}$$

and

$$I(\tau_0, -\mu) = 0, \qquad \mu > 0, \tag{2b}$$

can be expressed as

$$I(0, -\mu) = \frac{\omega}{2} \mu_0 \left(\frac{1}{\mu + \mu_0} \right) [X(\mu) X(\mu_0) - Y(\mu) Y(\mu_0)] \tag{3a}$$

and

$$I(\tau_0, \mu) = \delta(\mu - \mu_0) e^{-\tau_0/\mu} + \frac{\omega}{2} \mu_0 \left(\frac{1}{\mu - \mu_0} \right) [Y(\mu) X(\mu_0) - X(\mu) Y(\mu_0)], \tag{3b}$$

where $\mu \geq 0$. Note that by using $\mu > 0$ to specify propagation in the positive τ direction, we have a notational scheme slightly different from that of Chandrasekhar (1950) in our equations (1), (2), and (3). Chandrasekhar (1950) gave a system of nonlinear integral equations for the X and Y functions, and Mullikin (1962) later showed that the X and Y functions, for isotropic scattering, are the unique solutions to the singular integral equations

$$\lambda(\nu) X(\nu) - \frac{\omega}{2} \nu P \int_0^1 X(\mu) \frac{d\mu}{\mu - \nu} + \frac{\omega}{2} \nu e^{-\tau_0/\nu} \int_0^1 Y(\mu) \frac{d\mu}{\mu + \nu} = 1 \tag{4a}$$

and

$$\lambda(\nu) Y(\nu) - \frac{\omega}{2} \nu P \int_0^1 Y(\mu) \frac{d\mu}{\mu - \nu} + \frac{\omega}{2} \nu e^{-\tau_0/\nu} \int_0^1 X(\mu) \frac{d\mu}{\mu + \nu} = e^{-\tau_0/\nu} \tag{4b}$$

for $\nu \in [0,1]$ and the constraints, for $\omega < 1$,

$$\frac{\omega}{2}\nu_0 \int_0^1 X(\mu)\frac{d\mu}{\nu_0-\mu} + \frac{\omega}{2}\nu_0 e^{-\tau_0/\nu_0}\int_0^1 Y(\mu)\frac{d\mu}{\nu_0+\mu} = 1 \tag{5a}$$

and

$$\frac{\omega}{2}\nu_0 \int_0^1 Y(\mu)\frac{d\mu}{\nu_0-\mu} + \frac{\omega}{2}\nu_0 e^{-\tau_0/\nu_0}\int_0^1 X(\mu)\frac{d\mu}{\nu_0+\mu} = e^{-\tau_0/\nu_0} \tag{5b}$$

or, if $\omega = 1$,

$$\int_0^1 [X(\mu)+Y(\mu)]\, d\mu = 2 \tag{6a}$$

and

$$\int_0^1 [X(\mu)-Y(\mu)]\mu\, d\mu - \tau_0\int_0^1 Y(\mu)\, d\mu = 0. \tag{6b}$$

Here the symbol P is used to denote that the integrals are to be evaluated in the Cauchy principal-value sense, ν_0 is the positive zero of

$$\Lambda(z) = 1 + \frac{\omega}{2}z\int_{-1}^1 \frac{d\mu}{\mu - z}, \tag{7}$$

and

$$\lambda(\nu) = 1 + \frac{\omega}{2}\nu \ln\left(\frac{1-\nu}{1+\nu}\right). \tag{8}$$

II. ANALYSIS

In order to develop our F_N solutions (Siewert and Benoist 1979) for the X and Y functions, we substitute the approximations

$$X(\mu) = \Delta(\tau_0) + \sum_{\alpha=0}^{N} a_\alpha P_\alpha(2\mu - 1) \tag{9a}$$

and

$$Y(\mu) = \Delta(\tau_0)e^{-\tau_0/\mu} + \sum_{\alpha=0}^{N} b_\alpha P_\alpha(2\mu - 1), \tag{9b}$$

where the Legendre polynomials are denoted by $P_\alpha(z)$, into equations (4) and (5) to find, for $\xi \in \nu_0 \cup [0,1]$,

$$\sum_{\alpha=0}^{N} \{a_\alpha[B_\alpha(\xi) + \omega\delta_{\alpha,0}] - \omega e^{-\tau_0/\xi} b_\alpha[A_\alpha(\xi) - \delta_{\alpha,0}]\} = K_1(\xi) \tag{10a}$$

and

$$\sum_{\alpha=0}^{N} \{b_\alpha[B_\alpha(\xi) + \omega\delta_{\alpha,0}] - \omega e^{-\tau_0/\xi} a_\alpha[A_\alpha(\xi) - \delta_{\alpha,0}]\} = K_2(\xi). \tag{10b}$$

For the case $\omega = 1$ we substitute equations (9) into equations (6) to obtain

$$a_0 + b_0 = 2 - \Delta(\tau_0)[1 + E_2(\tau_0)] \tag{11a}$$

and

$$a_0+\tfrac{1}{3}a_1-(1+2\tau_0)b_0-\tfrac{1}{3}b_1=\Delta(\tau_0)[-1+2E_3(\tau_0)+2\tau_0E_2(\tau_0)]. \tag{11b}$$

Here

$$K_1(\xi)=2[1-\Delta(\tau_0)]+\omega\xi\,\Delta(\tau_0)\left[\ln\left(1+\frac{1}{\xi}\right)-e^{-\tau_0/\xi}\int_0^1 e^{-\tau_0/\eta}\frac{d\eta}{\eta+\xi}\right], \tag{12a}$$

$$K_2(\xi)=2[1-\Delta(\tau_0)]\,e^{-\tau_0/\xi}+\omega\xi\,\Delta(\tau_0)\int_0^1 C(\eta,\xi)\,d\eta, \tag{12b}$$

$$C(x,y)=\frac{e^{-\tau_0/x}-e^{-\tau_0/y}}{x-y}, \tag{13}$$

and $E_\alpha(z)$ denotes the exponential integral function. By defining

$$\Delta(\tau_0)=1, \qquad \tau_0<\tau_*,$$
$$\Delta(\tau_0)=0, \qquad \text{otherwise}, \tag{14}$$

we include in equations (9) terms that can improve the approximation as τ_0 becomes small (our choice of τ_* is discussed in the next section). The functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ are those introduced previously (Garcia and Siewert 1981), and reviewed in the Appendix.

For $\omega<1$ we can consider equations (10) at $\xi=\nu_0$ and N values of $\xi\in[0,1]$ and solve the resulting system of linear algebraic equations to find the constants $\{a_\alpha\}$ and $\{b_\alpha\}$ required in equations (9). In a similar way equations (11) and equations (10) evaluated at N values of $\xi\in[0,1]$ define, for the case $\omega=1$, a system of equations that yields the constants $\{a_\alpha\}$ and $\{b_\alpha\}$. Of course, we can now use the constants $\{a_\alpha\}$ and $\{b_\alpha\}$ in equations (9) to find the X and Y functions. However, we wish to use here a recently reported improvement (Garcia and Siewert 1982) in the F_N method for computing angular fluxes or, in this case, the X and Y functions. We note that we can write

$$B_\alpha(\xi)+\omega\delta_{\alpha,0}=2P_\alpha(2\xi-1)-\omega\xi\left[\ln\left(1+\frac{1}{\xi}\right)P_\alpha(2\xi-1)+M_\alpha(\xi)\right], \tag{15}$$

where the polynomials $M_\alpha(\xi)$, with $M_0(\xi)=0$, satisfy, for $\alpha\geq 1$,

$$M_\alpha(\xi)=\left(\frac{2\alpha-1}{\alpha}\right)(2\xi-1)M_{\alpha-1}(\xi)-\left(\frac{\alpha-1}{\alpha}\right)M_{\alpha-2}(\xi)+2\delta_{\alpha,1}. \tag{16}$$

Thus as an alternative to using equations (9) to compute $X(\mu)$ and $Y(\mu)$ once the constants $\{a_\alpha\}$ and $\{b_\alpha\}$ have been found, we can use equations (10) for $\xi=\mu\in[0,1]$ and equations (9) to find

$$X(\mu)=1+\frac{\omega}{2}\left\{\mu\sum_{\alpha=0}^{N}a_\alpha\left[M_\alpha(\mu)+\ln\left(1+\frac{1}{\mu}\right)P_\alpha(2\mu-1)\right]+e^{-\tau_0/\mu}\sum_{\alpha=0}^{N}b_\alpha[A_\alpha(\mu)-\delta_{\alpha,0}]+\Delta(\tau_0)\Xi(\mu)\right\} \tag{17a}$$

and

$$Y(\mu)=e^{-\tau_0/\mu}+\frac{\omega}{2}\left\{\mu\sum_{\alpha=0}^{N}b_\alpha\left[M_\alpha(\mu)+\ln\left(1+\frac{1}{\mu}\right)P_\alpha(2\mu-1)\right]+e^{-\tau_0/\mu}\sum_{\alpha=0}^{N}a_\alpha[A_\alpha(\mu)-\delta_{\alpha,0}]+\Delta(\tau_0)\Upsilon(\mu)\right\}, \tag{17b}$$

where

$$\Xi(\mu)=\mu\left[\ln\left(1+\frac{1}{\mu}\right)-e^{-\tau_0/\mu}\int_0^1 e^{-\tau_0/\eta}\frac{d\eta}{\eta+\mu}\right] \tag{18a}$$

and

$$\Upsilon(\mu)=\mu\int_0^1 C(\eta,\mu)\,d\eta. \tag{18b}$$

III. NUMERICAL RESULTS

For a given value of $\omega<1$ we first compute ν_0. We then use the collocation strategy (Garcia and Siewert 1981) $\xi_0=\nu_0$ and

$$\xi_\beta=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\beta-1}{2N}\pi\right),\qquad \beta=1,2,\ldots,N, \tag{19}$$

evaluate the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$, as discussed in the Appendix, and solve for $\beta=0,1,2,\ldots,N$ the system of linear algebraic equations

$$\sum_{\alpha=0}^{N}\left\{a_\alpha\left[B_\alpha(\xi_\beta)+\omega\delta_{\alpha,0}\right]-\omega e^{-\tau_0/\xi_\beta}b_\alpha\left[A_\alpha(\xi_\beta)-\delta_{\alpha,0}\right]\right\}=K_1(\xi_\beta) \tag{20a}$$

and

$$\sum_{\alpha=0}^{N}\left\{b_\alpha\left[B_\alpha(\xi_\beta)+\omega\delta_{\alpha,0}\right]-\omega e^{-\tau_0/\xi_\beta}a_\alpha\left[A_\alpha(\xi_\beta)-\delta_{\alpha,0}\right]\right\}=K_2(\xi_\beta) \tag{20b}$$

to find the constants $\{a_\alpha\}$ and $\{b_\alpha\}$. For the special case $\omega=1$ we use equations (11) and equations (20) for $\beta=1,2,\ldots,N$ to define the system of equations that yields the constants $\{a_\alpha\}$ and $\{b_\alpha\}$. Having found these constants, we compute the X and Y functions from equations (17). In Tables 1 and 2 we report some typical results we believe to be correct to ± 1 in the last digit given.

In our study of the computational aspects of the method we have found for $\omega\in[0,0.999999]$, $\omega=1$ and $\tau_0\in[10^{-5},\infty)$ that N typically in the range 20–30 is adequate to yield $X(\mu)$ and $Y(\mu)$, for $\mu\in[0,1]$, accurate to five or six significant figures. In trying to obtain results accurate to five or six significant figures, we also found that $\tau_*=0.5$ was sufficient, so that only for very thin layers were the leading terms in equations (9) actually used. To establish confidence in our results we have observed what appears to be numerical convergence in our calculations as N, in the F_N method, increases, and we have used equations (9) to verify to at least seven significant figures the moments

$$X_\alpha=\int_0^1 X(\mu)\mu^\alpha\,d\mu,\qquad \alpha=0,1,\ldots,10, \tag{21a}$$

TABLE 1

THE X FUNCTION FOR $\omega=1$

μ	$\tau_0=0.01$	$\tau_0=0.1$	$\tau_0=1$	$\tau_0=10$	$\tau_0=100$	$\tau_0=10^{10}$
0.......	1	1	1	1	1	1
0.01 ...	1.01718	1.02581	1.02989	1.03336	1.03416	1.03426
0.05 ...	1.02368	1.08288	1.11252	1.13160	1.13601	1.13657
0.10 ...	1.02475	1.11285	1.19449	1.23643	1.24612	1.24735
0.20 ...	1.02530	1.13530	1.32653	1.42495	1.44749	1.45035
0.30 ...	1.02549	1.14438	1.42940	1.59938	1.63766	1.64252
0.40 ...	1.02559	1.14927	1.51007	1.76521	1.82206	1.82928
0.50 ...	1.02565	1.15232	1.57403	1.92466	2.00286	2.01278
0.60 ...	1.02568	1.15441	1.62553	2.07886	2.18115	2.19413
0.70 ...	1.02571	1.15593	1.66767	2.22847	2.35759	2.37397
0.80 ...	1.02573	1.15708	1.70269	2.37389	2.53257	2.55270
0.90 ...	1.02575	1.15799	1.73220	2.51541	2.70636	2.73059
1.......	1.02576	1.15872	1.75737	2.65321	2.87914	2.90781

TABLE 2

THE Y FUNCTION FOR $\omega = 1$

μ	$\tau_0 = 0.01$	$\tau_0 = 0.1$	$\tau_0 = 1$	$\tau_0 = 10$	$\tau_0 = 100$	$\tau_0 = 10^{10}$
0.......	0	0	0	0	0	0
0.01 ...	3.83501(−1)	1.20175(−2)	4.54995(−3)	9.05590(−4)	1.01977(−4)	1.03426(−12)
0.05 ...	8.41962(−1)	1.97655(−1)	2.51138(−2)	4.97587(−3)	5.60326(−4)	5.68287(−12)
0.10 ...	9.29346(−1)	4.65105(−1)	5.55906(−2)	1.09217(−2)	1.22988(−3)	1.24735(−11)
0.20 ...	9.76411(−1)	7.32008(−1)	1.37304(−1)	2.53983(−2)	2.86006(−3)	2.90070(−11)
0.30 ...	9.92627(−1)	8.53815(−1)	2.49841(−1)	4.31453(−2)	4.85853(−3)	4.92757(−11)
0.40 ...	1.00084	9.22529(−1)	3.75967(−1)	6.40679(−2)	7.21459(−3)	7.31710(−11)
0.50 ...	1.00580	9.66513(−1)	5.00051(−1)	8.81186(−2)	9.92290(−3)	1.00639(−10)
0.60 ...	1.00912	9.97048(−1)	6.14943(−1)	1.15270(−1)	1.29804(−2)	1.31648(−10)
0.70 ...	1.01149	1.01947	7.18516(−1)	1.45505(−1)	1.63850(−2)	1.66178(−10)
0.80 ...	1.01328	1.03664	8.10880(−1)	1.78811(−1)	2.01355(−2)	2.04216(−10)
0.90 ...	1.01467	1.05020	8.92996(−1)	2.15179(−1)	2.42310(−2)	2.45753(−10)
1.......	1.01579	1.06118	9.66056(−1)	2.54591(−1)	2.86707(−2)	2.90781(−10)

and

$$Y_\alpha = \int_0^1 Y(\mu)\mu^\alpha \, d\mu, \qquad \alpha = 0, 1, \ldots, 10, \tag{21b}$$

tabulated by Burkart (1975).

We note that in comparing our results with the early calculations of Chandrasekhar, Elbert, and Franklin (1952), Sobouti (1963), Bellman, Kagiwada, Kalaba, and Ueno (1966), and Carlstedt and Mullikin (1966) we have found general agreement; however, we have observed what we believe to be errors in the third significant figure, for some cases, in all of these works. In regard to Caldwell's (1971) tabulations of the X and Y functions for $\omega_0 = 1$ and $\tau_0 = 0.5(0.5)4.5$, which we believe to be reported with a considerable number of digits in excess of the accuracy, we find general agreement to four or five significant figures; but we also find what we believe to be an error of 3 in the fourth significant figure of $Y(0.04)$ for $\tau_0 = 4.5$. In conclusion, we note that Karp (1981) has used the method of Carlstedt and Mullikin (1966) with a 150-point Gaussian representation of the integrals to find X and Y functions that agree perfectly with all but five of the entries in our Tables 1 and 2. For the five entries [$X(0.6)$ and $Y(0.6)$ for $\tau_0 = 0.01$, $X(0.8)$ and $Y(1)$ for $\tau_0 = 0.1$, and $Y(0.4)$ for $\tau_0 = 1$] we find a maximum difference of ± 5 in the eighth significant figure.

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APPENDIX

THE FUNCTIONS $A_\alpha(\xi)$ AND $B_\alpha(\xi)$

The functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ are defined for $\xi \in \nu_0 \cup [0, 1]$ by

$$A_\alpha(\xi) = \int_0^1 \mu P_\alpha(2\mu - 1) \frac{d\mu}{\mu + \xi}, \tag{A1}$$

$$B_\alpha(\nu_0) = \omega \int_0^1 \mu P_\alpha(2\mu - 1) \frac{d\mu}{\nu_0 - \mu}, \tag{A2a}$$

and

$$B_\alpha(\nu) = \omega P \int_0^1 \mu P_\alpha(2\mu - 1) \frac{d\mu}{\nu - \mu} + 2\lambda(\nu) P_\alpha(2\nu - 1), \tag{A2b}$$

for $\nu \in [0,1]$, and can be conveniently evaluated recursively from

$$A_\alpha(\xi) = -\left(\frac{2\alpha-1}{\alpha}\right)(2\xi+1)A_{\alpha-1}(\xi) - \left(\frac{\alpha-1}{\alpha}\right)A_{\alpha-2}(\xi) + \frac{1}{2}\delta_{\alpha,2} + \delta_{\alpha,1} \tag{A3a}$$

and

$$B_\alpha(\xi) = \left(\frac{2\alpha-1}{\alpha}\right)(2\xi-1)B_{\alpha-1}(\xi) - \left(\frac{\alpha-1}{\alpha}\right)B_{\alpha-2}(\xi) - \frac{\omega}{2}\delta_{\alpha,2} - \omega\delta_{\alpha,1} \tag{A3b}$$

with

$$A_0(\xi) = 1 - \xi \ln\left(1+\frac{1}{\xi}\right) \tag{A4}$$

and

$$B_0(\xi) = 2 - \omega\left[1 + \xi \ln\left(1+\frac{1}{\xi}\right)\right]. \tag{A5}$$

We note that while equations (A3) can for some values of ξ be used in a forward manner without loss of accuracy, backward recursion is sometimes necessary (Garcia and Siewert 1981).

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