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Analytical solutions to two matrix Riemann-Hilbert problems

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I. Introduction

In a recent paper [1] concerning the evaporation of a liquid into a half space [2] we reported our analysis of the system of partial-integro differential equations

$$(\mu + u)\frac{\partial}{\partial x}\boldsymbol{\Psi}(x,\mu) + \boldsymbol{\Psi}(x,\mu) = \pi^{-1/2}\int_{-\infty}^{\infty}[\boldsymbol{Q}(\mu)\,\boldsymbol{Q}^T(\mu') + 2\mu\mu'\boldsymbol{P}]\,\boldsymbol{\Psi}(x,\mu')\,\mathrm{e}^{-\mu'^2}\mathrm{d}\mu' \tag{1}$$

that describe temperature and density variations [3] in the kinetic theory of gases. Basic to our solution [1] of boundary-value problems relevant to Eq. (1) are the solutions to the matrix Riemann-Hilbert problems defined by

$$\boldsymbol{\Phi}^+(\mu) = \boldsymbol{G}(\mu)\,\boldsymbol{\Phi}^-(\mu), \qquad \mu \in [-u, \infty)\,, \tag{2}$$

and

$$\boldsymbol{\Theta}^+(\mu) = \boldsymbol{G}_*(\mu)\,\boldsymbol{\Theta}^-(\mu)\,, \qquad \mu \in (-\infty, -u]\,, \tag{3}$$

where

$$\boldsymbol{G}(\mu) = [\boldsymbol{\Lambda}^+(\mu)]^T[\boldsymbol{\Lambda}^-(\mu)]^{-T}, \tag{4}$$

$$\boldsymbol{G}_*(\mu) = \boldsymbol{\Lambda}^+(\mu)\,[\boldsymbol{\Lambda}^-(\mu)]^{-1} \tag{5}$$

and

$$\boldsymbol{\Lambda}(z) = \boldsymbol{I} + (z+u)\int_{-\infty}^{\infty}\boldsymbol{\Psi}(\mu)\frac{\mathrm{d}\mu}{\mu - z}\,. \tag{6}$$

In addition

$$\boldsymbol{\Psi}(\mu) = \pi^{-1/2}\,\boldsymbol{Q}^T(\mu)\,[\boldsymbol{Q}(\mu) - 2u\mu\,\boldsymbol{T}]\,\mathrm{e}^{-\mu^2}, \tag{7}$$

$$Q(\mu) = \begin{vmatrix} \left(\frac{2}{3}\right)^{1/2} (\mu^2 - \frac{1}{2}) & 1 \\ \left(\frac{2}{3}\right)^{1/2} & 0 \end{vmatrix}, \tag{8}$$

$$T = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \quad P = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \tag{9a, b}$$

and for evaporation we consider $u > 0$. We seek 2×2 matrices $\boldsymbol{\Phi}(z)$ and $\boldsymbol{\Theta}(z)$ such that $\boldsymbol{\Phi}(z)$ is analytic in the complex plane cut along the real axis from $-u$ to ∞, $\boldsymbol{\Theta}(z)$ is analytic in the complex plane cut from $-\infty$ to $-u$ along the real axis, $\det \boldsymbol{\Phi}(z) \neq 0$, $\det \boldsymbol{\Theta}(z) \neq 0$ and such that the boundary values of $\boldsymbol{\Phi}(z)$ and $\boldsymbol{\Theta}(z)$, i.e. $\boldsymbol{\Phi}^{\pm}(\mu)$ and $\boldsymbol{\Theta}^{\pm}(\mu)$, as z approaches the real axis from above and below satisfy Eqs. (2) and (3). We note that Siewert and Kelley [4], following the work of Darrozès [5] and Cercignani [6], have reported the desired solution for the case $u = 0$.

II. Analysis

First of all we note that we can write

$$\boldsymbol{\Lambda}(z) = \boldsymbol{Y}(z) + \boldsymbol{\Xi}(z)\, J(z) \tag{10}$$

where $\boldsymbol{Y}(z)$ and $\boldsymbol{\Xi}(z)$ are polynomial matrices and

$$J(z) = \pi^{-1/2}(z+u) \int_{-\infty}^{\infty} \mathrm{e}^{-\mu^2} \frac{\mathrm{d}\mu}{\mu - z}. \tag{11}$$

Using the explicit forms [1] for $\boldsymbol{Y}(z)$ and $\boldsymbol{\Xi}(z)$, we find we can rewrite Eq. (10) as

$$\boldsymbol{Q}^{-T}(z)\, \boldsymbol{\Lambda}(z)\, \boldsymbol{E}(z)\, \boldsymbol{Q}^{-1}(z) = \boldsymbol{\Pi}(z) + (1 - 2uz)\, J(z)\, \boldsymbol{I} \tag{12}$$

where

$$\boldsymbol{E}(z) = \operatorname{diag}\{1 - 2uz, 1\} \tag{13}$$

and

$$\boldsymbol{\Pi}(z) = \begin{vmatrix} 1 - 2u(z+u) & \frac{1}{2} - u^2 \\ \frac{1}{2} - u^2 & \frac{3}{2}(1 - 2uz) + (\frac{1}{2} - u^2)(\frac{1}{2} - z^2) \end{vmatrix}. \tag{14}$$

It is apparent that for the special case $u^2 = \frac{1}{2}$

$$\boldsymbol{W}(z) = \boldsymbol{Q}^{-T}(z)\, \boldsymbol{\Lambda}(z)\, \boldsymbol{E}(z)\, \boldsymbol{Q}^{-1}(z) \tag{15}$$

is diagonal. Devoting the Appendix to this special case and considering here that $u^2 \neq \frac{1}{2}$, we observe that $\boldsymbol{W}(z)$ can be diagonalized by a similarity transformation involving at worst

$$R(z) = \sqrt{q(z)} \tag{16}$$

where $q(z)$ is a polynomial. We find

$$\boldsymbol{S}(z)\,\boldsymbol{\Lambda}(z)\,\boldsymbol{E}(z)\,\boldsymbol{Q}^{-1}(z)\,\boldsymbol{Q}^{-T}(z)\,\boldsymbol{S}^{-1}(z) = \boldsymbol{\Omega}(z) \tag{17}$$

where

$$\boldsymbol{\Omega}(z) = \operatorname{diag}\{\Omega_1(z), \Omega_2(z)\}, \tag{18}$$

$$\Omega_{1,2}(z) = \tfrac{1}{4}\left[\tfrac{11}{2} - 5u^2 - 10uz - (1-2u^2)z^2 \pm R(z) + 4(1-2uz)J(z)\right], \tag{19}$$

$$\boldsymbol{S}(z) = \begin{vmatrix} \left(\dfrac{3}{2}\right)^{1/2}(1-2u^2) & \tfrac{1}{2}\left[R(z) - z^2 - \tfrac{1}{2} - 5u^2 + 2uz + 2u^2z^2\right] \\ -\left(\dfrac{3}{2}\right)^{1/2}(1-2u^2) & \tfrac{1}{2}\left[R(z) + z^2 + \tfrac{1}{2} + 5u^2 - 2uz - 2u^2z^2\right] \end{vmatrix} \tag{20}$$

and

$$q(z) = (1-2u^2)^2 z^4 + 4u(1-2u^2)z^3 + [4u^2(1+3u^2) - 3]z^2 - 6u(1+2u^2)z + 25u^4 - 7u^2 + \tfrac{25}{4}. \tag{21}$$

The sectionally analytic function $R(z)$ which we write as

$$R(z) = |1-2u^2|\,[(z-z_1)(z-z_2)(z-z_3)(z-z_4)]^{1/2} \tag{22}$$

has branch points at z_1, z_2, z_3, and z_4, the four zeros of $q(z)$. We note that van Hooff [7] has proved, for $u^2 \neq \frac{1}{2}$, that $q(z)$ has no real zeros. We consider here the branch of $R(z)$ that is analytic in the complex plane cut along $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 denotes a path between the two branch points in the upper half plane and Γ_2 is a path between the other two branch points.

Investigating now the Riemann-Hilbert problem defined by Eq. (2), we generalize the method reported by Siewert and Kelley [4] for the case $u = 0$. We let $\boldsymbol{\Delta}(z) = \operatorname{diag}\{1, 1-2uz\}$ and substitute

$$\boldsymbol{\Phi}_1(z) = \boldsymbol{\Delta}(z)\,\boldsymbol{S}^{-1}(z)\,\boldsymbol{U}(z)\,\boldsymbol{S}(z) \tag{23}$$

into Eq. (2) to find

$$\boldsymbol{U}^+(\mu) = \boldsymbol{\Omega}^+(\mu)\,[\boldsymbol{\Omega}^-(\mu)]^{-1}\,\boldsymbol{U}^-(\mu), \qquad \mu \in [-u, \infty). \tag{24}$$

We must also require

$$\boldsymbol{\Phi}^+(\tau) = \boldsymbol{\Phi}^-(\tau), \qquad \tau \in \Gamma, \tag{25}$$

and since

$$\boldsymbol{S}^-(\tau) = -\boldsymbol{B}\boldsymbol{S}^+(\tau) \tag{26}$$

where

$$B = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \tag{27}$$

we find that $U(z)$ must also satisfy

$$U^+(\tau)\, B = B\, U^-(\tau)\,, \qquad \tau \in \Gamma. \tag{28}$$

If we let

$$\gamma_\alpha(\mu) = \Omega_\alpha^+(\mu)/\Omega_\alpha^-(\mu)\,, \tag{29}$$

$$A(\mu) = \gamma_1(\mu)/\gamma_2(\mu) \tag{30}$$

and

$$B(\mu) = \gamma_1(\mu)\, \gamma_2(\mu) \tag{31}$$

then since $R^+(\tau) = -R^-(\tau)$, $\tau \in \Gamma$, we find that

$$U^*(z) = \operatorname{diag}\{U_1^*(z), U_2^*(z)\}\,, \tag{32}$$

where

$$U_\alpha^*(z) = \exp\left(\frac{1}{4\pi i}\int_{-u}^{\infty}\left\{\log B(x) - (-1)^\alpha \frac{R(z)}{R(x)} \cdot [\log A(x) + 2k\pi i\, \Delta(x)]\right\}\frac{\mathrm{d}x}{x-z}\right), \tag{33}$$

yields a solution to Eqs. (24) and (28). Here k is an *even* integer (Siewert and Kelley [4] failed to mention that k must be even) and

$$\Delta(x) = \begin{cases} 1, & x \in (x_0, x_1) \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

Since $A(x)$ and $B(x)$ have unit magnitudes we write Eq. (33) as

$$U_\alpha^*(z) = \exp\left(\frac{1}{4\pi}\int_{-u}^{\infty}\left\{\arg B(x) - (-1)^\alpha \frac{R(z)}{R(x)} \cdot [\arg A(x) + 2k\pi\, \Delta(x)]\right\}\frac{\mathrm{d}x}{x-z}\right) \tag{35}$$

and use continuous values of $\arg A(x)$ and $\arg B(x)$ such that $\arg A(\infty) = \arg B(\infty) = 0$. Since $R(z) \to z^2$ as $|z| \to \infty$ we must impose the condition [4] that

$$\int_{x_0}^{x_1} \frac{\mathrm{d}x}{R(x)} = -\frac{1}{2k\pi}\int_{-u}^{\infty} \arg A(x) \frac{\mathrm{d}x}{R(x)} \tag{36}$$

in order to avoid in $U_\alpha^*(z)$ an apparent essential singularity at infinity.

We now consider that $u^2 \in (0, 1/2) \cup (1/2, 5/6)$, i.e. the non-special values of u for which the strong evaporation problem was solved [1]. Now since $q(0) > 0$ it is clear that $q(x) > 0$ for all $x \in (-\infty, \infty)$. We therefore take $R(x) > 0$ for all $x \in (-\infty, \infty)$. We find that $\Omega_1\left(\frac{1}{2u}\right) = 0$, and so we write

$$\Omega_1(z) = (1 - 2uz)\, W(z) \tag{37}$$

and

$$\gamma_1(x) = W^+(x)/W^-(x)\,. \tag{38}$$

We can now write

$$\gamma_\alpha(x) = \mathrm{e}^{2i\vartheta_\alpha(x)} \tag{39}$$

where

$$\vartheta_1(x) = \tan^{-1}\left(\frac{4\pi^{1/2}(x+u)\exp(-x^2)}{w(x)}\right) \tag{40a}$$

and

$$\vartheta_2(x) = \tan^{-1}\left(\frac{4\pi^{1/2}(x+u)(1-2ux)\exp(-x^2)}{M(x)-R(x)}\right). \tag{40b}$$

In addition

$$M(x) = \tfrac{11}{2} - 5u^2 - 10ux - (1-2u^2)\,x^2 - 8(1-2ux)(x+u)\,\mathrm{e}^{-x^2}\int_0^x \mathrm{e}^{\tau^2}\,\mathrm{d}\tau \tag{41}$$

and

$$w(x) = \frac{M(x)+R(x)}{1-2ux}. \tag{42}$$

We take continuous values of $\vartheta_\alpha(x)$, with $\vartheta_\alpha(-u) = 0$, and note that $\vartheta_1(x) \in [0, \pi]$ for $x \in [-u, \infty)$ and that $\vartheta_1(\infty) = \pi$. Further we find that $\vartheta_2(x) \in [0, 2\pi]$ for $x \in [-u, \infty)$ and that $\vartheta_2(\infty) = \pi$. It follows that

$$\arg A(x) = 2[\vartheta_1(x) - \vartheta_2(x)] = -2\theta(x) \tag{43a}$$

and

$$\arg B(x) = 2[\vartheta_1(x) + \vartheta_2(x) - 2\pi] = 2\varphi(x)\,. \tag{43b}$$

Equation (36) can now be written as

$$\int_{x_0}^{x_1} \frac{\mathrm{d}x}{R(x)} = \frac{1}{k\pi}\int_{-u}^{\infty} \theta(x)\frac{\mathrm{d}x}{R(x)}, \tag{44}$$

and since $\theta(x) \in [0, 2\pi]$ for $x \in [-u, \infty)$ we can take $k = 2$, $x_0 = -u$ and solve

$$\int_{-u}^{x_1} \frac{\mathrm{d}x}{R(x)} = \frac{1}{2\pi}\int_{-u}^{\infty} \theta(x)\frac{\mathrm{d}x}{R(x)} \tag{45}$$

to find x_1. Some typical results are shown in Table 1. It now follows that

$$U_\alpha^*(z) = \exp\left(\frac{1}{2\pi}\int_{-u}^{\infty}\left[\varphi(x) + (-1)^\alpha \frac{R(z)}{R(x)}\theta(x)\right]\frac{\mathrm{d}x}{x-z} + (-1)^{\alpha+1} R(z)\int_{-u}^{x_1}\frac{1}{R(x)}\frac{\mathrm{d}x}{x-z}\right), \tag{46}$$

and to remove the singularity at $z = x_1$ we write

$$U_\alpha(z) = (z - x_1)\, U_\alpha^*(z)\,. \tag{47}$$

From Eq. (23) we observe that

$$\det \boldsymbol{\Phi}_1(z) = (1 - 2u\,z)\; U_1(z)\; U_2(z) \tag{48}$$

or

$$\det \boldsymbol{\Phi}_1(z) = (1 - 2u\,z)\;(z+u)^2 (z - x_1)^2 \left[\frac{1}{(z+u)^2} \exp\left(\frac{1}{\pi} \int_{-u}^{\infty} \varphi(x) \frac{\mathrm{d}x}{x - z}\right)\right]. \tag{49}$$

The solution $\boldsymbol{\Phi}_1(z)$ therefore is not a canonical solution since $\det \boldsymbol{\Phi}_1(z)$ vanishes at $z = x_1$, $z = -u$ and $z = 1/(2u)$. We thus let $\boldsymbol{\Phi}(z)$ denote a canonical solution and write

$$\boldsymbol{\Phi}_1(z) = \boldsymbol{\Phi}(z)\; \boldsymbol{P}_*(z) \tag{50}$$

or

$$\boldsymbol{\Phi}(z) = [(1 - 2u\,z)\;(z+u)^2 (z - x_1)^2]^{-1}\; \boldsymbol{\Phi}_1(z)\; \boldsymbol{P}(z)\,. \tag{51}$$

Here $\boldsymbol{P}_*(z)$ and $\boldsymbol{P}(z)$ are polynomials and

$$\det \boldsymbol{P}(z) \propto (1 - 2u\,z)\;(z - x_1)^2 (z+u)^2. \tag{52}$$

We must determine $\boldsymbol{P}(z)$ so that

$$\boldsymbol{\Phi}_1(\xi)\; \boldsymbol{P}(\xi) = \mathbf{0}\,, \qquad \xi = \frac{1}{2u}, \qquad \xi = -u \quad \text{and} \quad \xi = x_1\,, \tag{53a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\xi}[\boldsymbol{\Phi}_1(\xi)\; \boldsymbol{P}(\xi)] = \mathbf{0}\,, \qquad \xi = -u \quad \text{and} \quad \xi = x_1. \tag{53b}$$

On using Eqs. (23) and (46), we find that Eqs. (53) yield

$$\begin{vmatrix} k_1 & 1 \\ 0 & 0 \end{vmatrix} \boldsymbol{P}(-u) = \mathbf{0}\,, \tag{54a}$$

$$\begin{vmatrix} k_1 & 1 \\ 0 & 0 \end{vmatrix} \frac{\mathrm{d}}{\mathrm{d}z} \boldsymbol{P}(z) + \begin{vmatrix} 0 & k_2 \\ 0 & 0 \end{vmatrix} \boldsymbol{P}(z) = \mathbf{0}\,, \qquad z = -u\,, \tag{54b}$$

$$\begin{vmatrix} 0 & 0 \\ 1 & k_3 \end{vmatrix} \boldsymbol{P}(x_1) = \mathbf{0}\,, \tag{54c}$$

$$\begin{vmatrix} 0 & 0 \\ 1 & k_3 \end{vmatrix} \frac{\mathrm{d}}{\mathrm{d}x_1} \boldsymbol{P}(x_1) + \begin{vmatrix} 0 & 0 \\ 0 & k_4 \end{vmatrix} \boldsymbol{P}(x_1) = \mathbf{0}\,, \tag{54d}$$

and

$$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \boldsymbol{P}\left(\frac{1}{2u}\right) = \mathbf{0} \tag{54e}$$

where

$$k_1 = 2(1-2u^2)\left(\frac{3}{2}\right)^{1/2}[R(-u)+2u^4-8u^2-\tfrac{1}{2}]^{-1}, \tag{55a}$$

$$k_2 = 2u\,[2(1-u^2)-u^2(4u^4+8u^2+3)/R(-u)] \cdot [R(-u)+2u^4-8u^2-\tfrac{1}{2}]^{-1}, \tag{55b}$$

$$k_3 = -\left(\frac{2}{3}\right)^{1/2} S_{22}(x_1)/(1-2u^2)\,, \tag{55c}$$

$$k_4 = -\left(\frac{2}{3}\right)^{1/2} \frac{\mathrm{d}}{\mathrm{d}x_1} S_{22}(x_1)/(1-2u^2)\,. \tag{55d}$$

and $S_{22}(z)$ is labeled as an element of $\boldsymbol{S}(z)$. From Eq. (52) it is apparent that the matrix $\boldsymbol{P}(z)$ must be at least cubic in z, and it is also clear that Eqs. (54) can be satisfied by a cubic; thus in order to obtain at once a canonical solution that has normal form at infinity we let

$$\boldsymbol{P}(z) = \begin{vmatrix} a_{11}+b_{11}z+c_{11}z^2+l_{22}z^3 & a_{12}+b_{12}z+c_{12}z^2 \\ a_{21}+b_{21}z & a_{22}+b_{22}z-l_{11}z^2 \end{vmatrix}\,. \tag{56}$$

Here the two sets of constants $\{a_{11}, b_{11}, c_{11}, a_{21}, b_{21}\}$ and $\{a_{12}, b_{12}, c_{12}, a_{22}, b_{22}\}$ can be readily found by substituting Eq. (56) into Eqs. (54) and solving the two resulting sets of five linear algebraic equations. In writing Eq. (56) we have made use of the fact that Eqs. (20), (46) and (47) can be used in Eq. (23) to yield, for large $|z|$,

$$\boldsymbol{\Phi}_1(z) \to \begin{vmatrix} l_{11}z & l_{12}z^2 \\ l_{21} & l_{22}z^2 \end{vmatrix} + \dots \tag{57}$$

where

$$l_{11} = [b\,U_1^*(\infty) + a\,U_2^*(\infty)]\,(2\,|1-2u^2|)^{-1}, \tag{58a}$$

$$l_{22} = -u\,[a\,U_1^*(\infty) + b\,U_2^*(\infty)]\,(|1-2u^2|)^{-1} \tag{58b}$$

Table 1
Computed Values of x_1 and y_1

u	x_1	y_1
0.0	0.4232585948	–
0.1	0.3343463538	1.135463005
0.2	0.2672015586	1.068122027
0.3	0.2306651046	1.025512132
0.4	0.2509172223	1.010471675
0.5	0.3895740878	1.024144838
0.6	0.8002481006	1.063757221
0.7	1.973904503	1.123421895
0.8	1.824695075	1.197300176
0.9	1.582201876	1.281710422

and

$$U_\alpha^*(\infty) = \exp\left[(-1)^{\alpha+1}\,|1-2u^2|\left(\frac{1}{2\pi}\int_{-u}^{\infty}\frac{x\,\theta(x)}{R(x)}\,\mathrm{d}x - \int_{-u}^{x_1}\frac{x}{R(x)}\,\mathrm{d}x\right)\right]. \tag{59}$$

In addition

$$a = |1-2u^2| - 1 + 2u^2 \quad \text{and} \quad b = |1-2u^2| + 1 - 2u^2. \tag{60 and 61}$$

Considering that we can solve the mentioned systems of linear algebraic equations, we conclude that our desired canonical solution, that has normal form at infinity, is given by Eq. (51). For large $|z|$ we find from Eqs. (51), (56) and (57) that

$$\boldsymbol{\Phi}(z) \to \begin{vmatrix} \frac{1}{z} & \frac{1}{z}\,r_{12} \\ \frac{1}{z^2}\,r_{21} & -\frac{1}{z} \end{vmatrix} + \ldots. \tag{62}$$

From Eq. (62) we conclude that the partial indices here are $\varkappa_1 = \varkappa_2 = 1$, as anticipated in our previous paper [1]. We can now write one of the desired $\boldsymbol{H}$ matrices [1] as

$$\boldsymbol{H}_2(z) = \boldsymbol{\Phi}(-u)\,\boldsymbol{\Phi}^{-1}(z) \tag{63}$$

or

$$\boldsymbol{H}_2(z) = \boldsymbol{\Phi}(-u)\,\boldsymbol{I}(z)\,\boldsymbol{S}^{-1}(z)\,\boldsymbol{U}^{-1}(z)\,\boldsymbol{S}(z)\,\boldsymbol{\Delta}^{-1}(z) \tag{64}$$

where

$$\boldsymbol{I}(z) = \begin{vmatrix} -a_{22} - b_{22}z + l_{11}z^2 & a_{12} + b_{12}z + c_{12}z^2 \\ a_{21} + b_{21}z & -a_{11} - b_{11}z - c_{11}z^2 - l_{22}z^3 \end{vmatrix}. \tag{65}$$

We now consider the Riemann-Hilbert problem defined by Eq. (3) and write

$$\boldsymbol{\Theta}_1(z) = \boldsymbol{S}^{-1}(z)\,\boldsymbol{V}(z)\,\boldsymbol{S}(z) \tag{66}$$

where

$$\boldsymbol{V}^+(\mu) = \boldsymbol{\Omega}^+(\mu)\,[\boldsymbol{\Omega}^-(\mu)]^{-1}\,\boldsymbol{V}^-(\mu)\,, \qquad \mu \in (-\infty, -u]\,, \tag{67 a}$$

and

$$\boldsymbol{V}^+(\tau)\,\boldsymbol{B} = \boldsymbol{B}\boldsymbol{V}^-(\tau)\,, \qquad \tau \in \Gamma. \tag{67 b}$$

In a manner similar to the foregoing we find

$$\boldsymbol{V}(z) = (z + y_1)\,\mathrm{diag}\,\{V_1^*(z), V_2^*(z)\} \tag{68}$$

where

$$V_\alpha^*(z) = \exp\left(\frac{1}{2\pi}\int_{-\infty}^{-u}\left[\varphi(x) + (-1)^\alpha\frac{R(z)}{R(x)}\,\theta(x)\right]\frac{\mathrm{d}x}{x-z} - (-1)^\alpha R(z)\int_{-y_1}^{-u}\frac{1}{R(x)}\,\frac{\mathrm{d}x}{x-z}\right), \tag{69}$$

$$\int_{-y_1}^{-u} \frac{1}{R(x)}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{-u} \frac{\theta(x)}{R(x)}\,\mathrm{d}x, \tag{70}$$

$$\varphi(x) = \vartheta_1(x) + \vartheta_2(x) + \pi \tag{71a}$$

and

$$\theta(x) = \vartheta_2(x) - \vartheta_1(x) + \pi. \tag{71b}$$

Here the angles

$$\vartheta_1(x) = \tan^{-1}\left(\frac{4\pi^{1/2}(x+u)\,(1-2u\,x)\exp(-x^2)}{M(x)+R(x)}\right) \tag{72a}$$

and

$$\vartheta_2(x) = \tan^{-1}\left(\frac{4\pi^{1/2}(x+u)\,(1-2u\,x)\exp(-x^2)}{M(x)-R(x)}\right) \tag{72b}$$

are defined to be continuous for $x \in (-\infty, -u]$ and such that $\vartheta_1(-\infty) = \vartheta_1(-u) = \vartheta_2(-u) = 0$ and $\vartheta_2(-\infty) = -\pi$. It is clear that Eq. (70) can be solved, and thus we include in Table 1 computed values of y_1 for selected values of u. Now since Eqs. (66), (68) and (69) yield

$$\det \boldsymbol{\Theta}_1(z) = (z+y_1)^2 (z+u)\left[\frac{1}{(z+u)}\exp\left(\frac{1}{\pi}\int_{-\infty}^{-u} \varphi(x)\frac{\mathrm{d}x}{x-z}\right)\right] \tag{73}$$

we conclude that $\boldsymbol{\Theta}_1(z)$ is not a canonical solution. However, the desired canonical solution $\boldsymbol{\Theta}(z)$ is given by

$$\boldsymbol{\Theta}(z) = [(z+u)\,(z+y_1)^2]^{-1}\,\boldsymbol{S}^{-1}(z)\,\boldsymbol{V}(z)\,\boldsymbol{S}(z)\,\boldsymbol{B}(z) \tag{74}$$

where the polynomial matrix $\boldsymbol{B}(z)$, with $\det \boldsymbol{B}(z) \propto (z+u)\,(z+y_1)^2$, can be found from

$$\begin{vmatrix} 0 & 0 \\ 1 & l_1 \end{vmatrix} \boldsymbol{B}(-u) = \boldsymbol{0}, \tag{75a}$$

$$\begin{vmatrix} 1 & l_2 \\ 0 & 0 \end{vmatrix} \boldsymbol{B}(-y_1) = \boldsymbol{0} \tag{75b}$$

and

$$\begin{vmatrix} 1 & l_2 \\ 0 & 0 \end{vmatrix} \frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{B}(z) + \begin{vmatrix} 0 & l_3 \\ 0 & 0 \end{vmatrix} \boldsymbol{B}(z) = \boldsymbol{0}, \quad z = -y_1, \tag{75c}$$

where

$$l_1 = -\left(\frac{2}{3}\right)^{1/2} [R(-u) - 2u^4 + 8u^2 + \tfrac{1}{2}]\,[2(1-2u^2)]^{-1}, \tag{76a}$$

$$l_2 = \left(\frac{2}{3}\right)^{1/2} S_{12}(-y_1)/(1-2u^2) \tag{76b}$$

and

$$l_3 = \left(\frac{2}{3}\right)^{1/2} \left(\frac{\mathrm{d}}{\mathrm{d}z} S_{12}(z)\Big|_{z=-y_1}\right) \Big/ (1 - 2u^2) \,. \tag{76c}$$

For large $|z|$

$$\boldsymbol{\Theta}_1(z) \to \begin{vmatrix} m_{11} z & m_{12} z^2 \\ \frac{1}{z} m_{21} & m_{22} z \end{vmatrix} + \dots \tag{77}$$

where

$$m_{11} = [b\, V_1^*(\infty) + a\, V_2^*(\infty)]\, (2\, |1 - 2u^2|)^{-1}, \tag{78}$$

$$m_{22} \doteq [a\, V_1^*(\infty) + b\, V_2^*(\infty)]\, (2\, |1 - 2u^2|)^{-1} \tag{79}$$

and

$$V_\alpha^*(\infty) = \exp\left[(-1)^{\alpha+1}\, |1 - 2u^2| \left(\frac{1}{2\pi} \int_{-\infty}^{-u} \frac{x\,\theta(x)}{R(x)}\,\mathrm{d}x - \int_{-y_1}^{-u} \frac{x}{R(x)}\,\mathrm{d}x\right)\right]. \tag{80}$$

We thus can use

$$\boldsymbol{B}(z) = \begin{vmatrix} \alpha_{11} + \beta_{11} z - m_{22} z^2 & \alpha_{12} + \beta_{12} z + m_{22} z^2 \\ \alpha_{21} + m_{11} z & \alpha_{22} + m_{11} z \end{vmatrix} \tag{81}$$

and conclude that for large $|z|$

$$\boldsymbol{\Theta}(z) \to \begin{vmatrix} 1 & \frac{1}{z} s_{12} \\ \frac{1}{z} s_{21} & \frac{1}{z} \end{vmatrix} + \dots. \tag{82}$$

Here the constants $\{\alpha_{11}, \beta_{11}, \alpha_{21}\}$ and $\{\alpha_{12}, \beta_{12}, \alpha_{22}\}$ can be determined from Eqs. (75). It follows from Eq. (81) that the partial indices for this problem are $\varkappa_1^* = 0$ and $\varkappa_2^* = 1$, as suggested earlier [1]. We can now use the definition

$$\boldsymbol{H}_1^{-1}(z) = \boldsymbol{\Theta}(z)\, \boldsymbol{\Theta}^{-1}(-u) \tag{83}$$

and invoke the condition [1]

$$\boldsymbol{H}_1^{-T}(\infty) \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \boldsymbol{0} \tag{84}$$

to find our final result, *viz.*

$$\boldsymbol{H}_1(z) = \boldsymbol{\Theta}(-u)\, \boldsymbol{J}(z)\, \boldsymbol{S}^{-1}(z)\, \boldsymbol{V}^{-1}(z)\, \boldsymbol{S}(z) \tag{85}$$

where

$$\boldsymbol{J}(z) = (z + u)\, (z + y_1)^2\, \boldsymbol{B}^{-1}(z)\,. \tag{86}$$

We have evaluated Eqs. (64) and (A-16) for $z = -\tau$, $\tau \in [u, \infty)$, and Eqs. (85) and (A-26) for $z = \mu$, $\mu \in [-u, \infty)$, to obtain for selected values of

$u^2 \in (0, 5/6)$ numerical results that agree (to ten significant figures) with values of $\boldsymbol{H}_2(-\tau)$ and $\boldsymbol{H}_1(\mu)$ established previously [1] by solving in a strictly numerical way a set of coupled non-linear integral equations. In conclusion we note that although it clearly is satisfying to find that the developed results do in fact constitute computationally viable solutions of the considered Riemann-Hilbert problems, we believe it considerably more important that we have been able, as anticipated [9], to deduce the partial indices that play a vital role in the theory of systems of singular integral equations [8].

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Appendix: The special case $u^2 = \frac{1}{2}$

Here we write Eq. (12) as

$$\boldsymbol{Q}^{-T}(z)\,\boldsymbol{\Lambda}(z)\,\boldsymbol{E}(z)\,\boldsymbol{Q}^{-1}(z)\,\boldsymbol{\Delta}^{-1}(z) = \boldsymbol{F}(z) \tag{A-1}$$

where

$$\boldsymbol{F}(z) = \operatorname{diag}\{-2uz + (1-2uz)\,J(z), \tfrac{3}{2} + J(z)\} \tag{A-2}$$

and

$$\boldsymbol{\Delta}(z) = \operatorname{diag}\{1, 1-2uz\}. \tag{A-3}$$

Thus on introducing a diagonal matrix $\boldsymbol{U}(z)$, with elements $U_1(z)$ and $U_2(z)$, and substituting

$$\boldsymbol{\Phi}_1(z) = \boldsymbol{E}^{-1}(z)\,\boldsymbol{Q}^T(z)\,\boldsymbol{\Delta}(z)\,\boldsymbol{U}(z) \tag{A-4}$$

into Eq. (A-2), we find the diagonal system

$$\boldsymbol{U}^+(\mu) = \boldsymbol{F}^+(\mu)\,[\boldsymbol{F}^-(\mu)]^{-1}\,\boldsymbol{U}^-(\mu), \qquad \mu \in [-u, \infty), \tag{A-5}$$

which can be solved by the theory of Muskhelishvili [8] to yield

$$U_1(z) = (z+u)^{-2} \exp\left(\frac{1}{\pi} \int_{-u}^{\infty} [\vartheta_1(\mu) - 2\pi] \frac{d\mu}{\mu - z}\right) \tag{A-6a}$$

and

$$U_2(z) = \exp\left(\frac{1}{\pi} \int_{-u}^{\infty} \vartheta_2(\mu) \frac{d\mu}{\mu - z}\right). \tag{A-6b}$$

Here

$$\vartheta_\alpha(\mu) = \arg F_\alpha^+(\mu), \tag{A-7}$$

$$F_1(z) = -2uz + (1 - 2uz)(z+u)\,\pi^{-1/2} \int_{-\infty}^{\infty} e^{-x^2} \frac{dx}{x - z} \tag{A-8a}$$

and

$$F_2(z) = \frac{3}{2} + (z+u)\,\pi^{-1/2} \int_{-\infty}^{\infty} e^{-x^2} \frac{dx}{x - z}. \tag{A-8b}$$

We find that $\vartheta_1(\mu) \in [0, 2\pi]$ varies continuously from $\vartheta_1(-u) = 0$ to $\vartheta_1(\infty) = 2\pi$ and that $\vartheta_2(\mu) \in [0, \pi]$ varies continuously from $\vartheta_2(-u) = 0$ to $\vartheta_2(\infty) = 0$. Thus on using continuous values for the arctan function, we write

$$\vartheta_1(\mu) = \tan^{-1}\left\{\frac{\pi^{1/2}(\frac{1}{2} - \mu^2)\, e^{-\mu^2}}{(\frac{1}{2} - \mu^2) f(\mu) - \mu}\right\} \tag{A-9a}$$

and

$$\vartheta_2(\mu) = \tan^{-1}\left\{\frac{\pi^{1/2}(u+\mu)\, e^{-\mu^2}}{(u+\mu) f(\mu) + \frac{3}{2}}\right\} \tag{A-9b}$$

where

$$f(\mu) = \pi^{-1/2}\, P \int_{-\infty}^{\infty} e^{-x^2} \frac{dx}{x - \mu} = -2\, e^{-\mu^2} \int_0^{\mu} e^{x^2}\, dx. \tag{A-10}$$

Equation (A-4) yields the canonical solution [8]

$$\boldsymbol{\Phi}_1(z) = \begin{vmatrix} -3^{-1/2}(z+u)\, U_1(z) & \left(\frac{2}{3}\right)^{1/2} U_2(z) \\ U_1(z) & 0 \end{vmatrix}. \tag{A-11}$$

Since a column that vanishes faster than $1/z$ as $|z| \to \infty$ cannot be obtained from polynomial combinations of the columns of $\boldsymbol{\Phi}_1(z)$ we conclude that here the partial indices are $\varkappa_1 = \varkappa_2 = 1$. Though $\boldsymbol{\Phi}_1(z)$ is a canonical solution it is not in normal form [8], so we let

$$\boldsymbol{\Phi}_2(z) = \boldsymbol{\Phi}_1(z) \begin{vmatrix} 2u(z+\alpha) & 1 \\ 1 & 0 \end{vmatrix} \tag{A-12}$$

where α is a constant to obain

$$\boldsymbol{\Phi}_2(z) = \begin{vmatrix} \left(\frac{2}{3}\right)^{1/2} [U_2(z) - (z+\alpha)(z+u)\, U_1(z)] - 3^{-1/2}(z+u)\, U_1(z) \\ 2u(z+\alpha)\, U_1(z) \qquad U_1(z) \end{vmatrix}. \tag{A-13}$$

Finally the most general canonical solution with $\varkappa_1 = \varkappa_2 = 1$ and in normal form can be written as

$$\boldsymbol{\Phi}(z) = \boldsymbol{\Phi}_2(z)\, \boldsymbol{M} \tag{A-14}$$

where $\boldsymbol{M}$ is a non-singular constant matrix, and thus one of the $\boldsymbol{H}$ matrices [1] we seek

$$\boldsymbol{H}_2^{-1}(z) = \boldsymbol{\Phi}(z)\, \boldsymbol{\Phi}^{-1}(-u) \tag{A-15}$$

is

$$\boldsymbol{H}_2^{-1}(z) = \begin{vmatrix} [U_2(z) - (z+u)^2\, U_1(z)]\, U_2^{-1}(-u) & -3^{-1/2}(z+u)\, U_1(z)\, U_1^{-1}(-u) \\ 3^{1/2}(z+u)\, U_1(z)\, U_2^{-1}(-u) & U_1(z)\, U_1^{-1}(-u) \end{vmatrix}. \tag{A-16}$$

Considering now the second Riemann-Hilbert problem, we let $\boldsymbol{V}(z)$ be diagonal with elements $V_1(z)$ and $V_2(z)$ and substitute

$$\boldsymbol{\Theta}_1(z) = \boldsymbol{Q}^T(z)\, \boldsymbol{V}(z) \tag{A-17}$$

into Eq. (3) to find

$$\boldsymbol{V}^+(\tau) = \boldsymbol{F}^+(\tau)\, [\boldsymbol{F}^-(\tau)]^{-1}\, \boldsymbol{V}^-(\tau)\,, \qquad \tau \in (-\infty, -u]\,, \tag{A-18}$$

which we can readily solve to obtain

$$V_1(z) = (z+u)^{-1} \exp\left(\frac{1}{\pi} \int_{-\infty}^{-u} [\vartheta_1(\tau) + \pi] \frac{d\tau}{\tau - z}\right) \tag{A-19a}$$

and

$$V_2(z) = \exp\left(\frac{1}{\pi} \int_{-\infty}^{-u} \vartheta_2(\tau) \frac{d\tau}{\tau - z}\right) \tag{A-19b}$$

with $\vartheta_1(-\infty) = -\pi$ and $\vartheta_1(-u) = \vartheta_2(-\infty) = \vartheta_2(-u) = 0$. On using Eqs. (A-19) in Eq. (A-17) we obtain a canonical solution. Finding that the solution is not in normal form, we write

$$\boldsymbol{\Theta}_2(z) = \boldsymbol{\Theta}_1(z) \begin{vmatrix} 0 & 1 \\ 1 & a-z \end{vmatrix} \tag{A-20}$$

where a is a constant. If we write

$$V_1(z) \to \frac{1}{z} + \frac{m}{z^2} + \ldots \tag{A-21a}$$

and

$$V_2(z) \to 1 + \frac{l}{z} + \ldots \tag{A-21b}$$

as $|z| \to \infty$ and take $a = l - m$ then

$$\boldsymbol{\Theta}_2(z) = \begin{vmatrix} \left(\frac{2}{3}\right)^{1/2} V_2(z) & \left(\frac{2}{3}\right)^{1/2} [(z^2 - \tfrac{1}{2})\, V_1(z) - (z-a)\, V_2(z)] \\ 0 & V_1(z) \end{vmatrix} \tag{A-22}$$

clearly implies that here the partial indices are $\varkappa_1^* = 0$ and $\varkappa_2^* = 1$. In addition $\boldsymbol{\Theta}_2(z)$ is now in normal form. If we let $\boldsymbol{\Theta}(z)$ denote a general canonical solution in normal form then we must have

$$\boldsymbol{\Theta}(z) = \boldsymbol{\Theta}_2(z) \begin{vmatrix} \alpha & 0 \\ \beta + \gamma z & \delta \end{vmatrix} \tag{A-23}$$

where α, β, γ and δ are constants. If we now substitute Eq. (A-23) into the definition [1]

$$\boldsymbol{H}_1^{-1}(z) = \boldsymbol{\Theta}(z)\, \boldsymbol{\Theta}^{-1}(-u) \tag{A-24}$$

and invoke the condition [1]

$$\boldsymbol{H}_1^{-T}(\infty) \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \mathbf{0} \tag{A-25}$$

we find that $\gamma \equiv 0$, and thus we obtain the final result

$$\boldsymbol{H}_1^{-1}(z) = \begin{vmatrix} V_2(z)\, V_2^{-1}(-u) & \left(\frac{2}{3}\right)^{1/2} [(z^2 - \frac{1}{2})\, V_1(z) - (z+u)\, V_2(z)]\, V_1^{-1}(-u) \\ 0 & V_1(z)\, V_1^{-1}(-u) \end{vmatrix}. \tag{A-26}$$

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Abstract

Two matrix Riemann-Hilbert problems derived from boundary-value problems in the kinetic theory of gases are solved analytically.

Riassunto

Vengono risolti analiticamente due problemi relativi alla matrice di Riemann-Hilbert, che si incontrano in problemi con condizioni al contorno nella teoria cinetica dei gas.

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