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A NEW APPROACH TO CHANDRASEKHAR'S SCATTERING MATRIX FOR A SEMI-INFINITE RAYLEIGH-SCATTERING ATMOSPHERE

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ABSTRACT

A set of *switched* scalar products between the half-range adjoint solutions and the eigenfunctions of the vector equation of transfer for two components of a Rayleigh-scattered radiation field is given. These scalar products are then used to develop Chandrasekhar's scattering matrix, which can be used to determine the emergent angular distribution for any of the standard half-space problems. As an example, the exit distribution for the Milne problem is obtained.

I. INTRODUCTION

In his classic book on radiative transfer, Chandrasekhar (1950) gives an exact solution for the intensity vector, $I(0,-\mu)$, where $\mu \in (0,1)$, reflected at the surface of a semi-infinite Rayleigh-scattering atmosphere. The solution given by Chandrasekhar (1950) is expressed in terms of a scattering matrix S. The form of this scattering matrix was deduced by analogy with similar scalar problems; the correctness of the form and the determination of the necessary constants were then established by substitution into the integral equation of which S is a solution (Chandrasekhar 1950).

In a recent paper (Siewert and Fraley 1967), Case's method of normal modes (Case and Zweifel 1967) has been used to construct the solution to the Milne problem for a Rayleigh-scattering atmosphere. The particular merit of the approach used by Siewert and Fraley (1967) is that it gives the exact solution for the two components of the radiation field at *any* optical depth. In their work, Siewert and Fraley (1967) obtained the exit angular distribution as a special case of their general solution; however, their free-surface result could have been obtained more readily if use had been made of Chandrasekhar's scattering matrix.

The purpose of this paper is twofold: first, to show how Chandrasekhar's scattering matrix can be derived from the adjoint solutions and eigenvectors introduced by Siewert and Fraley (1967) and, second, to demonstrate the use of the scattering matrix to determine the exit distribution in the Milne problem.

II. THE BASIC FORMALISM

The equation of transfer appropriate for the two components of the radiation field in a semi-infinite Rayleigh-scattering atmosphere is discussed in detail by Chandrasekhar (1950). It can be written as

$$\mu \frac{\partial}{\partial \tau} I(\tau,\mu) + I(\tau,\mu) = \tfrac{3}{8}\int_{-1}^{1} K(\mu,\mu') I(\tau,\mu') d\mu' , \tag{1}$$

where

$$K(\mu,\mu') = \begin{vmatrix} 2(1-\mu^2)(1-\mu'^2)+\mu^2\mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{vmatrix} . \tag{2}$$

Here $I(\tau,\mu)$ is a vector whose two components, $I_l(\tau,\mu)$ and $I_r(\tau,\mu)$, represent the radiation intensities of the two states of polarization. In addition, τ is the optical variable, and μ denotes the direction cosine of the propagation vector as measured from the *inward* normal to the free surface. (To be in agreement with Chandrasekhar [1950], it is necessary to change μ to $-\mu$.)

Siewert and Fraley (1967) found a general solution to equation (1) that could be written (without the constraints of whatever boundary conditions might be appropriate) as

$$I(\tau,\mu) = A_+\Phi_+ + A_-\Phi_-(\tau,\mu) + \int_{-1}^{1}\alpha(\eta)e^{-\tau/\eta}\Phi_1(\eta,\mu)d\eta + \int_{-1}^{1}\beta(\eta)e^{-\tau/\eta}\Phi_2(\eta,\mu)d\eta\,, \quad (3)$$

where

$$\Phi_+ = \begin{vmatrix}1\\1\end{vmatrix}\,, \quad \Phi - (\tau,\mu) = (\tau-\mu)\begin{vmatrix}1\\1\end{vmatrix}\,,$$

$$\Phi_1(\eta,\mu) = \begin{vmatrix}\frac{3\eta}{2}(1-\mu^2)\frac{P}{\eta-\mu} + \lambda_1(\eta)\delta(\eta-\mu)\\ 0\end{vmatrix}\,, \quad (4)$$

$$\Phi_2(\eta,\mu) = \begin{vmatrix}-\frac{3\eta}{2}(\eta+\mu)\\ \frac{3\eta}{2}(1-\eta^2)\frac{P}{\eta-\mu} + \lambda_2(\eta)\delta(\eta-\mu)\end{vmatrix}\,.$$

Also,

$$\lambda_i(\eta) = (-1)^i + 3(1-\eta^2)(1-\eta\tanh^{-1}\eta)\,. \quad (5)$$

The quantities A_+, A_-, $\alpha(\eta)$, and $\beta(\eta)$ are the arbitrary expansion coefficients and can be obtained once the boundary conditions are specified. In addition, the symbol P is used to indicate that integrals involving these functions are to be evaluated in the Cauchy principal-value sense; the Dirac delta function is denoted by $\delta(x)$.

If we impose the condition

$$\lim_{\tau\to\infty} e^{-\tau}I(\tau,\mu) = 0\,, \quad (6a)$$

then, in the solution given by equation (3), we must take $\alpha(\eta) = \beta(\eta) = 0$ for $\eta < 0$. The distribution incident on the free surface of the half-space is denoted by $I_{\text{inc}}(\mu)$, i.e.,

$$I(0,\mu) = I_{\text{inc}}(\mu)\,, \quad \mu \in (0,1)\,. \quad (6b)$$

Applying these two boundary conditions to equation (3), we find

$$I(\mu) = A_+\Phi_+ + \int_0^1\alpha(\eta)\Phi_1(\eta,\mu)d\eta + \int_0^1\beta(\eta)\Phi_2(\eta,\mu)d\eta\,, \quad \mu\in(0,1)\,, \quad (7)$$

where

$$I(\mu) \triangleq I_{\text{inc}}(\mu) - A_-\Phi_-(0,\mu)\,, \quad \mu\in(0,1)\,. \quad (8)$$

For the albedo problem, the solution must vanish as τ tends to infinity (in addition to satisfying eq. [6a]). Thus,

$$A^a_- = 0 \quad \text{and} \quad I^a{}_{\text{inc}}(\mu) = \tfrac{1}{2}\begin{vmatrix}F_l\\F_r\end{vmatrix}\delta(\mu-\mu_0)\,, \quad \mu,\mu_0\in(0,1)\,. \quad (9)$$

In the Milne problem,

$$A^m{}_- = \tfrac{3}{8}F \quad \text{and} \quad I^m{}_{\text{inc}}(\mu) = 0\,, \quad \mu\in(0,1)\,. \quad (10)$$

The quantities F_l, F_r, and F are the normalization constants of the respective problems.

The half-range completeness theorem given by Siewert and Fraley (1967) states that equation (7) is a valid expansion for an arbitrary $I(\mu)$. The adjoint vectors (Siewert and Fraley 1967),

$$\Phi_+^\dagger(\mu) = \frac{\sqrt{5}}{2q}\mu \begin{vmatrix} qH_l(\mu) \\ (c+\mu)H_r(\mu) \end{vmatrix} , \tag{11a}$$

$$\Phi_1^\dagger(\eta,\mu) = \frac{\sqrt{5}}{2}\mu \begin{vmatrix} H_l(\mu)\left[\frac{3\eta}{2}(1-\mu^2)\frac{P}{\eta-\mu} + \lambda_1(\eta)\delta(\eta-\mu) + \frac{3\eta}{2}(c-\mu)\right] \\ -\frac{3\eta}{4}qH_r(\mu) \end{vmatrix} , \tag{11b}$$

and

$$\Phi_2^\dagger(\eta,\mu) = \frac{\sqrt{2}}{4}\mu \begin{vmatrix} -\frac{3\eta}{2}qH_l(\mu) \\ H_r(\mu)\left[\frac{3\eta}{2}(1-\mu^2)\frac{P}{\eta-\mu} + \lambda_2(\eta)\delta(\eta-\mu) - \frac{3\eta}{2}(c+\mu)\right] \end{vmatrix} , \tag{11c}$$

were so constructed that

$$\langle i|j\rangle \triangleq \int_0^1 \tilde{\Phi}_i^\dagger(\eta',\mu)\Phi_j(\eta,\mu)d\mu = 0 , \quad i \neq j = + , \quad 1 , \quad \text{or} \quad 2 , \quad \eta,\eta' > 0 , \tag{12a}$$

$$\langle + | + \rangle = \frac{2\sqrt{10}}{3q} , \tag{12b}$$

$$\langle 1 | 1 \rangle = \frac{\sqrt{5}}{2}\eta N_1(\eta)H_l(\eta)\delta(\eta-\eta') , \tag{12c}$$

and

$$\langle 2 | 2 \rangle = \frac{\sqrt{2}}{4}\eta N_2(\eta)H_r(\eta)\delta(\eta-\eta') . \tag{12d}$$

The superscript tilde is used here to denote the transpose operation, $H_l(\eta)$ and $H_r(\eta)$ are Chandrasekhar's two H-functions for this problem, and

$$N_i(\eta) = \lambda_i^2(\eta) + \tfrac{9}{4}\pi^2\eta^2(1-\eta^2)^2 . \tag{13}$$

Further, the constants q and c were calculated by Bond and Siewert (1967) to be $q = 0.689891054$ and $c = 0.872940529$.

The expansion coefficients A_+, $\alpha(\eta)$, and $\beta(\eta)$ can thus be obtained at once by taking scalar products of equation (7) with the appropriate adjoint vectors, i.e.,

$$A_+ = \frac{3\sqrt{10}}{20}q\int_0^1 \tilde{\Phi}_+^\dagger(\mu)I(\mu)d\mu ,$$

$$\alpha(\eta) = \frac{2\sqrt{5}}{5}\int_0^1 \tilde{\Phi}_1^\dagger(\eta,\mu)I(\mu)d\mu/\eta H_l(\eta)N_1(\eta) , \tag{14}$$

and

$$\beta(\eta) = 2\sqrt{2}\int_0^1 \tilde{\Phi}_2^\dagger(\eta,\mu)I(\mu)d\mu/\eta H_r(\eta)N_2(\eta) .$$

III. THE SCATTERING MATRIX

In the previous section, it was indicated that the expansion coefficients in equation (7) could be obtained by taking scalar products. If, however, we are interested only in the surface quantity, $I(0,-\mu)$ for $\mu \in (0,1)$, the result can be obtained without the necessity of determining A_+, $\alpha(\eta)$, or $\beta(\eta)$. In order to illustrate this, a procedure used by McCormick and Kuščer (1965) for a scalar transfer equation is followed. Consider the following set of six *switched* scalar products:

$$(i,j) \triangleq \int_0^1 \tilde{\Phi}_i^{\dagger}(-\mu,\mu')\Phi_j(\eta,\mu')d\mu' \,, \quad i = 1 \text{ or } 2 \,, \quad j = + \,, \ 1 \,, \text{ or } 2 \,, \quad \eta,\mu > 0 \,. \tag{15}$$

With the above restrictions on the variables, the two continuum adjoint vectors are no longer singular and can be written in the more tractable forms

$$\Phi_1^{\dagger}(-\mu,\mu') = \frac{3\sqrt{5}}{8}\mu\mu' \begin{vmatrix} 2H_l(\mu')\left[\dfrac{(1-\mu'^2)}{\mu+\mu'} + \mu' - c\right] \\ qH_r(\mu') \end{vmatrix} \,, \quad \mu,\mu' \in (0,1) \,, \tag{16a}$$

and

$$\Phi_2^{\dagger}(-\mu,\mu') = \frac{3\sqrt{2}}{8}\mu\mu' \begin{vmatrix} qH_l(\mu') \\ H_r(\mu')\left[\dfrac{(1-\mu'^2)}{\mu+\mu'} + \mu' + c\right] \end{vmatrix} \,, \quad \mu,\mu' \in (0,1) \,. \tag{16b}$$

The six integrals defined by equation (15) can be evaluated explicitly; however, since the calculations are tedious, only the results are given here. These results can be summarized as

$$(1,j) = \frac{\sqrt{5}}{H_l(\mu)}\mu\phi_{j1}(\eta,-\mu) \,, \qquad j = +, 1, \text{ or } 2 \,, \tag{17a}$$

and

$$(2,j) = \frac{\sqrt{2}}{H_r(\mu)}\mu\phi_{j2}(\eta,-\mu) \,, \qquad j = +, \ 1 \,, \text{ or } 2 \,, \tag{17b}$$

where $\phi_{ji}(\eta,-\mu)$ denotes the ith component of $\Phi_j(\eta,-\mu)$.

If we change μ to μ' in equation (7), multiply by $\tilde{\Phi}_i^{\dagger}(-\mu,\mu')$ where $i = 1$ or 2, and integrate over μ' from 0 to 1, we find

$$\int_0^1 \tilde{\Phi}_1^{\dagger}(-\mu,\mu')I(\mu')d\mu' = \frac{\sqrt{5}}{H_l(\mu)}\mu\Big[A_+\phi_{+1} + \int_0^1 \alpha(\eta)\phi_{11}(\eta,-\mu)d\eta + \int_0^1 \beta(\eta)\phi_{21}(\eta,-\mu)d\eta\Big] \,, \quad \mu \in (0,1) \,, \tag{18a}$$

and

$$\int_0^1 \tilde{\Phi}_2^{\dagger}(-\mu,\mu')I(\mu')d\mu' = \frac{\sqrt{2}}{H_r(\mu)}\mu\Big[A_+\phi_{+2} + \int_0^1 \alpha(\eta)\phi_{12}(\eta,-\mu)d\eta + \int_0^1 \beta(\eta)\phi_{22}(\eta,-\mu)d\eta\Big] \,, \quad \mu \in (0,1) \,. \tag{18b}$$

Equations (18) can be written in a more manageable form by constructing a 2×2 S matrix from the two row vectors:

$$\frac{\sqrt{5}}{5\mu} H_l(\mu)\tilde{\Phi}_1^{\dagger}(-\mu,\mu') \quad \text{and} \quad \frac{\sqrt{2}}{2\mu} H_r(\mu)\tilde{\Phi}_2^{\dagger}(-\mu,\mu') .$$

Thus, equations (18) become

$$\int_0^1 S(-\mu,\mu')I(\mu')d\mu' = A_+\Phi_+ + \int_0^1 \alpha(\eta)\Phi_1(\eta,-\mu)d\eta + \int_0^1 \beta(\eta)\Phi_2(\eta,-\mu)d\eta , \quad \mu \in (0,1) , \tag{19}$$

where

$$S(-\mu,\mu') = \tfrac{3}{8}\mu' \begin{vmatrix} 2H_l(\mu)H_l(\mu')\left[\dfrac{(1-\mu'^2)}{\mu+\mu'}+\mu'-c\right] & qH_l(\mu)H_r(\mu') \\ qH_r(\mu)H_l(\mu') & H_r(\mu)H_r(\mu')\left[\dfrac{(1-\mu'^2)}{\mu+\mu'}+\mu'+c\right] \end{vmatrix} . \tag{20}$$

We note that

$$S(-\mu,\mu') = \frac{3}{8\mu} S^{(0)}(\mu,\mu') ,$$

where $S^{(0)}(\mu,\mu')$ is the scattering matrix given by Chandrasekhar (1950).

Noting that the right-hand side of equation (19) is the analytic continuation to negative μ of the right-hand side of equation (7), we obtain the desired result, namely, if

$$I(\mu) = A_+\Phi_+ + \int_0^1 \alpha(\eta)\Phi_1(\eta,\mu)d\eta + \int_0^1 \beta(\eta)\Phi_2(\eta,\mu)d\eta , \quad \mu \in (0,1) , \tag{21a}$$

then

$$I(-\mu) = \int_0^1 S(-\mu,\mu')I(\mu')d\mu' , \quad \mu \in (0,1) , \tag{21b}$$

or, alternatively,

$$I(-\mu) = \frac{3}{8\mu}\int_0^1 S^{(0)}(\mu,\mu')I(\mu')d\mu' , \quad \mu \in (0,1) . \tag{21c}$$

IV. THE MILNE PROBLEM

As has been discussed (Siewert and Fraley 1967), the solution to the Milne problem can be written as

$$I_m(\tau,\mu) = A_+\Phi_+ + A_-\Phi_-(\tau,\mu) + \int_0^1 \alpha(\eta)e^{-\tau/\eta}\Phi_1(\eta,\mu)d\eta + \int_0^1 \beta(\eta)e^{-\tau/\eta}\Phi_2(\eta,\mu)d\eta . \tag{22}$$

The expansion coefficients A_+, $\alpha(\eta)$, and $\beta(\eta)$ are determined from the expansion given by equation (21a), where

$$I(\mu) = -A_-\Phi_-(0,\mu) = \tfrac{3}{8}\mu F\Phi_+ . \tag{23}$$

Equating τ to zero in equation (22) and noting equations (21a) and (23), we find the emergent distribution to be

$$I_m(0,-\mu) = I(-\mu) + \tfrac{3}{8}\mu F\Phi_+ , \quad \mu \in (0,1) . \tag{24}$$

The quantity $I(-\mu)$ can be obtained immediately from equation (21b), i.e.,

$$I(-\mu) = \tfrac{3}{8}F\int_0^1 S(-\mu,\mu')\Phi_+\mu' d\mu' , \quad \mu \in (0,1) . \tag{25}$$

The integral above is easily evaluated; the result,

$$I(-\mu) = \tfrac{3}{8}F\begin{vmatrix} \frac{\sqrt{2}}{2}\, qH_l(\mu) - \mu \\ \frac{\sqrt{2}}{2}\,(c+\mu)H_r(\mu) - \mu \end{vmatrix} , \tag{26}$$

is substituted into equation (24) to give

$$I_m(0,-\mu) = \frac{3\sqrt{2}}{16}F\begin{vmatrix} qH_l(\mu) \\ (c+\mu)H_r(\mu) \end{vmatrix} , \quad \mu \in (0,1) . \tag{27}$$

This expression is in complete agreement with the solution given by Chandrasekhar (1950).

It has been shown that the scattering matrix developed by Chandrasekhar (1950) for the solution to the albedo problem can be derived from the results of Siewert and Fraley (1967). Further, a particular merit of the S matrix has been illustrated by using it to calculate the exit distribution in the Milne problem. This property of the scattering matrix follows from the fact that the expansion vector $I(\mu)$ for the albedo problem is the angular Green's function for expansions of the type given by equation (7).

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