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ON THE INTEGRAL FORM OF THE EQUATION OF TRANSFER FOR A HOMOGENEOUS SPHERE

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Abstract—Basic considerations are used to derive the integral form of the equation of transfer relevant to a homogeneous sphere with internal sources as well as external illumination and a specularly and diffusely reflecting surface.

1. INTRODUCTION

The equation of transfer basic to plane-parallel media can be readily converted from the integro-differential form for the specific intensity of radiation to the integral form for the energy density even for the general case where radiation is incident on the two surfaces and when the boundaries reflect both specularly and diffusely. The resulting integral form of the equation of transfer has been used to advantage, for example, by workers who use projection techniques to develop solutions[1] or basic analysis to prove existence theorems.[2] However for the case of spheres this transformation has not, to our knowledge, been reported for the general case we consider. This transformation from the integro-differential form to the integral form has, for spheres, an additional feature in that, for some problems,[3–5] the integral formulation for the given spherical problem can be actually solved in terms of a pseudo-slab problem. These potential uses provide, we believe, good reasons for the following brief development.

2. ANALYSIS

We consider, for $r \in (0, R]$ and $\mu \in [-1, 1]$, the equation of transfer

$$\left\{\mu \frac{\partial}{\partial r} + \frac{1}{r}(1-\mu^2)\frac{\partial}{\partial \mu} + 1\right\} I(r, \mu) = \frac{\omega}{2}\int_{-1}^{1} I(r, \mu')\,\mathrm{d}\mu' + Q(r) \tag{1}$$

and the boundary condition, for $\mu \in [0, 1]$,

$$I(R, -\mu) = K(\mu) + \alpha I(R, \mu) + \beta\chi(\mu)\int_0^1 I(R, \mu')\mu'\,\mathrm{d}\mu'. \tag{2}$$

Here $I(r, \mu)$ is the specific intensity of radiation, r is the optical variable, R is the radius (in optical units) of the sphere, μ is the direction cosine (measured with respect to the radial variable r) of the propagating radiation, and $Q(r)$ represents an inhomogeneous source of radiation. In addition, α and β are coefficients for specular and diffuse reflection, $K(\mu)$ describes the radiation incident on the surface, and the redistribution function $\chi(\mu)$ is normalized so that

$$\int_0^1 \chi(\mu)\mu\,\mathrm{d}\mu = 1. \tag{3}$$

We can use the idea of integrating back along the propagation ray, discussed for example by Case, de Hoffmann and Placzek,[6] and Fig. 1 to write

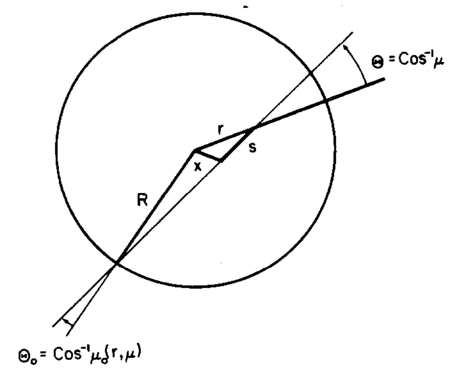

Fig. 1. The geometry used.

$$I(r,\mu)=I[R,-\mu_0(r,\mu)]\,\mathrm{e}^{-S_0(r,\mu)}+\int_0^{S_0(r,\mu)}\left[\frac{\omega}{2}I(\surd(r^2+s^2-2rs\mu))+Q(\surd(r^2+s^2-2rs\mu))\right]\mathrm{e}^{-s}\,\mathrm{d}s, \tag{4}$$

where

$$\mu_0(r,\mu)=\cos\theta_0=[1-(r/R)^2(1-\mu^2)]^{1/2}, \tag{5}$$

$$S_0(r,\mu)=r\mu+[R^2-r^2(1-\mu^2)]^{1/2}, \tag{6}$$

and, aside from a factor of the speed of light, the energy density of the radiation field is

$$I(r)=\int_{-1}^{1}I(r,\mu)\,\mathrm{d}\mu. \tag{7}$$

Changing the integration variable to x, as shown in Fig. 1, we rewrite Eq. (4) as

$$I(r,\mu)=I[R,-\mu_0(r,\mu)]\,\mathrm{e}^{-S_0(r,\mu)}+(LI)(r,\mu),\ \mu\geq 0, \tag{8a}$$

and

$$I(r,-\mu)=I[R,-\mu_0(r,\mu)]\,\mathrm{e}^{-S_0(r,-\mu)}+(GI)(r,\mu),\ \mu\geq 0, \tag{8b}$$

where

$$(LI)(r,\mu)=\int_{r\surd(1-\mu^2)}^{r}\left[\frac{\omega}{2}I(x)+Q(x)\right]\pi^{-1}(x,r,\mu)\exp[-r\mu+\pi(x,r,\mu)]x\,\mathrm{d}x$$
$$+\int_{r\surd(1-\mu^2)}^{R}\left[\frac{\omega}{2}I(x)+Q(x)\right]\pi^{-1}(x,r,\mu)\exp[-r\mu-\pi(x,r,\mu)]x\,\mathrm{d}x \tag{9a}$$

and

$$(GI)(r,\mu)=\int_r^R\left[\frac{\omega}{2}I(x)+Q(x)\right]\pi^{-1}(x,r,\mu)\exp[r\mu-\pi(x,r,\mu)]x\,\mathrm{d}x. \tag{9b}$$

Here

$$\pi(x,r,\mu)=[x^2-r^2(1-\mu^2)]^{1/2}. \tag{10}$$

If we now integrate Eqs. (8) over μ and add the two resulting equations, we find

$$rI(r) = \int_0^R x\left[\frac{\omega}{2} I(x) + Q(x)\right][E_1(|r-x|) - E_1(r+x)]\,dx + 2r\int_0^1 I[R, -\mu_0(r,\mu)]\cosh(r\mu)\,e^{-\tau(R,r,\mu)}\,d\mu. \quad (11)$$

In order to reduce Eq. (11) to an integral equation for $I(r)$ we must express $I[R, -\mu_0(r, \mu)]$ in terms of $I(r)$. Setting $r = R$ in Eq. (8a) and entering the resulting equation into Eq. (2) we find

$$I(R, -\mu) = T(\mu)[K(\mu) + \alpha(LI)(R, \mu) + \beta\chi(\mu)J], \quad (12)$$

where

$$J = \int_0^1 I(R, \mu)\mu\,d\mu \quad (13)$$

and

$$T(\mu) = [1 - \alpha\,e^{-2R\mu}]^{-1}. \quad (14)$$

We can now multiply Eq. (2) by μ and integrate to find

$$(\alpha+\beta)J = \int_0^1 [I(R, -\mu) - K(\mu)]\mu\,d\mu; \quad (15)$$

thus on multiplying Eq. (12) by μ, integrating and using Eq. (15), we deduce that

$$(1 - \beta\chi_*)J = K_* + \int_0^1 T(\mu)(LI)(R, \mu)\mu\,d\mu, \quad (16)$$

where

$$\chi_* = \int_0^1 T(\mu)\chi(\mu)\,e^{-2R\mu}\mu\,d\mu \quad (17)$$

and

$$K_* = \int_0^1 T(\mu)K(\mu)\,e^{-2R\mu}\mu\,d\mu. \quad (18)$$

We can now substitute Eq. (9a) evaluated at $r = R$ into Eq. (16) to find

$$J = (1 - \beta\chi_*)^{-1}\left\{K_* + \int_0^R x^2\left[\frac{\omega}{2} I(x) + Q(x)\right]L(x)\,dx\right\}, \quad (19)$$

where

$$L(x) = \frac{2}{R^2}\int_0^1 T[\mu_0(x, t)]\cosh(xt)\,e^{-\tau(R,x,t)}\,dt. \quad (20)$$

We can now change μ to $\mu_0(r, \mu)$ in Eq. (12) and substitute the resulting equation into Eq. (11) to find our final result, viz.

$$rI(r) = rG(r) + \int_0^R x\left[\frac{\omega}{2} I(x) + Q(x)\right][E_1(|r-x|) - E_1(r+x) + \alpha F_1(r, x) + \beta F_2(r, x)]\,dx, \quad (21)$$

where the known term is

$$G(r) = 2\int_0^1 T[\mu_0(r,\mu)]\{K[\mu_0(r,\mu)] + \beta(1-\beta\chi_*)^{-1}K_*\chi[\mu_0(r,\mu)]\}\cosh(r\mu)\,e^{-\pi(R,r,\mu)}\,d\mu. \tag{22}$$

In addition to the exponential integral functions appearing in Eq. (21), we have

$$F_1(r,x) = 4x\int_0^1 T[\mu_0(x,\mu)]\pi^{-1}(r,x,\mu)\cosh(x\mu)\cosh[\pi(r,x,\mu)]\,e^{-2R\mu_0(x,\mu)}\,d\mu,\ x \le r, \tag{23}$$

$$F_1(r,x) = 4r\int_0^1 T[\mu_0(r,\mu)]\pi^{-1}(x,r,\mu)\cosh(r\mu)\cosh[\pi(x,r,\mu)]\,e^{-2R\mu_0(r,\mu)}\,d\mu,\ x \ge r, \tag{24}$$

and

$$F_2(r,x) = 2(1-\beta\chi_*)^{-1}rxL(x)\int_0^1 T[\mu_0(r,\mu)]\chi[\mu_0(r,\mu)]\cosh(r\mu)\,e^{-\pi(R,r,\mu)}\,d\mu. \tag{25}$$

It is clear that Eq. (21) is the desired integral equation for the energy density in a homogeneous sphere with an isotropic internal source distribution for the general case of external illumination with partial specular and diffuse reflection at the surface. To conclude we note that the method of characteristics has been used to provide an independent verification of Eq. (21).

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