Journal of Applied Mathematics and Physics (ZAMP)
Vol. 33, November 1982

0044-2275/82/006801-06 $ 2.70/0

On the dispersion function in particle transport theory

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I. Introduction

We consider the particle transport equation written as

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu, \varphi) + I(\tau, \mu, \varphi) = \frac{c}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\cos \Theta)\, I(\tau, \mu', \varphi')\, \mathrm{d}\varphi'\, \mathrm{d}\mu', \tag{1}$$

where $c > 0$ is the mean number of secondary particles per collision, μ is the direction cosine of the propagating radiation and φ is the azimuthal angle. In addition Θ is the scattering angle, and we consider scattering laws that can be adequately represented by a finite Legendre expansion, i.e.

$$p(\cos \Theta) = \sum_{l=0}^{L} (2l+1) f_l P_l(\cos \Theta), \quad f_0 = 1. \tag{2}$$

Using the addition theorem to write Eq. (2) as

$$p(\cos \Theta) = \sum_{l=0}^{L} \sum_{m=0}^{l} (2 - \delta_{0,m})\, \beta_l^m\, P_l^m(\mu)\, P_l^m(\mu') \cos m(\varphi - \varphi'), \tag{3}$$

where

$$\beta_l^m = \frac{(l-m)!}{(l+m)!} (2l+1) f_l \tag{4}$$

and

$$P_l^m(\mu) = (1-\mu^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_l(\mu), \tag{5}$$

we find [1, 2] that the components in a Fourier representation of $I(\tau, \mu, \varphi)$ must satisfy for $0 \leqq m \leqq L$, the equations

$$\mu \frac{\partial}{\partial \tau} I^m(\tau, \mu) + I^m(\tau, \mu) = \frac{c}{2} \sum_{l=m}^{L} \beta_l^m\, P_l^m(\mu) \int_{-1}^{1} P_l^m(\mu')\, I^m(\tau, \mu')\, \mathrm{d}\mu'. \tag{6}$$

II. The dispersion function

If we now follow the work of McCormick and Kuščer [3] and substitute

$$I^m(\xi : \tau, \mu) = \varphi^m(\xi, \mu)\, \mathrm{e}^{-\tau/\xi} \tag{7}$$

into Eq. (6) we find

$$(\xi-\mu)\,\varphi^m(\xi,\mu)=\frac{c}{2}\,\xi\sum_{l=m}^{L}\beta_l^m\,P_l^m(\mu)\,g_l^m(\xi)\,, \tag{8}$$

where

$$g_l^m(\xi)=\int_{-1}^{1}P_l^m(\mu)\,\varphi^m(\xi,\mu)\,\mathrm{d}\mu\,. \tag{9}$$

Considering that $\xi\notin[-1,1]$, we write Eq. (8) as

$$\varphi^m(\xi,\mu)=\frac{c}{2}\,\xi\left(\frac{1}{\xi-\mu}\right)\sum_{l=m}^{L}\beta_l^m\,P_l^m(\mu)\,g_l^m(\xi)\,. \tag{10}$$

We normalize the solutions $\varphi^m(\xi,\mu)$ such that

$$g_m^m(\xi)=(2m-1)!!\,. \tag{11}$$

Thus on multiplying Eq. (10) by $P_m^m(\mu)$ and integrating we find

$$g_m^m(\xi)=\frac{c}{2}\,\xi\sum_{l=m}^{L}\beta_l^m\,g_l^m(\xi)\int_{-1}^{1}P_m^m(\mu)\,P_l^m(\mu)\frac{\mathrm{d}\mu}{\xi-\mu} \tag{12}$$

or

$$\Lambda^m(\xi)\,(2m-1)!!=0 \tag{13}$$

where

$$\Lambda^m(z)=1+\frac{c}{2}\,z\sum_{l=m}^{L}\beta_l^m\,g_l^m(z)\int_{-1}^{1}(1-\mu^2)^{m/2}\,P_l^m(\mu)\frac{\mathrm{d}\mu}{\mu-z}\,. \tag{14}$$

It is apparent that the zeros $\nu_\alpha\notin[-1,1]$ of the dispersion function $\Lambda^m(z)$ lead, by way of Eqs. (7) and (10), to solutions of Eq. (6).

From Chandrasekhar's work [1] it is clear that the zeros $\nu_\alpha\notin[-1,1]$ of $\Lambda^m(z)$ occur in $\pm$ pairs, and we note that Shultis and Hill [4] have argued in the manner of Case [5] that the zeros are real and simple for $1-cf_l>0$, $l=0,1,\ldots,L$. For the special case $m=0$, Case [5] and Hangelbroek [6] have argued that the limiting values $\Lambda^0(\nu\pm io)$ of $\Lambda^0(z)$ cannot vanish for $\nu\in(-1,1)$ and Lekkerkerker [7] has extended that result to include the endpoints ±1. Here we wish to show for the general case that the limiting values $\Lambda^m(\nu\pm io)$ of $\Lambda^m(z)$ cannot vanish for $\nu\in(-1,1)$ and further that $\Lambda^m(z)$ and

$$\psi^m(z)=\frac{c}{2}\sum_{l=m}^{L}\beta_l^m\,g_l^m(z)\,(1-z^2)^{m/2}\,P_l^m(z) \tag{15}$$

cannot have a common zero for any $z\notin[-1,1]$.

We define

$$Q_l^m(z)=\frac{1}{2}\int_{-1}^{1}(1-\mu^2)^{m/2}\,P_l^m(\mu)\frac{\mathrm{d}\mu}{z-\mu}\,, \tag{16}$$

and rewrite Eq. (14) as

$$\Lambda^m(z) = 1 - cz \sum_{l=m}^{L} \beta_l^m \, g_l^m(z) \, Q_l^m(z) \,. \tag{17}$$

Now since the associated Legendre functions $P_l^m(z)$, with

$$P_m^m(z) = (1 - z^2)^{m/2} (2m - 1)!! \,, \tag{18}$$

satisfy, for $l \geqq m$, the recursion formula

$$(l - m + 1) P_{l+1}^m(z) = (2l + 1) z \, P_l^m(z) - (l + m)(1 - \delta_{m,l}) \, P_{l-1}^m(z) \tag{19}$$

we can readily deduce, for $l \geqq m$, that

$$(l - m + 1) \, Q_{l+1}^m(z)$$
$$= (2l + 1) z \, Q_l^m(z) - (l + m)(1 - \delta_{m,l}) \, Q_{l-1}^m(z) - 2^m m! \, \delta_{m,l} \,. \tag{20}$$

In addition the polynomials $g_l^m(z)$ satisfy [1], for $l \geqq m$,

$$(l - m + 1) \, g_{l+1}^m(z) = h_l \, z \, g_l^m(z) - (l + m)(1 - \delta_{m,l}) \, g_{l-1}^m(z) \,, \tag{21}$$

where

$$h_l = (2l + 1)(1 - c f_l) \,. \tag{22}$$

We can multiply Eq. (21) by $(l - m)! \, Q_l^m(z)/(l + m)!$, multiply Eq. (20) by $(l - m)! \, g_l^m(z)/(l + m)!$, subtract the two equations one from the other and sum the resulting equation from $l = m$ to $l = L$ to find

$$\Lambda^m(z) = \frac{(L - m + 1)!}{(L + m)!} \{ Q_L^m(z) \, g_{L+1}^m(z) - Q_{L+1}^m(z) \, g_L^m(z) \} \,. \tag{23}$$

Similarly we can multiply Eq. (21) by $(l - m)! \, (1 - z^2)^{m/2} P_l^m(z)/(l + m)!$, multiply Eq. (19) by $(l - m)! \, (1 - z^2)^{m/2} g_l^m(z)/(l + m)!$, subtract and sum the resulting equation from $l = m$ to $l = L$ to find, after using Eq. (15),

$$z \, \psi^m(z) = \frac{1}{2} (1 - z^2)^{m/2} \frac{(L - m + 1)!}{(L + m)!} \{ P_{L+1}^m(z) \, g_L^m(z) - P_L^m(z) \, g_{L+1}^m(z) \} . \tag{24}$$

In a similar way we can find from Eqs. (19) and (20) that

$$(1 - z^2)^{m/2} = \frac{(L - m + 1)!}{(L + m)!} \{ P_{L+1}^m(z) \, Q_L^m(z) - P_L^m(z) \, Q_{L+1}^m(z) \} \,. \tag{25}$$

Finally we can multiply Eq. (23) by $(1 - z^2)^{m/2} P_{L+1}^m(z)$ and use Eqs. (24) and (25) to find

$$(1 - z^2)^{m/2} P_{L+1}^m(z) \, \Lambda^m(z) = (1 - z^2)^m \, g_{L+1}^m(z) - 2z \, \psi^m(z) \, Q_{L+1}^m(z) \,. \tag{26}$$

At this point we would like to generalize our previously given proof [8] and show that $\Lambda^m(z)$ and $\psi^m(z)$ cannot have a common zero for $z \notin [-1, 1]$.

By contradiction, if we suppose that there exists $z_0 \notin [-1, 1]$ such that $\Lambda^m(z_0) = \psi^m(z_0) = 0$ we must conclude from Eq. (26) that $g^m_{L+1}(z_0) = 0$. Since $P^m_{L+1}(z_0) \neq 0$ as shown by Robin [9], we see that Eq. (24) would require $g^m_L(z_0) = 0$ which is not possible because Eq. (21) would yield $g^m_l(z_0) = 0$, for all $l \geqq m$, and clearly $g^m_m(z_0) \neq 0$. We must then conclude that there is no such z_0.

As the dispersion function can also be written as

$$\Lambda^m(z) = 1 + z \int_{-1}^{1} \psi^m(\mu) \frac{d\mu}{\mu - z} \tag{27}$$

we can use the Plemelj formulas [10] to express the limiting values of $\Lambda^m(z)$ as z approaches the branch cut from above (+) and below (−) as

$$[\Lambda^m(\tau)]^{\pm} = \lambda^m(\tau) \pm \pi i \tau \psi^m(\tau), \qquad \tau \in (-1, 1), \tag{28}$$

where

$$\lambda^m(\tau) = 1 + \tau P \int_{-1}^{1} \psi^m(\mu) \frac{d\mu}{\mu - \tau}. \tag{29}$$

We can also use the Plemelj formulas with Eqs. (23), (25) and (26) to obtain

$$\lambda^m(\tau) = \frac{(L - m + 1)!}{(L + m)!} \{q^m_L(\tau)\, g^m_{L+1}(\tau) - q^m_{L+1}(\tau)\, g^m_L(\tau)\}, \tag{30}$$

$$(1 - \tau^2)^{m/2} = \frac{(L - m + 1)!}{(L + m)!} \{P^m_{L+1}(\tau)\, q^m_L(\tau) - P^m_L(\tau)\, q^m_{L+1}(\tau)\} \tag{31}$$

and

$$(1 - \tau^2)^{m/2} P^m_{L+1}(\tau)\, \lambda^m(\tau) = (1 - \tau^2)^m g^m_{L+1}(\tau) - 2\tau\, \psi^m(\tau)\, q^m_{L+1}(\tau), \tag{32}$$

where

$$q^m_l(\tau) = \frac{1}{2} P \int_{-1}^{1} (1 - \mu^2)^{m/2} P^m_l(\mu) \frac{d\mu}{\tau - \mu}. \tag{33}$$

Now since $\lambda^m(\tau_0)$ and $\psi^m(\tau_0)$ are real, we see from Eq. (28) that $[\Lambda^m(\tau_0)]^{\pm} = 0$, $\tau_0 \in (-1, 1)$, implies that $\lambda^m(\tau_0) = \psi^m(\tau_0) = 0$. Equation (32) thus yields $g^m_{L+1}(\tau_0) = 0$, and Eq. (24) yields $P^m_{L+1}(\tau_0)\, g^m_L(\tau_0) = 0$. It is clear from Eq. (21) that $g^m_L(\tau_0)$ and $g^m_{L+1}(\tau_0)$ cannot both be zero, and we conclude that $P^m_{L+1}(\tau_0) = 0$. We thus have a contradiction since Eqs. (30) and (31) clearly show that $P^m_{L+1}(\tau_0)$ and $g^m_{L+1}(\tau_0)$ cannot both be zero. It therefore follows that $[\Lambda^m(\tau_0)]^{\pm} \neq 0$ for $\tau_0 \in (-1, 1)$.

Finally we consider $z = \pm 1$. We note that for the case $m = 0$, the left-hand side of Eq. (31) is unity, and thus the arguments used to show that $\lambda^0(\tau)$ and $\psi^0(\tau)$ do not have a common zero for $\tau \in (-1, 1)$ clearly are valid for $\tau \in [-1, 1]$. For $m \geqq 1$ it is apparent from Eq. (15) that $\psi^m(\pm 1) = 0$, and we can use the fact that $Q^m_l(1) = 2^{m-1}(m-1)!$, $l \geqq m$, in Eq. (23) to deduce the condition for $\Lambda^m(\pm 1) = 0$:

$$g^m_{L+1}(1) = g^m_L(1). \tag{34}$$

Equation (34) can be readily used to check, for specific values of c and f_l, $l = 0, 1, 2, \ldots, L$, if $\Lambda^m(\pm 1) = 0$. We consider two simple examples. For linearly anisotropic scattering and $m = L = 1$, the condition given by Eq. (34) reduces to

$$c f_1 = \tfrac{2}{3}. \tag{35}$$

Since the physical restriction that $p(\cos\Theta) \geqq 0$, $0 \leqq \Theta \leqq \pi$, implies $|f_1| \leqq 1/3$ we see that Eq. (35) can be satisfied only for $0 < f_1 \leqq 1/3$ and $c \geqq 2$. In the case of quadratically anisotropic scattering, the requirement that $p(\cos\Theta) \geqq 0$, $0 \leqq \Theta \leqq \pi$, implies [11]

$$|f_1| \leqq \frac{1}{3}(1 + 5f_2), \quad -\frac{1}{5} \leqq f_2 \leqq \frac{1}{10}, \tag{36a}$$

and

$$f_1^2 \leqq \frac{5}{3} f_2 (2 - 5f_2), \quad \frac{1}{10} \leqq f_2 \leqq \frac{2}{5}. \tag{36b}$$

For $m = 1$ and $L = 2$, Eq. (34) reduces to

$$5 f_1 f_2 c^2 - (3f_1 + 5f_2)c + 2 = 0. \tag{37}$$

Thus, for a given c, the values of f_1 and f_2 that yield $\Lambda^m(\pm 1) = 0$, $m = 1$, comprise, in this case, the hyperbolic segment given by Eq. (37) and contained in the region defined by Eqs. (36) in the $f_1 - f_2$ plane. It should be pointed out here that this can happen even for $c \leqq 1$, the case of interest for radiative transfer applications. The set $f_1 = 1/(2c)$, $f_2 = 1/(5c)$, $c = 0.95$, for example, satisfies Eq. (36b) and yields $\Lambda^m(\pm 1) = 0$, $m = 1$. For $m = 2$ and $L = 2$, Eq. (34) reduces to

$$c f_2 = \tfrac{4}{5} \tag{38}$$

which can be satisfied only for $0 < f_2 \leqq \frac{2}{5}$ and $c \geqq 2$. In conclusion we note that, in contrast to the $m = 0$ case where even for those situations for which $\psi^0(\pm 1) = 0$ we cannot have $\Lambda^0(\pm 1) = 0$, we have found and demonstrated by the foregoing examples the surprising result that $\Lambda^m(\pm 1)$ can be zero for $m \geqq 1$.

Acknowledgement

The authors are grateful to C. van der Mee for several helpful suggestions concerning this work, to D. H. Roy for the interest shown in this study and to the Babcock & Wilcox Company for partial support of this research, which was also supported in part by the National Science Foundation. One of the authors (R.D.M.G.) wishes also to acknowledge the financial support of Comissão Nacional de Energia Nuclear and Instituto de Pesquisas Energéticas e Nucleares, both of Brazil.

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Abstract

Elementary considerations are used to show in regard to particle transport theory that the dispersion function relevant to each of the Fourier-component problems resulting from a finite Legendre expansion of the scattering law cannot have a zero for $\nu \in (-1, 1)$, and a condition for the endpoints ± 1 to be zeros is reported. It is also shown that the dispersion function and the characteristic function cannot, for a given Fourier-component problem, vanish simultaneously for any value of $z \notin [-1, 1]$.

Zusammenfassung

Mit elementaren Betrachtungen der Teilchentransport-Theorie wird gezeigt, daß die Dispersionsfunktion für jedes Fourier-Komponentenproblem, welches durch eine endliche Legendre-Entwicklung des Streuungsproblems entsteht, keine Nullstellen für $\nu \in (-1, 1)$ hat; eine Bedingung für Nullstellen an den Endpunkten ± 1 wird angegeben. Es wird gezeigt, daß die Dispersionsfunktion und die charakteristische Funktion für ein gegebenes Fourier-Komponenten-Problem nicht gleichzeitig verschwinden können für Werte $z \notin [-1, 1]$.

(Received: February 23, 1982.)