The Critical Problem for an Infinite Cylinder

J. R. Thomas, Jr. and J. D. Southers

Virginia Polytechnic Institute and State University
Nuclear Engineering Group, Blacksburg, Virginia 24061

and

C. E. Siewert

North Carolina State University
Departments of Mathematics and Nuclear Engineering
Raleigh, North Carolina 27650

Received November 18, 1982
Accepted January 1, 1983

The $F_N$ method is used to compute the critical radius and the flux distribution for a bare cylinder of infinite length. With modest computational effort, the developed solution technique, though approximate, yields results accurate to at least six significant figures.

I. INTRODUCTION

The integral equation for the neutron flux distribution $\phi(r)$ in a bare homogeneous right circular cylinder of infinite length and radius $R$ was written by Mitsis$^1$ for the case of no inhomogeneous source term and no incident neutrons as

$$
\phi(r) = c \int_0^1 \left[ K_0(r/\mu) \int_0^r t\phi(t)I_0(t/\mu)dt 
+ I_0(r/\mu) \int_r^R t\phi(t)K_0(t/\mu)dt \right] \frac{d\mu}{\mu^2},
$$

(1)

where $I_0(x)$ and $K_0(x)$ denote modified Bessel functions$^2$ and $c$ is the mean number of secondary neutrons per collision. Equation (1) is, of course, based on a one-speed model, and we have assumed that the redistribution of secondary neutrons is isotropic. In this work we seek, for a given value of $c > 1$, the critical radius $R$ and the resulting non-negative neutron flux $\phi(r)$, $r \in [0,R]$ that satisfies Eq. (1).

Following Mitsis,$^1$ we let

$$
\Phi(r,\mu) = c \left[ K_0(r/\mu) \int_0^r t\phi(t)I_0(t/\mu)dt 
+ I_0(r/\mu) \int_r^R t\phi(t)K_0(t/\mu)dt \right]
$$

(2)

so that

$$
\phi(r) = \int_0^1 \Phi(r,\mu) \frac{d\mu}{\mu^2}.
$$

(3)

Differentiating Eq. (2), we find that $\Phi(r,\mu)$ for $\mu \in [0,1]$ and $r \in [0,R]$ satisfies

$$
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{\mu^2} \right) \Phi(r,\mu) = -c \int_0^1 \Phi(r,\mu') \frac{d\mu'}{\mu'}
$$

(4)

subject to the conditions$^1$ that $\Phi(0,\mu)$ is bounded and

$$
K_1(R/\mu)\Phi(R,\mu) + \mu K_0(R/\mu) \frac{\partial}{\partial r} \Phi(r,\mu) \bigg|_{r=R} = 0,
$$

$$
\mu \in [0,1].
$$

(5)
We deduce a solution (bounded at \( r = 0 \)) to the pseudo-problem defined by Eqs. (4) and (5). Continuing to follow Mitsis,\(^1\) we write

\[
\Phi(r, \mu) = \mu^2 \left\{ A(\nu_0)[\phi(\nu_0, \mu) + \phi(-\nu_0, \mu)]I_0(r/\nu_0) + \int_0^1 A(\nu)[\phi(\nu, \mu) + \phi(-\nu, \mu)]I_0(r/\nu)d\nu \right\},
\]

(6)

where

\[
\phi(\pm \nu, \mu) = \frac{c}{2} \nu \nu_0 \left( \frac{1}{\nu_0 + \mu} \right) + (1 - c \nu \tanh^{-1} \nu) \delta(\nu \mp \mu),
\]

\( \nu \in (0,1) \),

(7)

and

\[
\phi(\pm \nu_0, \mu) = \frac{c}{2} \nu_0 \left( \frac{1}{\nu_0 + \mu} \right)
\]

(8)

are the familiar (generalized) functions appropriate to one-speed neutron-transport theory\(^3\) in plane geometry. Here the discrete eigenvalue \( \nu_0 \) is the “positive” solution of

\[
1 + \frac{c \nu_0}{2} \int_1^1 \frac{d\mu}{\mu - \nu_0} = 0.
\]

(9)

Note that at this point in his analysis, Mitsis\(^1\) substituted Eq. (6) into Eq. (5) and investigated the resulting equation for the expansion coefficients \( A(\nu_0) \) and \( A(\nu) \). In more recent work, Westfall and Metcalf\(^5\) and Westfall\(^6\) carried the Mitsis analysis to completion and deduced accurate numerical results for \( R \) and \( \Phi(r) \), \( r \in [0, R] \). We take a considerably different approach here. We use\(^4\) the full-range orthogonality condition

\[
(\xi - \xi') \int_1^1 \mu \phi(\xi, \mu) \phi(\xi', \mu) d\mu = 0
\]

(10)

to deduce from Eq. (6) that

\[
\int_0^1 [\phi(\xi, \mu) - \phi(-\xi, \mu)] \Phi(r, \mu) \frac{d\mu}{\mu} = A(\xi)I_0(r/\xi)N(\xi)
\]

(11a)

and

\[
\int_0^1 [\phi(\xi, \mu) - \phi(-\xi, \mu)] \frac{\partial}{\partial r} \Phi(r, \mu) \frac{d\mu}{\mu} = \frac{1}{\xi} A(\xi)I_1(r/\xi)N(\xi)
\]

(11b)

for \( \xi \in P = \nu_0 \cup [0,1] \). Here

\[
N(\nu_0) = \frac{c}{2} \nu_0^3 \left( \frac{c}{\nu_0^2 - 1} - \frac{1}{\nu_0^2} \right)
\]

(12a)

and

\[
N(\nu) = \nu \left[ (1 - c \nu \tanh^{-1} \nu)^2 + \frac{c^2 \nu^2 \pi^2}{4} \right].
\]

(12b)

We can eliminate between Eqs. (11a) and (11b) to obtain, for \( \xi \in P \),

\[
\int_0^1 [\phi(\xi, \mu) - \phi(-\xi, \mu)] \times \left[ \Phi(r, \mu) - \xi \Upsilon(r/\xi) \frac{\partial}{\partial r} \Phi(r, \mu) \right] \frac{d\mu}{\mu} = 0,
\]

(13)

where

\[
\Upsilon(x) = I_0(x)/I_1(x)
\]

(14)

We can now set \( r = R \) in Eq. (13) and use Eq. (5) to deduce that

\[
\int_0^1 [\phi(\xi, \mu) - \phi(-\xi, \mu)] \times \left[ \mu + \xi \Upsilon(R/\xi) \Xi(R/\mu) \Phi(R, \mu) \right] \frac{d\mu}{\mu^2} = 0
\]

(15)

for \( \xi \in P \). Here

\[
\Xi(x) = K_0(x)/K_0(\xi)
\]

(16)

Note that Eq. (15), which has been derived without approximation, represents a singular-integral equation and constraint for the unknown function \( \Phi(R, \mu) \).

In Sec. II we use approximate analysis to deduce \( R \) from Eq. (15).

II. THE \( F_N \) SOLUTION

Following previous work with the \( F_N \) method,\(^7\) we introduce the approximation

\[
\Phi(R, \mu) = \mu^2 \sum_{a=0}^N a_a \mu^a
\]

(17)

into Eq. (15) to find, for \( \xi \in P \),

\[
\sum_{a=0}^N a_a [E_a(\xi) + \Upsilon(R/\xi)D_a(\xi)] = 0,
\]

(18)

where

\[
E_a(\xi) = \frac{1}{\xi} \int_0^1 \mu^{a+1} [\phi(\xi, \mu) - \phi(-\xi, \mu)] d\mu
\]

(19)


and

\[ D_\alpha(\xi) = \int_0^1 \mu^\alpha \left[ \phi(\xi, \mu) - \phi(-\xi, \mu) \right] \Xi(R/\mu) d\mu . \]  

(20)

We can readily deduce that, for \( c > 1 \),

\[ E_\alpha(\xi) = 1 - c , \quad \xi \in P , \]  

(21a)

\[ E_\alpha(\nu) = -\frac{c}{2} \left[ 1 - |\nu|_0^2 \ln \left( 1 + \frac{1}{|\nu|_0^2} \right) \right] , \]  

(21b)

\[ E_\alpha(\nu_0) = -\frac{c}{2} \left[ 1 - |\nu_0|_0^2 \ln \left( 1 + \frac{1}{|\nu_0|_0^2} \right) \right] , \]  

(21c)

\[ D_\alpha(\nu_0) = -ic |\nu_0| \int_0^1 \mu \Xi(R/\mu) \frac{d\mu}{\mu^2 + |\nu_0|_0^2} , \]  

(22a)

\[ D_\alpha(\nu_0) = -ic |\nu_0| \int_0^1 \mu^2 \Xi(R/\mu) \frac{d\mu}{\mu^2 + |\nu_0|_0^2} , \]  

(22b)

\[ D_\alpha(\nu_0) = -ic |\nu_0| \int_0^1 \mu^2 \Xi(R/\mu) \frac{d\mu}{\mu^2 + |\nu_0|_0^2} , \]  

(22c)

\[ D_\alpha(\nu_0) = -ic |\nu_0| \int_0^1 \mu^2 \Xi(R/\mu) \frac{d\mu}{\mu^2 + |\nu_0|_0^2} , \]  

(22d)

We can subsequently use the recursion formulas

\[ E_\alpha(\xi) = \xi^2 E_{\alpha-2}(\xi) - \frac{c}{\alpha + 1} \]  

(23)

and

\[ D_\alpha(\xi) = \xi^2 D_{\alpha-2}(\xi) - c \xi \int_0^1 \mu^{\alpha-1} \Xi(R/\mu) d\mu \]  

(24)

to evaluate the functions \( E_\alpha(\xi) \) and \( D_\alpha(\xi) \) for all \( \xi \in P \).

As Eq. (18) is homogeneous in the desired coefficients \( \{a_\alpha\} \), we normalize our solution by taking \( a_0 = -1 \). Subsequently, we consider Eq. (18) at a set of collocation points \( \xi = \xi_\beta \) defined by \( \xi_0 = \nu_0 \) and

\[ \xi_\beta = \frac{1}{2} + \frac{1}{2} \cos \left( \frac{2\beta - 1}{2N} \pi \right) , \quad \beta = 1, 2, \ldots, N . \]

(25)

Thus, to find the desired constants \( a_\alpha, \alpha = 1, 2, \ldots, N \), and the critical radius, we must solve the system of linear algebraic equations

\[ \sum_{\alpha=1}^N a_\alpha [E_\alpha(\xi_\beta) + \gamma(R/\xi_\beta) D_\alpha(\xi_\beta)] = 1 - c + \gamma(R/\xi_\beta) D_\alpha(\xi_\beta) \]  

(26)

for \( \beta = 1, 2, \ldots, N \), subject to the critical condition

\[ \sum_{\alpha=1}^N a_\alpha [E_\alpha(\nu_0) + iD_\alpha(\nu_0) U(R/\nu_0)] = 1 - c + iD_\alpha(\nu_0) U(R/\nu_0) , \]

(27)

where

\[ U(R/\nu_0) = J_\delta(R/|\nu_0|)/J_\delta(R/|\nu_0|) . \]

(28)

It is apparent that once we have deduced the critical radius \( R \) and the constants \( \{a_\alpha\} \), we can readily compute the desired flux distribution \( \phi(r) \) from Eqs. (3), (6), and (11a). We find, for \( r \in [0,R] \),

\[ \phi(r) = \frac{\nu_0}{N(\nu_0)} J(r/\nu_0) \sum_{\alpha=0}^N a_\alpha E_\alpha(\nu_0) + \int_0^1 \frac{\nu}{N(\nu)} I(r/\nu) \sum_{\alpha=0}^N a_\alpha E_\alpha(\nu) d\nu , \]

(29)

where

\[ J(r/\nu_0) = J_\delta(r/|\nu_0|)/J_\delta(R/|\nu_0|) \]

(30a)

and

\[ I(r/\nu) = I_\delta(r/\nu)/I_\delta(R/\nu) . \]

(30b)

III. NUMERICAL RESULTS

For a given value of \( c \), we first compute \( \nu_0 \) and evaluate the functions \( E_\alpha(\xi) \) at the collocation points defined in Sec. II. We then assume an initial value of \( R \), evaluate the functions \( D_\alpha(\xi) \) at the collocation points, and solve (for fixed \( N \)) the linear system given by Eq. (26) to find the constants \( \{a_\alpha\} \). Using these constants, we then solve Eq. (27), the critical condition, to find a corrected value of \( R \). The procedure is then repeated until a converged

<table>
<thead>
<tr>
<th>( c )</th>
<th>( F_N )</th>
<th>Westfall (Ref. 6)</th>
<th>Sanchez (Ref. 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.01</td>
<td>13.12551647</td>
<td>9.043255</td>
<td>13.12551647</td>
</tr>
<tr>
<td>1.02</td>
<td>9.04325484</td>
<td>5.41128828</td>
<td>9.043255</td>
</tr>
<tr>
<td>1.05</td>
<td>5.41128828</td>
<td>3.57739129</td>
<td>5.411288</td>
</tr>
<tr>
<td>1.1</td>
<td>3.57739129</td>
<td>2.28720926</td>
<td>3.577391</td>
</tr>
<tr>
<td>1.2</td>
<td>2.28720926</td>
<td>1.72500292</td>
<td>2.287209</td>
</tr>
<tr>
<td>1.3</td>
<td>1.72500292</td>
<td>1.39697859</td>
<td>1.72500292</td>
</tr>
<tr>
<td>1.4</td>
<td>1.39697859</td>
<td>1.17834084</td>
<td>1.396979</td>
</tr>
<tr>
<td>1.5</td>
<td>1.17834084</td>
<td>1.02083901</td>
<td>1.17834084</td>
</tr>
<tr>
<td>1.6</td>
<td>1.02083901</td>
<td>0.80742662</td>
<td>1.020839</td>
</tr>
<tr>
<td>1.8</td>
<td>0.80742662</td>
<td>0.668613</td>
<td>0.807427</td>
</tr>
<tr>
<td>2.0</td>
<td>0.668613</td>
<td>0.66861287</td>
<td></td>
</tr>
</tbody>
</table>
value of $R$ is obtained. We then increase $N$ and repeat the calculation until we find our final results for the critical radius. Some final results obtained with $N \leq 19$ are shown in Table I along with what we believe to be particularly accurate results of Westfall\textsuperscript{6} and Sanchez.\textsuperscript{10}

We find that, for $1.01 \leq c \leq 1.6$, the critical radius is converged to six significant figures for $N \leq 7$; whereas, for $c$ of 1.8 or 2.0, $N = 9$ is required for the same convergence. We have also evaluated Eq. (29) for selected values of $c$ to obtain flux distributions that agree to all (four) of the decimal places given by Westfall.\textsuperscript{6} In Table II we report normalized flux distributions that are correct, we believe, to within $\pm 1$ in the last digit given.

\begin{table}[h]
\centering
\caption{The Normalized Flux Distribution $\phi(r)/\phi(0)$}
\begin{tabular}{|c|c|c|c|c|}
\hline
\multicolumn{2}{c}{r/R} & \multicolumn{3}{c}{c} \\
\hline & 1.05 & 1.1 & 1.6 & 2.0 \\
\hline 0 & 1.00 & 1.00 & 1.00 & 1.00 \\
0.25 & 0.929851 & 0.936052 & 0.954996 & 0.959783 \\
0.50 & 0.733990 & 0.756084 & 0.824845 & 0.842634 \\
0.75 & 0.452168 & 0.492189 & 0.621823 & 0.656963 \\
0.85 & 0.326662 & 0.371791 & 0.522344 & 0.564397 \\
0.91 & 0.249166 & 0.296040 & 0.456627 & 0.502561 \\
0.95 & 0.195805 & 0.243013 & 0.408837 & 0.457217 \\
0.98 & 0.153085 & 0.199922 & 0.368807 & 0.418991 \\
1 & 0.117908 & 0.164122 & 0.335065 & 0.386649 \\
\hline
\end{tabular}
\end{table}


\section*{ACKNOWLEDGMENTS}

One of the authors (CES) is grateful to D. H. Roy for the interest shown in this investigation.

This work was supported in part by the Babcock and Wilcox Company and the U.S. National Science Foundation.