The Critical Problem for an Infinite Cylinder

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Received November 18, 1982
Accepted January 1, 1983

The F_N method is used to compute the critical radius and the flux distribution for a bare cylinder of infinite length. With modest computational effort, the developed solution technique, though approximate, yields results accurate to at least six significant figures.

I. INTRODUCTION

The integral equation for the neutron flux distribution $\phi(r)$ in a bare homogeneous right circular cylinder of infinite length and radius R was written by Mitsis[1] for the case of no inhomogeneous source term and no incident neutrons as

$$\phi(r) = c \int_0^1 \left[K_0(r/\mu) \int_0^r t\phi(t) I_0(t/\mu) dt + I_0(r/\mu) \int_r^R t\phi(t) K_0(t/\mu) dt \right] \frac{d\mu}{\mu^2} , \quad (1)$$

where $I_0(x)$ and $K_0(x)$ denote modified Bessel functions[2] and c is the mean number of secondary neutrons per collision. Equation (1) is, of course, based on a one-speed model, and we have assumed that the redistribution of secondary neutrons is isotropic. In this work we seek, for a given value of $c > 1$, the critical radius R and the resulting non-negative neutron flux $\phi(r)$, $r \in [0,R]$ that satisfies Eq. (1).

Following Mitsis,[1] we let

$$\Phi(r,\mu) = c \left[K_0(r/\mu) \int_0^r t\phi(t) I_0(t/\mu) dt + I_0(r/\mu) \int_r^R t\phi(t) K_0(t/\mu) dt \right] \quad (2)$$

so that

$$\phi(r) = \int_0^1 \Phi(r,\mu) \frac{d\mu}{\mu^2} . \quad (3)$$

Differentiating Eq. (2), we find that $\Phi(r,\mu)$ for $\mu \in [0,1]$ and $r \in [0,R]$ satisfies

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{\mu^2} \right) \Phi(r,\mu) = -c \int_0^1 \Phi(r,\mu') \frac{d\mu'}{\mu'^2} \quad (4)$$

subject to the conditions[1] that $\Phi(0,\mu)$ is bounded and

$$K_1(R/\mu)\Phi(R,\mu) + \mu K_0(R/\mu) \frac{\partial}{\partial r} \Phi(r,\mu)\Big|_{r=R} = 0 , \quad \mu \in [0,1] . \quad (5)$$

[1]G. J. MITSIS, "Transport Solutions to the Monoenergetic Critical Problems," ANL-6787, Argonne National Laboratory (1963).

[2]*Handbook of Mathematical Functions*, AMS-55, M. ABRAMOWITZ and I. A. STEGUN, Eds., U.S. National Bureau of Standards (1964).

We deduce a solution (bounded at $r = 0$) to the pseudo-problem defined by Eqs. (4) and (5). Continuing to follow Mitsis,[1] we write

$$\Phi(r,\mu) = \mu^2\left\{A(\nu_0)[\phi(\nu_0,\mu) + \phi(-\nu_0,\mu)]I_0(r/\nu_0) + \int_0^1 A(\nu)[\phi(\nu,\mu) + \phi(-\nu,\mu)]I_0(r/\nu)d\nu\right\} , \quad (6)$$

where

$$\phi(\pm\nu,\mu) = \frac{c}{2}\nu Pv\left(\frac{1}{\nu \mp \mu}\right) + (1 - c\nu \tanh^{-1}\nu)\delta(\nu \mp \mu) , \quad \nu \in (0,1) , \quad (7)$$

and

$$\phi(\pm\nu_0,\mu) = \frac{c}{2}\nu_0\left(\frac{1}{\nu_0 \mp \mu}\right) \quad (8)$$

are the familiar (generalized) functions appropriate to one-speed neutron-transport theory[3,4] in plane geometry. Here the discrete eigenvalue ν_0 is the "positive" solution of

$$1 + \frac{c\nu_0}{2}\int_{-1}^{1}\frac{d\mu}{\mu - \nu_0} = 0 . \quad (9)$$

Note that at this point in his analysis, Mitsis[1] substituted Eq. (6) into Eq. (5) and investigated the resulting equation for the expansion coefficients $A(\nu_0)$ and $A(\nu)$. In more recent work, Westfall and Metcalf[5] and Westfall[6] carried the Mitsis analysis to completion and deduced accurate numerical results for R and $\phi(r)$, $r \in [0,R]$. We take a considerably different approach here. We use[4] the full-range orthogonality condition

$$(\xi - \xi')\int_{-1}^{1}\mu\phi(\xi,\mu)\phi(\xi',\mu)d\mu = 0 \quad (10)$$

to deduce from Eq. (6) that

$$\int_0^1[\phi(\xi,\mu) - \phi(-\xi,\mu)]\,\Phi(r,\mu)\frac{d\mu}{\mu} = A(\xi)I_0(r/\xi)N(\xi) \quad (11a)$$

and

$$\int_0^1[\phi(\xi,\mu) - \phi(-\xi,\mu)]\,\frac{\partial}{\partial r}\Phi(r,\mu)\frac{d\mu}{\mu} = \frac{1}{\xi}A(\xi)I_1(r/\xi)N(\xi) \quad (11b)$$

for $\xi \in P = \nu_0 \cup [0,1]$. Here

$$N(\nu_0) = \frac{c}{2}\nu_0^3\left(\frac{c}{\nu_0^2 - 1} - \frac{1}{\nu_0^2}\right) \quad (12a)$$

and

$$N(\nu) = \nu\left[(1 - c\nu\tanh^{-1}\nu)^2 + \frac{c^2\nu^2\pi^2}{4}\right] . \quad (12b)$$

We can eliminate between Eqs. (11a) and (11b) to obtain, for $\xi \in P$,

$$\int_0^1[\phi(\xi,\mu) - \phi(-\xi,\mu)] \times\left[\Phi(r,\mu) - \xi\Upsilon(r/\xi)\frac{\partial}{\partial r}\Phi(r,\mu)\right]\frac{d\mu}{\mu} = 0 , \quad (13)$$

where

$$\Upsilon(x) = I_0(x)/I_1(x) . \quad (14)$$

We can now set $r = R$ in Eq. (13) and use Eq. (5) to deduce that

$$\int_0^1[\phi(\xi,\mu) - \phi(-\xi,\mu)] \times[\mu + \xi\Upsilon(R/\xi)\Xi(R/\mu)]\Phi(R,\mu)\frac{d\mu}{\mu^2} = 0 \quad (15)$$

for $\xi \in P$. Here

$$\Xi(x) = K_1(x)/K_0(x) . \quad (16)$$

Note that Eq. (15), which has been derived without approximation, represents a singular-integral equation and constraint for the unknown function $\Phi(R,\mu)$. In Sec. II we use approximate analysis to deduce R from Eq. (15).

II. THE F_N SOLUTION

Following previous work with the F_N method,[7-9] we introduce the approximation

$$\Phi(R,\mu) = \mu^2\sum_{\alpha=0}^{N}a_\alpha\mu^\alpha \quad (17)$$

into Eq. (15) to find, for $\xi \in P$,

$$\sum_{\alpha=0}^{N}a_\alpha[E_\alpha(\xi) + \Upsilon(R/\xi)D_\alpha(\xi)] = 0 , \quad (18)$$

where

$$E_\alpha(\xi) = \frac{1}{\xi}\int_0^1\mu^{\alpha+1}[\phi(\xi,\mu) - \phi(-\xi,\mu)]d\mu \quad (19)$$

[3] B. DAVISON, *Neutron Transport Theory*, Oxford University Press, London (1957).

[4] K. M. CASE, *Ann. Phys.*, **9**, 1 (1960).

[5] R. M. WESTFALL and D. R. METCALF, *Trans. Am. Nucl. Soc.*, **15**, 266 (1972).

[6] R. M. WESTFALL, PhD Thesis, University of Virginia (1974).

[7] C. E. SIEWERT and P. BENOIST, *Nucl. Sci. Eng.*, **69**, 156 (1979).

[8] R. D. M. GARCIA and C. E. SIEWERT, *Nucl. Sci. Eng.*, **76**, 53 (1980).

[9] R. D. M. GARCIA and C. E. SIEWERT, *Nucl. Sci. Eng.*, **81**, 474 (1982).

and

$$D_\alpha(\xi) = \int_0^1 \mu^\alpha[\phi(\xi,\mu) - \phi(-\xi,\mu)]\Xi(R/\mu)d\mu \ . \quad (20)$$

We can readily deduce that, for $c > 1$,

$$E_0(\xi) = 1 - c \ , \quad \xi \in P \ , \quad (21a)$$

$$E_1(\nu) = \nu - \frac{c}{2} - c\nu^2 \ln\left(1 + \frac{1}{\nu}\right) \ , \quad \nu \in [0,1] \ , \quad (21b)$$

$$E_1(\nu_0) = -\frac{c}{2}\left[1 - |\nu_0|^2 \ln\left(1 + \frac{1}{|\nu_0|^2}\right)\right] \ , \quad (21c)$$

$$D_0(\nu_0) = -ic|\nu_0| \int_0^1 \mu\Xi(R/\mu)\frac{d\mu}{\mu^2 + |\nu_0|^2} \ , \quad (22a)$$

$$D_1(\nu_0) = -ic|\nu_0| \int_0^1 \mu^2\Xi(R/\mu)\frac{d\mu}{\mu^2 + |\nu_0|^2} \ , \quad (22b)$$

$$D_0(\nu) = \Xi(R/\nu) - c\nu \int_0^1 \frac{\nu\Xi(R/\nu) - \mu\Xi(R/\mu)}{\nu^2 - \mu^2} d\mu \ , \quad \nu \in [0,1] \ , \quad (22c)$$

and

$$D_1(\nu) = \nu\Xi(R/\nu) - c\nu \int_0^1 \frac{\nu^2\Xi(R/\nu) - \mu^2\Xi(R/\mu)}{\nu^2 - \mu^2} d\mu \ , \quad \nu \in [0,1] \ . \quad (22d)$$

We can subsequently use the recursion formulas

$$E_\alpha(\xi) = \xi^2 E_{\alpha-2}(\xi) - \frac{c}{\alpha + 1} \quad (23)$$

and

$$D_\alpha(\xi) = \xi^2 D_{\alpha-2}(\xi) - c\xi \int_0^1 \mu^{\alpha-1}\Xi(R/\mu)d\mu \quad (24)$$

to evaluate the functions $E_\alpha(\xi)$ and $D_\alpha(\xi)$ for all $\xi \in P$.

As Eq. (18) is homogeneous in the desired coefficients $\{a_\alpha\}$, we normalize our solution by taking $a_0 = -1$. Subsequently, we consider Eq. (18) at a set of collocation points $\xi = \xi_\beta$ defined[9] by $\xi_0 = \nu_0$ and

$$\xi_\beta = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\beta - 1}{2N}\pi\right) \ , \quad \beta = 1, 2, \ldots, N \ . \quad (25)$$

Thus, to find the desired constants a_α, $\alpha = 1, 2, \ldots, N$, and the critical radius, we must solve the system of linear algebraic equations

$$\sum_{\alpha=1}^N a_\alpha[E_\alpha(\xi_\beta) + \Upsilon(R/\xi_\beta)D_\alpha(\xi_\beta)] = 1 - c + \Upsilon(R/\xi_\beta)D_0(\xi_\beta) \quad (26)$$

for $\beta = 1, 2, \ldots, N$, subject to the critical condition

$$\sum_{\alpha=1}^N a_\alpha[E_\alpha(\nu_0) + iD_\alpha(\nu_0)U(R/\nu_0)] = 1 - c + iD_0(\nu_0)U(R/\nu_0) \ , \quad (27)$$

where

$$U(R/\nu_0) = J_0(R/|\nu_0|)/J_1(R/|\nu_0|) \ . \quad (28)$$

It is apparent that once we have deduced the critical radius R and the constants $\{a_\alpha\}$, we can readily compute the desired flux distribution $\phi(r)$ from Eqs. (3), (6), and (11a). We find, for $r \in [0,R]$,

$$\phi(r) = \frac{\nu_0}{N(\nu_0)} J(r/\nu_0) \sum_{\alpha=0}^N a_\alpha E_\alpha(\nu_0) + \int_0^1 \frac{\nu}{N(\nu)} I(r/\nu) \sum_{\alpha=0}^N a_\alpha E_\alpha(\nu) d\nu \ , \quad (29)$$

where

$$J(r/\nu_0) = J_0(r/|\nu_0|)/J_0(R/|\nu_0|) \quad (30a)$$

and

$$I(r/\nu) = I_0(r/\nu)/I_0(R/\nu) \ . \quad (30b)$$

III. NUMERICAL RESULTS

For a given value of c, we first compute ν_0 and evaluate the functions $E_\alpha(\xi)$ at the collocation points defined in Sec. II. We then assume an initial value of R, evaluate the functions $D_\alpha(\xi)$ at the collocation points, and solve (for fixed N) the linear system given by Eq. (26) to find the constants $\{a_\alpha\}$. Using these constants, we then solve Eq. (27), the critical condition, to find a corrected value of R. The procedure is then repeated until a converged

TABLE I

The Critical Radius in Mean-Free-Paths

c	F_N	Westfall (Ref. 6)	Sanchez (Ref. 10)
1.01	13.12551647		13.12551647
1.02	9.04325484	9.043255	
1.05	5.41128828	5.411288	
1.1	3.57739129	3.577391	3.57739129
1.2	2.28720926	2.287209	
1.3	1.72500292		1.72500292
1.4	1.39697859	1.396979	
1.5	1.17834084		1.17834085
1.6	1.02083901	1.020839	
1.8	0.80742662	0.807427	
2.0	0.66861286	0.668613	0.66861287

value of R is obtained. We then increase N and repeat the calculation until we find our final results for the critical radius. Some final results obtained with $N \leqslant 19$ are shown in Table I along with what we believe to be particularly accurate results of Westfall[6] and Sanchez.[10]

We find that, for $1.01 \leqslant c \leqslant 1.6$, the critical radius is converged to six significant figures for $N \leqslant 7$; whereas, for c of 1.8 or 2.0, $N = 9$ is required for the same convergence. We have also evaluated Eq. (29) for selected values of c to obtain flux distributions that agree to all (four) of the decimal places given by Westfall.[6] In Table II we report normalized flux distributions that are correct, we believe, to within ±1 in the last digit given.

[10]R. SANCHEZ, "Generalisation of Asaoka's Method to Linearly Anisotropic Scattering: Benchmark Data in Cylindrical Geometry," CEA-N-1831, Centre d'Etudes Nucléaires de Saclay, Gif-sur-Yvette, France (1975).

TABLE II

The Normalized Flux Distribution $\phi(r)/\phi(0)$

r/R	$c = 1.05$	$c = 1.1$	$c = 1.6$	$c = 2.0$
0	1	1	1	1
0.25	0.929851	0.936052	0.954996	0.959783
0.50	0.733990	0.756084	0.824845	0.842634
0.75	0.452168	0.492189	0.621823	0.656963
0.85	0.326662	0.371791	0.522344	0.564397
0.91	0.249166	0.296040	0.456627	0.502561
0.95	0.195805	0.243013	0.408837	0.457217
0.98	0.153085	0.199922	0.368807	0.418991
1	0.117908	0.164122	0.335065	0.386649

ACKNOWLEDGMENTS

One of the authors (CES) is grateful to D. H. Roy for the interest shown in this investigation.

This work was supported in part by the Babcock and Wilcox Company and the U.S. National Science Foundation.